Representability of finite algebras of relations

by
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Abstract

Cayley's Theorem shows that groups correspond precisely to the isomorphism class of algebraic structures arising as systems of permutations on some set, under the operations of composition and inverse. This elementary fact underpins much of the application of group theory in modern mathematics: as algebras of symmetries. However, there are very many processes that cannot in general be faithfully reversed. We study situations such as this, in which the role of permutations is replaced by more general binary relations.

We survey the representability of reducts of Tarski's relation algebras as algebras of binary relations. In particular, we develop necessary conditions for the representability of semigroups as disjoint transformations. We also prove undecidable the problem of determining whether or not algebras in some reducts of Tarski's signature are representable.

Finally, we explore qualitative representability of nonassociative algebras. These are a broader class of algebras which includes Tarski's relation algebras. We determine the constraint satisfaction properties of small nonassociative algebras, and determine the representability of all nonassociative algebras on up to four atoms.

Statement of Authorship

This thesis includes work by the author that has been published or accepted for publication as described in the text. Except where reference is made in the text of the thesis, this thesis contains no other material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution. The author sought professional editing from Phillipa Telford in preparing this thesis.

Murray David Neuzerling
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Finally, I'd like to thank the mathematical community of Australia for so readily welcoming me as one of their own.

Notation and terminology

The most general algebraic notion considered here is that of an *algebra of relations*. It is intended that this phrase be rather ambiguous. The problem of deciding whether or not an algebra of relations is isomorphic to an algebra of binary relations is often undecidable, depending on the signature under consideration. Indeed, this is the motivation of much of this thesis. As such, we refer to a structure as an *algebra of relations* whenever we don't want to commit to anything more specific, even a particular signature.

One class of algebras of relations is specially defined. *Relation algebras* are defined by Tarski [86] with a specific signature and a short list of axioms. We will reserve the phrase *relation algebra* to refer to algebras of relations in the signature of and satisfying the axioms of Tarski.

We denote algebras by calligraphic letters with the underlying set denoted by the non-calligraphic letter. For example, an algebra $\mathcal{A}$ is defined over a set or domain A. We will usually adopt lower case latin letters $a, b, c, \ldots$ to represent elements. Where possible, we will denote idempotents by e or f.

When we discuss algebras of binary relations, which can be viewed as digraphs, we will reuse this notation by referring to the nodes of the graph as x, y, z, subscripting as necessary. We will reserve R, S, T for binary relations.

We denote the underlying associative binary operation of an algebra of relations with ;. In the world of binary relations, this becomes an associative composition operation $\circ$. In either case, we will often omit the notation, denoting $ab := a \cdot b$ for a semigroup and $RS := R \circ S$ for binary relations.

If R is a binary relation that relates element x to element y, we denote this $(x, y) \in R$ or xRy. This latter notation is particularly convenient when composing relations, so that if R and S are relations such that xRy and yRz, then $xRySz$ and so $x(RS)z$.

Define $\mathbb{N}$ to exclude 0. Finally, we denote the *proper* subset relation by $\subset$, as distinct from $\subseteq$.

Chapter 1

Relation algebras

1.1 Permutations and binary relations

Group theory did not begin with the axioms of a group. Group theory began with the study of permutations.

A *permutation* on a set X is a special case of a *binary relation*, which we view as a subset of the power set $\wp(X \times X)$. For example, the group S_3 can be thought of as a group of binary relations on the set $\{1, 2, 3\}$. The permutation (123), for example, can be viewed as $\{(1, 2), (2, 3), (3, 1)\}$.

From these binary relations, we can construct an edge-labelled digraph. We use the set X as vertices, and interpret an element (x, y) of $X \times X$ belonging to a binary relation r as an r-labelled edge from vertex x to vertex y. Figure 1.1 demonstrates this by interpreting the group S_3 as a digraph on $\{1, 2, 3\}$.

Groups are algebras, and so are equipped with operations. In particular, we have a binary function of composition, a unary function of inverse, and a nullary function (or constant) which is the identity. These operations can be interpreted in Figure 1.1. We see that (123) labels an arrow from vertex 1 to vertex 2, and its converse (132) labels an arrow going in the opposite direction. The identity permutation labels a loop on every vertex. We also see composition of group elements being represented in a very intuitive way as composition of binary relations: (12) maps vertex 1 to vertex 2, and (23) maps vertex 2 to vertex 3, so their composition (13) maps vertex 1 to vertex 3.

We can consider a group of permutations, both its elements and its operations, as an edge-labelled digraph. This is because a permutation is a special case of a binary relation. We'll be relying heavily on this interpretation throughout this entire thesis, so much so that we treat a binary relation and all edges having a given label in a digraph as one and the same.

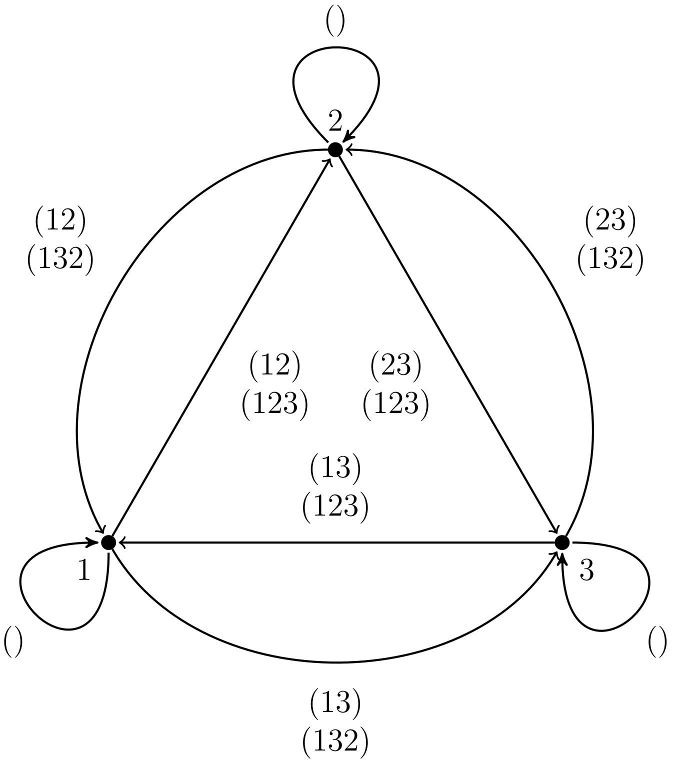

Figure 1.1: S_3 represented as permutations on the set $\{1, 2, 3\}$

Looking beyond groups as being simply systems of permutations, Walter von Dyck [23] was one of the first to treat groups with an axiomatic approach. He begins his paper with the following[1]:

> "To define a group of discrete operations, which are applied to a certain object, while abstracting from any special form of representation of the single objects and supposing the operations to be given only by those properties that are essential for the formation of the group."
>
> (Walter Dyck, 1882)

We see two 'worlds' here: the world of abstract groups, and the concrete world of groups of permutations. Von Dyck was interested in separating the worlds so that an abstract group could be considered without its "special form of representation" — its associated group of permutations or, as we consider it here, edge-labelled digraph of binary relations. The axioms that perfectly capture the world of abstract groups are, of course, the group axioms: associativity of composition, the presence of an identity element, and the existence of inverses. These group axioms capture the correspondence between the abstract world and the concrete world perfectly.

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[1] English translation available in [90].

Indeed, Cayley's Theorem tells us that the distinction between the world of abstract groups and the concrete world of groups of permutations is largely one of perspective. Cayley's Theorem[2] tells us that every abstract group is isomorphic to a group of permutations. We're viewing permutations as binary relations, so this gives us a specific interpretation of Cayley's Theorem as shown in Theorem 1.1.1. A proof may be found in any elementary group theory or abstract algebra textbook, such as [26, Theorem 6.1].

Theorem 1.1.1. *Every group is isomorphic to a group of permutations (binary relations).*

The situation is very neat for permutations. But why consider only permutations? What happens if we consider binary relations in general? Can we axiomatise some general abstract world of algebras in such a way that the correspondence between the abstract world and the concrete world is maintained? Can we construct a class of abstract algebras with a suitable signature of operations and with a corresponding form of Cayley's Theorem? Is there such a thing as a class of 'abstract algebras of relations' which are always isomorphic to a concrete algebra of binary relations?

We'll be constructing both the abstract and concrete worlds of algebras of relations side-by-side. We'll then be able to ask if an analogue to Cayley's Theorem exists.

The group signature—composition, inverse (or converse), and identity—capture an intuitive notion of what one would want to do with permutations. The same operations can be used with binary relations, but we can think of a few more 'natural' operations. Consider simple spatial relations like 'to my left'. I claim that the operations in Table 1.2 capture an intuitive notion of what one would want to do with binary relations, using spatial relations as an example. We'll also be introducing the symbols used for these operations.

Note that some of these operations can be derived from the others, as per Table 1.14. These operations exist in the abstract world, but they have corresponding notions in the concrete world of algebras of binary relations. This is useful because we will use our intuitions of the concrete world to understand the axioms of the abstract world. The intended interpretations of these operations are listed in Table 1.3.

We have an idea now of what our operations are for algebras of binary relations. There's nothing particularly special about these operations — one could use any signature — but they are considered 'natural' and have historical precedent. We know what these operations are in the abstract world and how they should look in the

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[2] Cayley did indeed show that the underlying set of a group is in bijection with that of a set of permutations [16], so this attribution is valid. He failed to show, however, that the bijection is an isomorphism [67]. This fact would be offered 16 years later by Jordan [47].

operation	symbol	intuition
composition	;	to the left of my left
join	$+$	to my left **or** right
meet	$\cdot$	to my left **and** ahead of me
order	$\leqslant$	“to my left” is less than “to my left or right”
converse	$\smile$	the converse of “to my left” is “to my right”
complement	$-$	not to my left
identity	$1'$	is equal to, is where I am
zero	0	constant that relates nothing to nothing
top	1	constant that is the largest relation in the domain

Table 1.2: The operations of an algebra of relations and their intuitive interpretation

abstract world	concrete world of binary relations
composition	composition of binary relations, $R \circ S = \{(x,y)\colon (\exists z)(x,z) \in R \text{ and } (z,y) \in S\}$
join	union of binary relations, $R \cup S$
meet	intersection of binary relations, $R \cap S$
order	set inclusion, $R \subseteq S$
converse	relational converse, $R^{-1} = \{(y,x)\colon (x,y) \in R\}$
complement	set complement $U \backslash R$, where U is as below[3]
identity	identity relation, $\{(x,x)\colon x \in X\}$
zero	empty set, $\mathrm{id}_X = \varnothing$
top	the biggest binary relation U in $\mathcal{A}$

Table 1.3: Interpreting the relation operations in an algebra of binary relations $\mathcal{A}$ over a set X

concrete world. Now let us consider what an actual algebra of binary relations would look like, with all the details. These exist in the concrete world, like groups of permutations.

Definition 1.1.2 ([36]). Let X be a set. A *proper relation algebra* with base set X is an algebra $\mathcal{A}$ with nonempty domain $A \subseteq \wp(X \times X)$ and signature $\{\circ, \cup, \cap, \subseteq, {}^{-1}, \backslash, \mathrm{id}_X, \mathsf{U}\}$ such that the following hold:

- A together with the operations in $\{\cup, \cap, \backslash, \varnothing, \mathsf{U}\}$ form a field of sets. That is, if $R, S \in A$ then $R \cup S, R \cap S, \mathsf{U} \backslash S \in A$. It follows that $\varnothing, \mathsf{U} \in A$. Also, U is the biggest binary relation in A, and so $\mathsf{U} = \cup A$,
- $\mathrm{id}_X \coloneqq \{(x,x)\colon x \in X\} \in A$, the identity relation over X,
- $\mathcal{A}$ is closed under taking converses: $R \in A$ implies $R^{-1} \in A$, where $R^{-1} = \{(y,x)\colon (x,y) \in R\}$,

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[3] Although complement is a unary operation, we can use it as a binary operation by letting $R \backslash S = R \cap (\mathsf{U} \backslash S)$. The same is true for the abstract operation.

- $\mathcal{A}$ is closed under composition of binary relations: $R, S \in A$ implies $R \circ S \in A$ where
$$R \circ S = \{(x, y) \colon (\exists z)(x, z) \in R \text{ and } (z, y) \in S\}.$$

In 1941 Tarski [86] offered a class of abstract algebras in the signature above along with a finite set of axioms[4]. Tarski referred to an algebra in this signature and satisfying these axioms as a *relation algebra*. This language is not ideal for two reasons:

1. Not every relation algebra is isomorphic to a proper relation algebra, as we will soon see.
2. Not every algebra of relations is a relation algebra. There are other algebras of relations with different signatures or axioms.

We will always refer to an algebra in the signature we have described and satisfying Tarski's axioms as a *relation algebra*, and reserve the more general term *algebra of relations* when we wish to be less specific with respect to either signature or axioms.

Definition 1.1.3. An algebra $\mathcal{A}$ over domain A with signature $\{;, +, \cdot, \leqslant, \breve{}, -, 1', 0, 1\}$ is a *relation algebra* if for all $a, b, c \in A$ the following hold:

1. $a + b = b + a,$
2. $a + (b + c) = (a + b) + c,$
3. $-(-a + -b) + -(-a + b) = a,$
4. $a \mathbin{;} (b \mathbin{;} c) = (a \mathbin{;} b) \mathbin{;} c,$
5. $(a + b) \mathbin{;} c = a \mathbin{;} c + b \mathbin{;} c,$
6. $a \mathbin{;} 1' = a,$
7. $(a^{\smile})^{\smile} = a,$
8. $(a + b)^{\smile} = a^{\smile} + b^{\smile},$
9. $(a \mathbin{;} b)^{\smile} = b^{\smile} \mathbin{;} a^{\smile},$
10. $(a^{\smile} \mathbin{;} -(a \mathbin{;} b)) + -b = -b,$
11. $1 = a + -a,$
12. $0 = -(a + -a).$

Tarski intended that proper relation algebras would be to relation algebras what permutation groups are to groups. That is, he intended that relation algebras would form the abstract counterpart to proper relation algebras.

Most of these axioms are fairly straightforward, and are easily seen to be satisfied in a proper relation algebra. The first three are those of a Boolean algebra (Defini-

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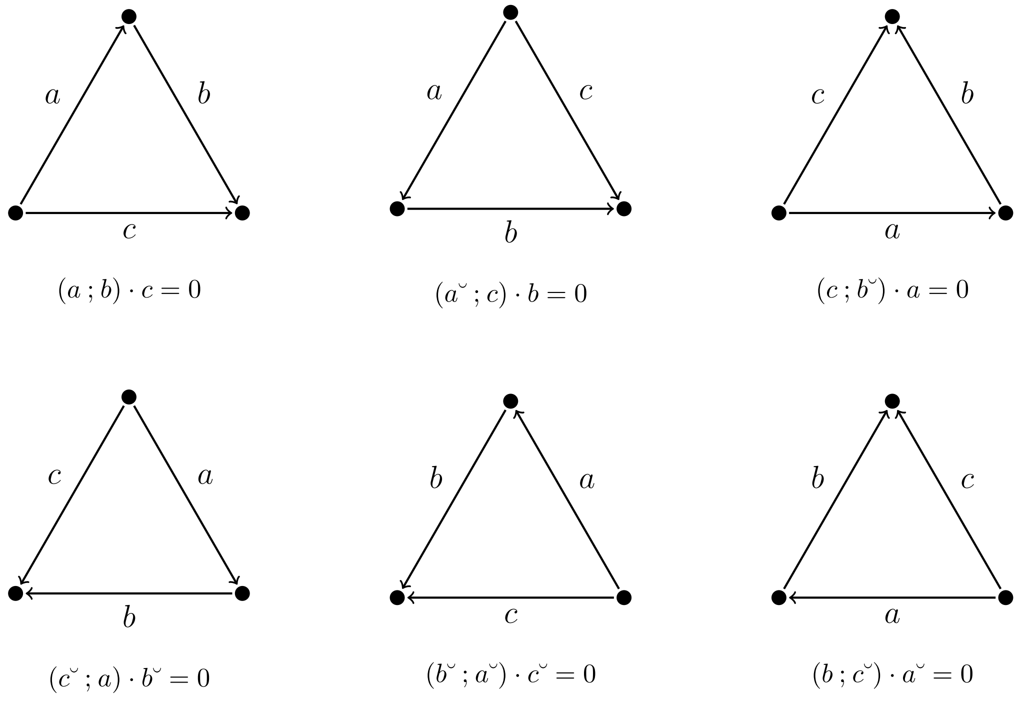
[4] Tarski developed his algebras in the 1941 paper, but the equational axioms were not developed until the mid '50s [87, 88]. We will be using the axioms as presented in [58, pp. 233, 289].

tion 1.2.1), and most of the others cover things like associativity or commutativity. Axiom 9 gives an interpretation of a particularly satisfying intuition we have regarding converse—we put our socks on before we put our shoes on, but we take our shoes off before we take our socks off. A proper treatment of the axioms can be found in Chapter 6 of [58]. Of all of these axioms, the only one which is not intuitively clear is the tenth.

Axiom 10, which we call the *Tarski axiom*, deals with the interaction between composition and converse in triangles. Consider an element $a \,;\, b$ which is disjoint from c; that is, $(a \,;\, b) \cdot c = 0$. Although we're working in the abstract world, Tarski's axioms are inspired by the concrete world of binary relations, so we also look there for inspiration. This condition excludes certain 'triangles' from appearing in a proper relation algebra. Note that each of the six triangles below is just a rotation of each of the others. These are called *De Morgan's equivalences*, and can be found in [21], although we present them as given in [58]. Each expression corresponds to a triangle which is *forbidden.*



$(a \,;\, b) \cdot c = 0$ $(a^{\smile} \,;\, c) \cdot b = 0$ $(c \,;\, b^{\smile}) \cdot a = 0$

$(c^{\smile} \,;\, a) \cdot b^{\smile} = 0$ $(b^{\smile} \,;\, a^{\smile}) \cdot c^{\smile} = 0$ $(b \,;\, c^{\smile}) \cdot a^{\smile} = 0$

Figure 1.4: De Morgan's equivalences.

Using all axioms in Definition 1.1.3 except for 4 and 6, one can derive De Morgan's equivalences [58, pp. 309], which is a statement of the equivalence of the six triangles above. Thus, the Tarski axiom is necessary to reproduce the intuitive notion that the triangles above are just rotations of each other, and so present the same information.

Lemma 1.1.4 (De Morgan's equivalences). *Let a, b, c be any three elements in a relation algebra. Then*

$$a\,;b \leqslant c \iff a^{\smile}\,;-c \leqslant -b \iff -c\,;b^{\smile} \leqslant -a.$$

The equivalence of the triangles in Figure 1.4 is also referred to by Hirsch and Hodkinson in [36] as the *Peircean law*. They go on to say that from this law it follows that "any 'triangle' of three elements of a relation algebra can be equivalently looked at in any of the six ways resulting from applying symmetries to it". This will be particularly useful to us when describing which compositions are allowed in a given relation algebra.

Call a *triple* (a, b, c) of relation algebra atoms *consistent* if $c \leqslant a\,;b$. In a concrete algebra of binary relations, this would be equivalent to seeing one of the triangles in Figure 1.4; the triangle is not forbidden but instead must be witnessed. If a triple is not consistent, call it *inconsistent* or *forbidden*. By considering symmetries, we can generate the six equivalent triples

$$\{(a, b, c), (a^{\smile}, c, b), (b, c^{\smile}, a^{\smile}), (b^{\smile}, a^{\smile}, c^{\smile}), (c^{\smile}, a, b^{\smile}), (c, b^{\smile}, a)\}.$$

These triples are *Peircean transforms* of one another. If one is consistent, then so are all the others. If one is forbidden, then so are all the others. We call all six triples taken together a *cycle* [58].

We can save time by specifying only one of the six triples of a cycle as a representative. It may even be expedient, as in Section 4.4, to write *abc* instead of (a, b, c). Note that a cycle contains six triples at most; for example, some of the atoms could be *self-converse*, also known as *symmetric*, giving fewer than six triples. Since $1'$ acts as an identity element, this forces certain cycles to exist in any relation algebra, for example $(1', a, a)$ for any element a. Since these cycles always exist we tend to ignore them in favour of cycles not guaranteed to exist, which we call *diversity cycles*.

We have defined the abstract and concrete worlds fairly well, so now we can explore the link between the two. Every proper relation algebra satisfies the axioms in Definition 1.1.3. But what about the other direction? Can we translate from the abstract to the concrete as easily as we can from the concrete to the abstract?

This question is at the core of the concept of a *representation* of a relation algebra. A given relation algebra is *representable* if it is isomorphic to a proper relation algebra, which is called a representation of that relation algebra. This is much like asking if a group is isomorphic to a group of permutations. In asking for a representation, we are asking if every operation in the signature can be interpreted as per Table 1.3 over *some* set.

Again, we exploit in our choice of language the fact that every representation can be viewed as an edge-labelled digraph. We will occasionally use the word *respect* here when referring to individual operations. For example, a representation respects identity when every vertex has an $1'$-labelled loop, and these are the only edges labelled $1'$. The formal notion of a representable relation algebra is given in Definition 1.1.5.

Definition 1.1.5. A relation algebra $\mathcal{A}$ over a domain A is *representable* if there exists a proper relation algebra $\mathcal{B}$ over a set X such that $\mathcal{A}$ is isomorphic to $\mathcal{B}$. The isomorphism $h : A \to \wp(X \times X)$ is called a *representation.*

Note that a representation is always a faithful function. The concept of a representation used here should not be confused with the representations of group theory.

We can break the representation of composition down into two major components: *composition moves* and *witness moves*[5]. Whenever we see a situation as in Figure 1.5 in a representation we must see it completed to Figure 1.6. This makes sense intuitively, since if we can relate x to y via a, and y to z via b, then we should be able to relate x to z via $a ; b$ directly.

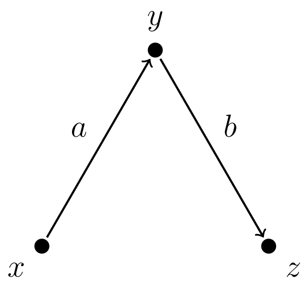


Figure 1.5: Composition move

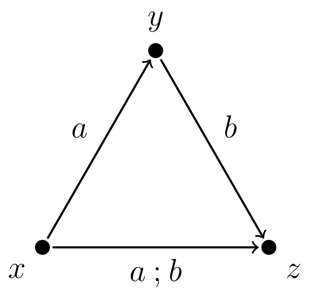


Figure 1.6: Composition response

If $c \leqslant a ; b$, then whenever we can relate x to z by c we should also be able to relate them by $a ; b$ through some third point y. This is a *witness move*, shown in Figures 1.7 and 1.8.

We can also be a bit stricter about the definition of the top element, 1. We are demanding that this is represented as the largest binary relation $U \subseteq X \times X$. Consider a relation algebra on two elements, $\{0, 1'\}$, and in which $1' = 1$. We can represent this on a single point x by a representation h such that $h(0) = \varnothing$ and $h(0) = \{(x, x)\}$. We could also add a second point y, and now make $h(1') = \{(x, x), (y, y)\}$. In this representation on 2 points, there is no label from x to y, and the top element, $1' = 1$,

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[5] The language of composition and witness moves is borrowed from [33], and will be explored in greater detail in Chapter 3.

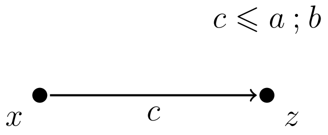

Figure 1.7: Witness move

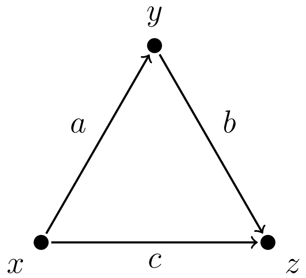

Figure 1.8: Witness response

is still the largest binary relation. While there is nothing wrong with a representation like this, we may wish to restrict ourselves to representations in which the top element relates any point to any point. That is, we may wish to enforce the requirement that the top element is a *universal relation.*

We call representations that meet this stricter requirement *square representations*. We also refer to a proper relation algebra in which U is the universal relation as *square.* In enforcing this requirement, we may suddenly find it impossible to represent certain relation algebras that were previously representable. Recall that an algebra is *simple* if it has only two congruence relations. The following lemma shows that if we are considering only square representations then we can restrict our view to simple relation algebras.

Lemma 1.1.6. *Every square proper relation algebra is simple.*

Proof. Let $\mathcal{A}$ be a square proper relation algebra with base set A. Let $\sim$ be a congruence relation[6] of $\mathcal{A}$ that is not trivial, that is, $\sim$ does not put each element into its own unique congruence class. We will show that $\sim$ must be the universal[7] congruence relation, that is, $a \sim b$ for all $a, b \in A$.

Relation algebras are also Boolean algebras (Definition 1.2.1). We have that any nontrivial congruence on a Boolean algebra has a nontrivial ideal as one of its blocks [13, Chapter IV, Theorem 3.5]. In particular, a nontrivial congruence relation on a square proper relation algebra must identify some nonempty R with the empty relation. Now we have that U relates everything to everything, so that $\mathsf{U} \,;\, R \,;\, \mathsf{U} = \mathsf{U}$. Similarly, $\mathsf{U} \,;\, \varnothing \,;\, \mathsf{U} = \varnothing$. So $\mathsf{U} \sim \varnothing$ and so everything is congruent to everything. ☐

We now consider some examples of representable relation algebras, before turning to the possibility of nonrepresentable relation algebras.

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[6] Note here that congruence relations are relations on the domain of the algebra A itself.

[7] There is potential confusion here because the elements of the algebra are themselves relations, and one of these is the 'universal relation' U, but these are not of concern to us right now.

Example 1.1.7 (The point algebra)**.** *The point algebra* is a simple relation algebra that describes a simple dense linear order. It has only one cycle not involving identity, $(<, <, <)$. There are 8 elements, but we only need 3. This is because we can use joins to develop the full algebra. These 3 basic elements, called *atoms*, are identity $1'$, and orders $<$ and $>$. As one would expect, $<\breve{} = >$. The composition table for atoms is shown in Figure 1.9.

;	$1'$	$<$	$>$
$1'$	$1'$	$<$	$>$
$<$	$<$	$<$	1
$>$	$>$	1	$>$

Figure 1.9: Atom composition table of the point algebra

If we wanted to determine, say, $(< + >) \; ; <$, we would use the axioms in Definition 1.1.3 as follows:

$$\begin{aligned}(< + >); < &= (< ; <) + (> ; <)\\ &= < +1\\ &= 1\end{aligned}$$

By using atoms to describe a relation algebra, we can see the order of the elements a little more clearly. First we note that atoms are *jointly exhaustive and pairwise disjoint*. This means that the meet of any two distinct atoms is empty, and that together the atoms give the top element. In this case, we have that

$$1' \cdot < = 0 \text{ and } < \cdot > = 0 \text{ and } 1' \cdot > = 0,$$

and that $1' + < + >= 1$. The order of the elements of the algebra then describe a lattice in which the atoms are at the bottom, above only 0. By considering every element as a join of atoms, we can then see the order more clearly by set inclusion.

Not every relation algebra can be expressed as atoms, but every finite relation algebra can. This is because the lattice operations of a finite relation algebra ($\cdot$ and $+$) form a finite bounded lattice, so we can take the minimal nonzero elements as the atoms. One benefit of discussing relation algebras in terms of atoms is that we can think of their representations as being edge-labelled digraphs in which every label is an atom. This is because the meet of any two distinct atoms is 0, and so at most one atom can label any edge. Suppose a and b are atoms and $a + b$ labels an edge (x, y). Then either a relates x to y, in which case b does not, or $-a$ relates x to y and so b does as well. That is, every edge is labelled by exactly one atom.

The point algebra describes a dense linear order. If we have a representation h such that $(x, y) \in h(<)$, then we can witness the composition $<$ below $< ; <$ on this edge. So we can expect to see a third point z in the representation such that $(x, z), (z, y) \in h(<)$. We can 'perform' this witness move an infinite number of times, and so any representation must be infinite; that is, on an infinite set. In particular, there is only one dense linear order on a countable set (up to isomorphism), the rationals $(\mathbb{Q}, <)$. This is a representation of the point algebra. Later on we will see that for a countable algebra it suffices to consider only finite or countably infinite representations, since the existence of an uncountable representation implies the existence of a countable one.

Example 1.1.8 (Allen's Interval Algebra)**.** One of the best examples of a relation algebra is Allen's Interval Algebra [1]. Its atoms are shown in Table 1.10. Note that there are 13 atoms, rather than 14, since the converse of identity is identity[8].

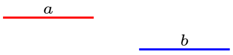
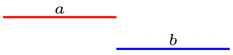
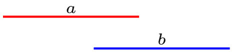
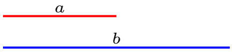
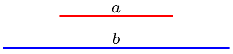
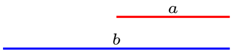
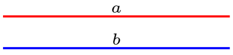
Example	Relation	Converse
	a before b	b after a
	a meets b	b is met by a
	a overlaps with b	b is overlapped by a
	a starts b	b is started by a
	a during b	b contains a
	a finishes b	b is finished by a
	a equals b	b equals a

Table 1.10: The atoms of Allen's Interval Algebra

This relation algebra is intended to capture the intuition behind how intervals of time interact with one another. We can see the operations of a relation algebra being used

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[8] To check this, suppose $(1')^{\smile}$ is some nonidentity element, a.

$$\begin{aligned} a^{\smile} &= (1' ; a)^{\smile} && \text{(axiom 6)} \\ &= a^{\smile} ; (1')^{\smile} && \text{(axiom 9)} \\ &= a^{\smile} ; a && \text{(assumption)} \\ &= 1' ; a && \text{(axiom 7)} \\ &= a. && \text{(axiom 6)} \end{aligned}$$

And so we conclude that $a = a^{\smile} = 1'$.

here in a very intuitive way. The converses are clear, and the composition of the atoms are easy to guess from the example intervals given in Table 1.10, although a full composition table is offered in Allen's original paper.

The example offered in this paper is quite elegant, and warrants repeating here. Consider the story:

> John was not in the room when I touched the switch to turn on the light.

Let S be the time of touching the switch, L the time the light was on, and R the time that John was in the room. Presumably, the light is turned on either exactly as the switch is touched, or slightly after. Hence, *S overlaps or meets L.* John may well have been in the room before or after any of these events occured. He may also have left the room as the switch was being touched, or entered the room as the switch was no longer being touched. Hence, *S is before, meets, is met by, or is after R.*

We are looking at an algebra which is to be interpreted as intervals of time, so intervals of the rationals $\mathbb{Q}$ or reals $\mathbb{R}$ are a natural choice. Either will suffice, but we'll stick wiith $\mathbb{Q}$ for now because it's countable. These intervals, along with the set-theoretic signature in Table 1.3, are a representation of Allen's Interval Algebra.

We began this chapter by discussing the world of abstract groups and the concrete world of permutation groups. Cayley's Theorem shows that every group is isomorphic to a group of permutations, which connects these two worlds. We then generalised permutations to binary relations and, for a certain signature, constructed a concrete world of proper relation algebras. We also have the axioms of Tarski, which characterise the abstract world of relation algebras. So the natural question to ask is whether or not there is an analogue to Cayley's Theorem for relation algebras? That is, is every relation algebra isomorphic to a proper relation algebra?

Tarski [86, p. 88] along with Jónsson [45] asked this very question. Specifically, they asked if every model of the axioms in Definition 1.1.3 was isomorphic to an algebra of binary relations. In 1950, Lyndon [54] offered an infinite family of nonrepresentable relation algebras that satisfied Tarski's axioms. Another example of a nonrepresentable relation algebra was later offered by McKenzie, and we reproduce it here.

Example 1.1.9 (The McKenzie Algebra)**.** This algebra was constructed by McKenzie [62, pp. 286] as an example of a nonrepresentable relation algebra. On only four atoms it is a minimal (but not unique) algebra with this property. Its atoms and their compositions are shown in Figure 1.11. The elements $<$ and $>$ are converses of each other, and $\#$ is self-converse. Composition is associative, although we omit the proof here.

;	$1'$	$<$	$>$	$\#$
$1'$	$1'$	$<$	$>$	$\#$
$<$	$<$	$<$	1	$< +\#$
$>$	$>$	1	$>$	$> +\#$
$\#$	$\#$	$< +\#$	$> +\#$	$< + > +1'$

Figure 1.11: Atom composition table of the McKenzie Algebra

.

The proof of its nonrepresentability is taken from [36, pp. 140–141]. Denote the McKenzie Algebra by $\mathcal{A}$ and suppose h is a representation of the McKenzie Algebra onto some set X. That is, suppose for contradiction that h is an injective function $h : \mathcal{A} \to \wp(X \times X)$. Since $\#$ is nonzero it cannot be represented as the empty relation, and so there exists $x, y \in X$ such that $(x, y) \in h(\#)$. As $(>, <, \#)$ and $(<, >, \#)$ are consistent cycles, we use witness moves and composition moves to build a partial representation as shown in Figure 1.12. By a *partial* representation, we mean a representation which respects everything except witness moves, and may not (yet) feature every element of $\mathcal{A}$.

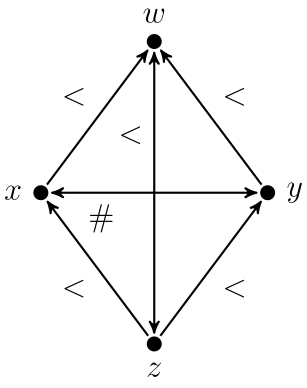

Figure 1.12: A partial representation of the McKenzie Algebra

We also have that $(<, \#, \#)$ is a consistent cycle, so we use a witness move here to deduce that there exists some $v \in X$ such that $(w, v), (z, v) \in h(\#)$. Since $\#^{\smile} = \#$ it follows that $(v, w), (v, z) \in h(\#)$. This situation is drawn in Figure 1.13.

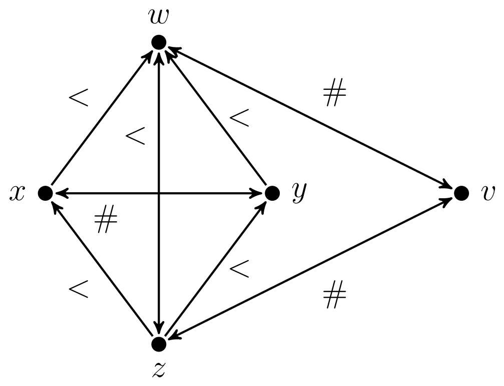

Figure 1.13: A partial representation of the McKenzie Algebra with a fifth point

Now we have four unlabelled edges, (x, v), (y, v), and their inverses. Suppose that $(x, v) \in h(a)$ and $(y, v) \in h(b)$ for some atoms a and b. Considering the triangles (x, w, v) and (x, z, v) we have that $(<, \#, a)$ and $(>, \#, a)$ must be consistent. The only choice here is that $a = \#$. Similarly reasoning about the point y shows that $b = \#$.

Now, looking at the triangle (x, y, v), we have that $(a, b^{\smile}, \#)$ must also be consistent. But $(\#, \#, \#)$ is a forbidden cycle. So no representation can exist.

If Tarski's axioms do not capture representable relation algebras, it is natural to consider whether or not we can add additional axioms that will fix this, or even consider a different set of axioms that might be more appropriate. However, Monk [65] later showed that no finite set of axioms can abstractly capture the concrete world. Finally, in 2001 Hirsch and Hodkinson [35] showed that representability is undecidable for finite relation algebras. So we have nonfinite axiomatisability of representability and, for finite relation algebras, undecidability of representability.

The representability of groups is a neat and tidy matter; the abstract world and the concrete world are identified with one another. But for binary relations, the abstract world of relation algebras and the concrete world of proper relation algebras are split. There can be no elementary Cayley's Theorem for relation algebras.

What are we to do when the situation is so negative? How can we explore this gap between the abstract and the concrete worlds?

This thesis considers three approaches. In Chapter 2 we consider signatures smaller than the relation algebra signature presented here. These are called *reducts*. We recover undecidability results for a range of reducts. In Chapter 3 we explore the representability of a particular reduct but with additional conditions on the representation. Specifically, we investigate semigroups with meet in which the meet semilattice is *flat*; that is, $a \cdot b = a$ if $a = b$ and 0 otherwise. We demand that the representations of these semigroups be *disjoint*. We prove a necessary condition for this representability, and conjecture that it is also sufficient.

In Chapter 4, the final and most substantial chapter, we explore notions of representability weaker than that of a proper relation algebra, in which witness moves are not fully respected. The two types of representations we consider are *weak* and *qualitative representations*. The appropriate abstract world for these representations is that of *nonassociative algebras*, which satisfy all axioms of a relation algebra, except maybe associativity of composition. We survey representability of all nonassociative algebras on four or fewer atoms. We survey computational properties of all nonassociative algebras on three or fewer atoms.

1.2 History and known results

In 1854 George Boole [8] presented "the world's first mathematical treatment of logic" [18], specifically an algebra of logic. Boole intended his formal treatment of logic to cover and extend that of Aristotle. The modern approach to Boole's work covers the interaction of *true* (1) and *false* (0) values as we would today use in programming. There are three operations: *conjunction* $(\cdot)$, *disjunction* $(+)$, and *negation* $(-)$. Informally, we could call these *and*, *or*, and *not*. The modern definition of a Boolean algebra is offered below.

Definition 1.2.1 ([42, 43, 44])**.** An algebra $\mathcal{A}$ on set A with signature $\{+, \cdot, -\}$ $\langle 2, 2, 1\rangle$ is a Boolean algebra if for all $a, b, c \in A$ the following hold:

1. $a + b = b + a$.
2. $a + (b + c) = (a + b) + c$.
3. $-(-a + -b) + -(-a + b) = a$.

We define $\cdot$ by DeMorgan's laws, with $-(a + b) = -a \cdot -b$ and $-(a \cdot b) = -a + -b$.

These axioms are the same as the first three of Definition 1.1.3. That is, every relation algebra is also a Boolean algebra. If we consider a 'representation' of a Boolean algebra as being like that of a relation algebra, but with concern only for the operations in $\{+, \cdot, -, 0, 1\}$ then every Boolean algebra is representable. In fact, they can be represented as algebras of unary relations [85], since we do not need to worry about composition.

A few years later, De Morgan became the first person in the world to consider a calculus of binary relations [20]. De Morgan also referenced Aristotle, who had denied that every relation has a converse. Aristotle's example was that "rudder of the ship" lacked the converse "ship of the rudder". De Morgan challenged this, arguing "Surely the question, 'What ship does this rudder belong to?' must sometimes have been heard in an Athenian dockyard." [70].

The birth of algebras of relations as we understand them today is largely due to C. S. Peirce. Peirce states his motivation clearly, while paying homage to the work of Boole.

> "Boole's logical algebra has such singular beauty, so far as it goes, that it is interesting to inquire whether it cannot be extended over the whole realm of formal logic, instead of being restricted to that simplest and least useful part of the subject, the logic of absolute terms, which, when he wrote, was the only formal logic known." (C. S. Peirce, 1873 [68])

Peirce compared the logical operations of Boole with the other operations in the relation algebra signature [68]. Denote by $0'$ the diversity relation, that is, $0' = -1'$. In an algebra of binary relations, this would be represented as a relation that relates every vertex to every other vertex. We also mention, but do not elaborate on, the *relative sum*[9] $\dagger$. With modern notation, the following table from [70] expresses Peirce's connection between the logical and other operations of the relation algebra signature.

logical:	0	1	$-a$	$a + b$	$a \cdot b$
relative:	$0'$	$1'$	$a^{\smile}$	$a \dagger b$	$a \mathbin{;} b$

A treatment of the calculus of relations was also offered by Schröder [83], who gave us much of the notation we use today for the relation algebra operations [57]. Bertrand Russell was next to take up the study of the calculus of relations. In particular, he noted its importance with the following:

> "The subject of symbolic logic is formed by three parts: the calculus of propositions, the calculus of classes, and the calculus of relations."
>
> (Russell [76])

Löwenheim was also fond of the calculus of relations [53]. Indeed, the first proof of the downward Löwenheim-Skolem theorem [52] is a work of the calculus of relations. A consequence of this theorem is that if a countable first-order theory has an infinite model, then it has a countable model. For our purposes, this means that if a countable relation algebra has an infinite representation, then it has a countably infinite

---

[9] In an algebra of binary relations over a set X, the relative sum would be defined $R \dagger S = \{(x, y) : \forall z \in X, \text{ either } (x, z) \in R \text{ or } (z, y) = S\}$. This operation is not frequently discussed in recent literature, relative to the other operations in the relation algebra signature, so we omit a proper discussion of it here.

representation. This justifies us in ignoring uncountably infinite representations, since it suffices to look only at the finite and countable cases.

The study of the calculus of relations was picked up by Tarski in 1941 [86]. In this paper, Tarski listed some axioms for a general calculus and stated without proof that the calculus was undecidable [70]. The equational form, as presented in Definition 1.1.3, would later appear in a paper with Tarki's student, Jónsson, in 1948 [45], thus defining the variety of relation algebras. In this same paper, Tarski and Jónsson asked if every relation algebra was isomorphic to a proper relation algebra, leading to the negative results described previously.

Tarski's motivations for studying the calculus of relations were not just mathematical but philosophical [70]. From the calculus of relations he and Givant created a language for doing set theory [89]. In particular, the language can be used to do set theory without variables. This might seem like a serious restriction, but the expressive and deductive powers of this language are equivalent to those of a system of first-order logic with only three variables [28].

So what does one do after the negative results of representability of relation algebras? There were several efforts to look into representability of other algebras of relations with a different signature. Of particular interest to us are *reducts* of the relation algebra signature. This is where we omit some of the operations. For example, the Boolean algebra signature can be considered a reduct of the relation algebra signature. For a signature τ we may refer to a reduct as a τ-algebra.

In taking a reduct of a relation algebra, we also take some of the relation algebra axioms as given in Definition 1.1.3. Specifically, we can always assume as necessary any of Tarski's axioms that remain expressible in the signature of the reduct. For example, semigroups contain a binary operation, which we take as composition, and which is associative. By axiom 4, when we take a reduct of the relation algebra signature involving composition, the underlying $\{;\}$-structure of the reduct is a semigroup.

Cayley's Theorem, or an analogue thereof, can be used to show that all semigroups are representable as transformations, which are binary relations[10]. As we noted earlier, all Boolean algebras are representable as algebras of unary relations, and unary relations are easily turned into binary relations. Somewhere between these smaller signatures and the full relation algebra signature, something changes and representability becomes nonfinitely axiomatisable, or even undecidable for finite algebras. Where is this boundary between representability and nonrepresentability?

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[10] It does not follow from Cayley's Theorem that all $\{;, {}^{\smile}, 1'\}$-reducts are representable because these are not necessarily groups. Specifically, it does not follow from the relation algebra axioms that $a\,;a^{\smile} = 1'$.

Zaretskiĭ [91], for example, found in 1959 that the class of representable ordered semigroups, that is, with signature $\{;, \leqslant\}$, is finitely axiomatisable. Yet in 2005 Hirsch discovered that by simply turning the semigroup into a monoid, that is, by including a $1'$ in the signature, the situation changes dramatically. In fact, the class of representable ordered monoids is nonfinitely axiomatisable. The boundary for this particular case is very sharp.

We survey the representability of all reducts of the relation algebra signature. We begin by noting that some operations can be defined by others. For example, strictly speaking $\leqslant$ is not in the relation algebra signature, but can be derived from $\cdot$ or $+$. That is, $a \leqslant b$ if and only if there exists a c such that $a = c \cdot b$, or $b = a + c$. Table 1.14, adapted from [34], describes how some operations can be defined by others.

Symbol	Defined by
$\leqslant$	$\{+\}, \{\cdot\}$
$\cdot$	$\{-, +\}$
$+$	$\{-, \cdot\}$
1	$\{0, -\}$, $\{+, -\}$
0	$\{1, -\}$, $\{+, -\}$

Table 1.14: Signature completion rules

Recall earlier that we represent 1 as U, which we can require to be the biggest binary relation or, as a stronger requirement, the universal relation. In the absence of order, the concept of a 'biggest' relation may not make much sense. To make matters even more complicated, if we only require that 1 be represented as the biggest binary relation, then any representation of an ordered algebra will easily meet this condition. As such, we take the stronger requirement in this survey that 1 must be represented as the universal relation.

There is another difficulty that arises when the signature gets smaller. We usually represent $-$ as set complementation $\setminus$ relative to 1. In the absence of 1, this definition falls apart. We can have two options: we can take *universal complementation*, in which we complement relative to a universal relation which may or may not be in the algebra, or we can take *relative complementation*, in which we complement relative to a top relation which may or may not be in the algebra. In this latter case, for a representation h of an algebra $\mathcal{A}$ over a set X, we have that $(x, y) \notin h(a)$ implies $(x, y) \in h(-a)$ but only if there exists some other nonzero element b such that $(x, y) \in h(b)$. That is, we maintain the possibility that some vertices in the representation may not be related at all, and relative complementation is not relevant to those points. This is discussed further in Chapter 2.

We survey finite axiomatisability of representability and decidability of representability (for finite algebras) for every subsignature of the relation algebra signature. We also survey the finite representation for finite algebras (FRFA) property. This property asserts that every representable finite algebra in that signature is representable over a finite set. If a signature τ has the FRFA property then representability of algebras with signature τ is in the class of recursively enumerable problems, denoted RE. This is because one can enumerate all of the potential finite representations. If a finite algebra is representable, one will eventually discover the representation. What is not so obvious is that the FRFA property also implies decidability of representability. This is a consequence of the Fundamental Theorem of Algebras of Relations, which we will now briefly cover. This discussion follows that given by Schein [80].

A *structure* is an object $\mathfrak{A}$ with domain A, a set of operations $(o_i)_{i \in I}$, and a set of relations $(\rho_j)_{j \in J}$. Let K be an axiomatisable class of structures. We will consider binary relations on a structure in K. Let M be a set of first-order formulas which impose properties of binary relations on structures in K, and let G be a set of first-order formulas defining an arbitrary signature τ of operations and relations in the language of binary relations on objects from K. Together K, M and G give a class of concrete algebra of relations—specifically, algebras of binary M-relations over K satisfying the formulas in G. We also have a corresponding abstract class of all algebras isomorphic to a member of this concrete class. Denote this abstract class by $R(K, M, G)$.

As an example, take K to be the class of all nonempty sets and let M be empty. Let $G = (O, P)$ where O defines the binary operation of composition $\circ$ of binary relations, and P defines the binary relation of inclusion of binary relations $\subseteq$. Then $R(K, M, G)$ is the isomorphism class of all inclusion-ordered semigroups of binary relations.

There are other choices for K and M. For example, one could let K be the class of all abelian groups and let M define the condition "is an endomorphism of". Furthermore, let G define composition of binary relations as well as the operations of pointwise addition and subtraction of endomorphisms. Then $R(K, M, G)$ is an isomorphism class of rings of endomorphisms of abelian groups. Cases such as this are beyond the scope of this survey, and so we will only consider cases in which K is the class of all nonempty sets and M is empty. The following theorem and its corollary assume a knowledge of recursively axiomatisable sets (see, for example, [9]).

Theorem 1.2.2 (Fundamental Theorem of Algebras of Relations[11] [78])**.** *If K is axiomatisable then $R(K, M, G)$ is universal, that is, it may be characterised by a set of universal first-order formulas. Moreover, if K is recursively axiomatisable and M and G are recursive, then $R(K, M, G)$ is recursively universally axiomatisable.*

Corollary 1.2.3. *Suppose we are interested in representing algebras with signature τ as members of $R(K, M, G)$. If K is recursively axiomatisable, M and G are recursive, and τ has the FRFA property, then representability of finite algebras is decidable.*

Proof. If τ has the FRFA property then for a finite representable algebra $\mathcal{A}$ with this signature one can enumerate all finite members of $R(K, M, G)$ to find a representation. Hence, representability is in RE. Likewise since $R(K, M, G)$ is recursively axiomatisable, one can generate these axioms and check a finite algebra against them, and at least one axiom will fail if the algebra is not representable. Hence, representability is also in $\mathsf{co\text{-}RE}$, and so is decidable. $\square$

We surveyed representability properties by first generating all possible signatures. This was done with the Python code given in Section C.5. These methods generate a signature, fill in the definable operations as per Table 1.14, and order the operations in the signature in a consistent manner. This produced a large table of signatures which was then collapsed according to known results. The concise version is given as Table 1.15.

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[11] This is also referred to as the *Fundamental Theorem of Relation Algebras*, but we refer to relation algebras in the sense of Tarski. Since this theorem applies to more than just Tarski relation algebras, we are using the more general name.

signature	finitely axiomatisable	decidability of representability	finite representations for finite algebras
$\tau \subseteq \{\leqslant, +, \cdot, \smile, -, 1', 0, 1\}$	yes [81]	yes	yes [34]
$\{;\} \subseteq \tau \subseteq \{;, 1', 0\}$	yes	yes	yes
$\{;, 1\} \subseteq \tau \subseteq \{;, \leqslant, 1', 0, 1\}$			
$\tau = \{;, \smile\}$	no [10]		
$\{;, \smile\} \subset \tau \subseteq \{;, \smile, 1', 0, 1\}$			
$\{;, -\} \subseteq \tau \subseteq \{;, -, \smile, 1', 0, 1\}$			
$\{;, \leqslant\} \subseteq \tau \subseteq \{;, \leqslant, 0\}$	yes [81, 91]	yes	yes [81, 91]
$\{;, \leqslant, 1'\} \subseteq \tau \subseteq \{;, \leqslant, 1', 0, 1\}$	no [33]		
$\{;, \leqslant, \smile\} \subseteq \tau \subseteq \{;, \leqslant, \smile, -, 1', 0, 1\}$			
$\{;, \leqslant, -\} \subseteq \tau \subseteq \{;, +, \cdot, \leqslant, -, 1', 0, 1\}$	no	no (Thm 2.2.6)	no
$\{;, \cdot, \leqslant\} \subseteq \tau \subseteq \{;, \cdot, \leqslant, 0, 1\}$	yes [11]	yes	no (Cor 3.1.9)
$\{;, \cdot, \leqslant, 1'\} \subseteq \tau \subseteq \{;, \cdot, \leqslant, 1', 0\}$	no [39]		no (Cor 3.1.9)
$\{;, \cdot, \leqslant, 1', 1\} \subseteq \tau \subseteq \{;, \cdot, \leqslant, 1', 0, 1\}$			no (Cor 3.1.9)
$\{;, +, \leqslant\} \subseteq \tau \subseteq \{;, +, \leqslant, \smile, 1', 0\}$	no [2, 5]		
$\{;, +, \leqslant, 1\} \subseteq \tau \subseteq \{;, +, \leqslant, \smile, 1', 0, 1\}$			
$\{;, +, \cdot, \leqslant\} \subseteq \tau \subseteq \{;, +, \cdot, \leqslant, -, 1', 0, 1\}$	no	no (Thm 2.2.6)	no
$\{;, \cdot, \leqslant, \smile\} \subseteq \tau \subset \{;, +, \cdot, \leqslant, \smile, -, 1', 0, 1\}$	no	no [37]	no
$\tau = \{;, +, \cdot, \leqslant, \smile, -, 1', 0, 1\}$	no [65]	no [35]	no

Table 1.15: Representability and finite representability of subsignatures of the relation algebra signature, with 1 to be represented as the universal relation and $-$ to be taken with respect to the universal relation in the absence of a top element.

Chapter 2

Undecidability results

This chapter is based on material appearing in [66]. Note that we often omit the symbol ; for composition, writing ab for $a \,;\, b$.

2.1 Reducts of the Tarski Signature

In the full relation algebra signature we have non-finite axiomatisability of representability and undecidability of representability for finite algebras. A natural question then is whether or not these results hold for reducts of the full signature.

A survey on this topic is offered by Schein [81]. In particular, Schein remarks that "it would be interesting to describe 'complemented semigroups'...This problem may be more treatable for ordered complemented semigroups." These are algebras with signature $\{;, \leqslant, -\}$. In this chapter, we show that representability in this signature is undecidable if complements are to be representated with universal complementation.

Representability over a finite base set is also shown to be undecidable, a result we are able to extend to a weaker notion of complementation. We also prove undecidability of finite representability for lattice-ordered semigroups, which are those with signature $\{;, +, \cdot\}$. Furthermore, these results regarding either representability or finite representability apply to any signature between one of these and that of a Boolean monoid, $\{;, +, \cdot, -, 1', 0, 1\}$.

In order to prove these results we adapt a construction of Boolean monoids used by Hirsch and Jackson [37], correcting issues that arise from weakening the signature.

2.2 Partial groups and Boolean monoids

A *partial group* is a system $\mathcal{A}$ with signature $\langle A; *, e\rangle$ of type $\langle 2, 1\rangle$ where $*$ is a partial binary operation such that whenever $(a * b) * c$ and $a * (b * c)$ are defined, they are equal, and such that e acts as an identity whenever $*$ is defined.

Furthermore, $\mathcal{A}$ is a *square partial group*[1] if in addition there is a subset $\sqrt{A}$ of A containing the identity e such that:

1. $a * b$ is defined if and only if $a, b \in \sqrt{A}$, and
2. $\sqrt{A} * \sqrt{A} = A$; that is, for every $c \in A$ there are $a, b \in \sqrt{A}$ such that $a * b = c$.

A partial group $\mathcal{A}$ is *cancellative* if it satisfies the cancellation laws

$$\begin{aligned} x * y = x * z &\implies y = z \\ \text{and } x * y = z * y &\implies x = z. \end{aligned}$$

The authors of [37] construct a finite Boolean monoid denoted $\mathbf{M}(\mathcal{A})$ and with signature $\{;, +, \cdot, -, 1', 0, 1\}$ from a finite square cancellative partial group $\mathcal{A}$. Consider such a group $\mathcal{A} = (A, *, e)$. We define a Boolean monoid (a Boolean algebra with composition) $\mathbf{M}(\mathcal{A})$ with atoms

$$\{e_{ii} \colon i \in \{1, 2, 3\}\} \cup \{w_{ij} \colon i, j \in \{1, 2, 3\}\} \cup \{a_{12}, a_{23} \colon a \in \sqrt{A}\} \cup \{b_{13} \colon b \in A\}.$$

The remaining elements of $\mathbf{M}(\mathcal{A})$ are arbitrary joins of these atoms, giving $\mathbf{M}(\mathcal{A})$ a total of $2^{3+9+2\times|\sqrt{A}|+|A|}$ elements. Composition between atoms is determined by the following:

1. $x_{ij}y_{j'k} = 0$ if $j \neq j'$.
2. $e_{ii}x_{ij} = x_{ij}e_{jj} = x_{ij}$.
3. $a_{12}b_{13} = (a * b)_{13}$, for $a, b \in \sqrt{A}$.
4. We can now define $A_{ij} := 1_{ij} - w_{ij}$ for all $i, j \in \{1, 2, 3\}$. If $i < j$, then $a_{ij}w_{jk} = 1_{ik} - a_{ij}A_{jk}$ for $a \in \sqrt{A}$ if $j - i = 1$ and $a \in A$ if $j - i = 2$.
5. If $j < k$, then $w_{ij}a_{jk} = 1_{ik} - A_{ij}a_{jk}$ for $a \in \sqrt{A}$ if $k - j = 1$ and $a \in A$ if $k - j = 2$.
6. $w_{ij}w_{jk} = 1_{ik}$.

---

[1] Not to be confused with a *square representation.*

We denote by 1_{ij} the sum of all atoms with subscript ij. We also define the constants $1' = e_{11} + e_{22} + e_{33}$, $1 = \sum_{i,j \in 3} 1_{ij}$, and 0 as the empty sum of atoms. The construction of $\mathbf{M}(\mathcal{A})$ partitions the algebra into three parts, with these parts and some relevant atoms illustrated in Figure 2.1.

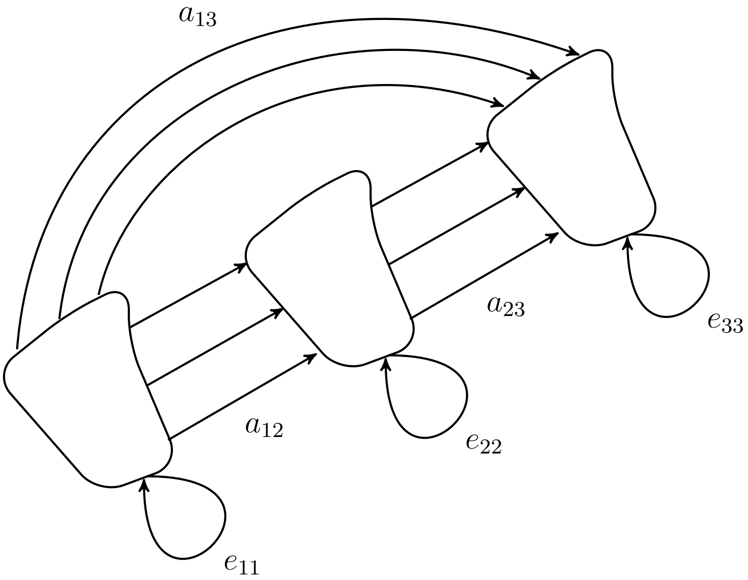

Figure 2.1: An illustration of some atomic elements of $\mathbf{M}(\mathcal{A})$

This does indeed define a Boolean monoid with signature $\{;, +, \cdot, -, 1', 0, 1\}$, as justified by Hirsch and Jackson [37, Lemma 4.1].

Define the unary term operations $\mathrm{D}(x) = (x;1) \cdot 1'$ and $\mathrm{R}(x) = (1;x) \cdot 1'$. The resultant $\mathbf{M}(\mathcal{A})$ is a *normal* Boolean monoid. That is, if $\mathrm{D}(a) = (a1) \cdot 1'$ and $\mathrm{R}(a) = (1a) \cdot e$ then $\mathrm{D}(a)a = a = a\,\mathrm{R}(a)$. In a representation of a normal Boolean monoid, $\mathrm{D}(a)$ and $\mathrm{R}(a)$ will be represented as a restriction of the identity relation to the domain and range of a, respectively. Note also that $\mathrm{D}(a)$ and $\mathrm{R}(a)$ are idempotent in $\mathbf{M}(\mathcal{A})$.

Lemma 2.2.1. $\mathbf{M}(\mathcal{A})$ *is a finite, simple, normal Boolean monoid with* $3 + 9 + 2 \times |\sqrt{A}| + |A|$ *distinct atoms.*

This construction from a partial group references the *partial group embedding problem* for a class of groups $\mathcal{K}$. This problem takes a finite partial group $\mathcal{A}$ and returns YES if there is a group $\mathcal{G} \in \mathcal{K}$ and an injective map $\phi\colon A \to G$ that respects all products defined in $\mathcal{A}$. Evans [24] showed that this problem is decidable for a class $\mathcal{K}$ if and only if the uniform word problem for $\mathcal{K}$ is decidable. In particular, this problem is undecidable if $\mathcal{K}$ is either the class of groups or class of finite groups, and $\mathcal{A}$ is a finite cancellative square partial group [37, Lemma 3.4].

One of the key concepts required to prove undecidability of representability is a formal means of referring to all elements that act as injective partial maps, hereafter called *injective functions*. In the full signature of relation algebras, one can consider a unary relation `i` as in Definition 2.2.2 to capture these elements.

Definition 2.2.2. Define a unary relation `i` in the language of relation algebras by

$$x \in \mathtt{i} \iff xx^{\smile} \leqslant 1' \text{ and } x^{\smile}x \leqslant 1'.$$

In a representation respecting converse, composition and identity, elements in `i` are exactly those relations that would be represented as injective functions. By considering the diversity relation $0' = -1'$, we can also view `i` as the set of elements satisfying the following formula in a signature containing $\{;, \cdot, 0'\}$.

Lemma 2.2.3 ([37, Lemma 2.12])**.** *Let* $\mathbf{R}$ *be a relation algebra. Then* $a \in \mathtt{i}^{\mathbf{R}}$ *if and only if*

$$(a0') \cdot a = 0 = (0'a) \cdot a.$$

The final concepts needed are those of domain and range equivalence. Binary relations r and s in an algebra over base set X are domain equivalent, denoted $r \sqcap s$, if

$$\{x \in X \mid (\exists y \in X)(x, y) \in r\} = \{x \in X \mid (\exists y \in X)(x, y) \in s\}.$$

We use the same notation for the abstraction of this concept in a Boolean monoid, with $x \sqcap y$ if $\mathrm{D}(x) = \mathrm{D}(y)$. Range equivalence $\sqcup$ is defined similarly. Note that in signatures weaker than that of a Boolean monoid a representation may preserve $\sqcap$ or $\sqcup$ without necessarily preserving D or R, respectively.

The following theorem is a combination of Propositions 5.1 and 6.3 from [37].

Theorem 2.2.4. *Let* $\mathcal{A}$ *be a finite cancellative square partial group. The following are equivalent, with the statements in square brackets giving a separate set of equivalences.*

1. $\mathbf{M}(\mathcal{A})$ *is representable [over a finite base set].*
2. *There is a* $\{;, \mathtt{i}, \sqcap, \sqcup\}$*-embedding of* $\mathbf{M}(\mathcal{A})$ *into* $\wp(X \times X)$ *for some [finite] set* X*.*
3. $\mathcal{A}$ *embeds into a [finite] group* $\mathcal{G}$*.*

We have already observed that both versions of (3) are known to be undecidable, and so too are (1) and (2). In this chapter we will introduce equivalent statements for decidability of representability of signatures weaker than that of a Boolean monoid.

In considering signatures without converse, we cannot be certain that 1 is represented as an equivalence relation. It turns out that a representable normal Boolean monoid can always be represented in such a way that 1 acts as an equivalence relation [37, Lemma 2.2]. This does not necessarily hold for weaker signatures. While this requirement on 1 is not without precedent for reducts of relation algebras (see [64, 81]), we may wish to remove it, or even consider algebras in which no top element exists, and so we will always state when this assumption is in use. With this in mind, we introduce statements in Theorem 2.2.5 equivalent to those in Theorem 2.2.4, but regarding representability of lattice-ordered semigroups and ordered complemented semigroups, thus proving undecidability of these problems as well.

Theorem 2.2.5. *Let $\mathcal{A}$ be a finite, cancellative, square partial group. The following are equivalent.*

1. $\mathbf{M}(\mathcal{A})$ *is representable [over a finite base set].*
2. $\mathbf{M}(\mathcal{A})$ *is representable [over a finite base set] as a lattice-ordered semigroup with* 1 *represented as an equivalence relation.*
3. $\mathbf{M}(\mathcal{A})$ *is representable [over a finite base set] as an ordered complemented semigroup with* 1 *represented as an equivalence relation.*
4. *There is a* $\{;, \mathtt{i}, \sqcap, \sqcup\}$*-embedding of* $\mathbf{M}(\mathcal{A})$ *into* $\wp(X \times X)$ *for some [finite] set* X*.*
5. $\mathcal{A}$ *embeds into a [finite] group* $\mathcal{G}$*.*

We note that a representation of $\mathbf{M}(\mathcal{A})$ as a lattice-ordered semigroup would respect the operations in $\{;, +, \cdot\}$, while a representation as an ordered complemented semigroup would respect those in $\{;, \leqslant, -\}$. Since both signatures are weaker than that of a Boolean monoid we can see that (1) $\implies$ (2) and (1) $\implies$ (3). Similarly we note that these results apply to any signature between one of these reducts and that of a Boolean monoid.

The remaining implications are (2) $\implies$ (4) and (3) $\implies$ (4). Since composition is preserved in a representation of a semigroup, this aspect is trivial. We must prove that relations in $\mathtt{i}$ are preserved as injective functions under a representation in either signature. We do this for lattice-ordered semigroups in Lemma 2.3.2 and ordered complemented semigroups in Lemma 2.3.3. In Lemma 2.3.4, we prove that a representation of $\mathbf{M}(\mathcal{A})$ in either reduct preserves domain and range equivalence.

Recall from Chapter 1 that there is some ambiguity here as to the definition of complementation in a reduct. In the full signature of relation algebras complementation is taken with respect to the top element. In the absence of a top element, one can declare that if x is related to y by an element of the algebra, then for all relations a we have that (x, y) belongs to just one of $\{-a, a\}$. This mimics the behaviour of complementation when a top element is present by taking complements with respect to the union of all elements. We call this *relative complementation.* A stronger definition would take complements with respect to a universal relation, demanding that every (x, y) belongs to just one of $\{-a, a\}$. We refer to this as *universal complementation.*

For lattice-ordered semigroups and ordered complemented semigroups with relative complementation, the requirement on 1 can be removed if the representation is to be over a finite base set, and we show this in Lemmas 2.3.6 and 2.3.7. A representation of an ordered complemented semigroup with universal complementation will always represent 1 as an equivalence relation if it exists, as shown in Lemma 2.3.8, and so representability of algebras in this signature is undecidable. These results are stated in Theorem 2.2.6.

Theorem 2.2.6. *Let τ be a signature such that $\tau \subseteq \{;, +, \cdot, -, 1', 0, 1\}$. The following problems are undecidable:*

- *Finite representability of algebras with signature τ where $\{;, +, \cdot\} \subseteq \tau$.*
- *Finite representability of algebras with signature τ where $\{;, \leqslant, -\} \subseteq \tau$ and $-$ is to be represented as relative complementation.*
- *Representability and finite representability of algebras with signature τ where $\{;, \leqslant, -\} \subseteq \tau$ and $-$ is to be represented as universal complementation.*

Theorem 2.2.6 also yields results about non-finite axiomatisability for the same signatures. If there exists a finite set of first-order axioms characterising representability of a class of algebras, then one can consider an algorithm that checks a finite algebra against each of these axioms to determine representability. Hence, finite axiomatisability implies decidability of representability, giving us the results in Corollary 2.2.7.

Corollary 2.2.7. *Let τ be a signature such that $\tau \subseteq \{;, +, \cdot, -, 1', 0, 1\}$. The following classes of algebras are not finitely axiomatisable in first order logic:*

- *Any class whose finite members are the finitely representable algebras with signature τ where $\{;, +, \cdot\} \subseteq \tau$.*
- *Any class whose finite members are the finitely representable algebras with signature τ where $\{;, \leqslant, -\} \subseteq \tau$ and $-$ is to be represented as relative complementation.*

- *Representable algebras with signature τ where $\{;, \leqslant, -\} \subseteq \tau$ and $-$ is to be represented as universal complementation.*

We note that non-finite axiomatisability of representability of algebras with signature τ where $\{;, +, \cdot\} \subseteq \tau \subseteq \{;, +, \cdot, \breve{}, 1', 0, 1\}$ was shown by Andréka [3]; see also Andréka and Mikulás [5].

2.3 Proofs of undecidability results

Let $\mathcal{S} = \{S; ;, 1'\}$ be a monoid. We also consider meet $\cdot$, join $+$, complementation $-$, a partial order relation $\leqslant$, and constants 0 and 1.

Let $h\colon \mathcal{S} \to \wp(X \times X)$ be a representation of $\mathcal{S}$ on a base set X preserving at least composition. Define an equivalence relation $\sim$ on X such that for all $x, y \in X$, $x \sim y$ if $x = y$ or $1'$ acts as the universal relation on the set $\{x, y\}$, a situation illustrated in Figure 2.2. Define a new representation $\hat{h}\colon \mathcal{S} \to \wp(X/{\sim} \times X/{\sim})$ such that for $a \in S$,

$$\hat{h}\colon a \mapsto \{[x], [y] \mid (\exists x' \in [x])(\exists y' \in [y])\ (x', y') \in h(a)\}.$$

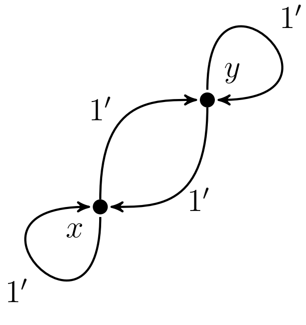

Figure 2.2: $1'$ acting as the universal relation on $\{x, y\}$

Lemma 2.3.1. *If h is a representation of $\mathcal{S}$ preserving composition then so too is $\hat{h}$. Furthermore, $\hat{h}$ preserves Boolean operations and constants 0, 1 and $1'$, if they are correctly represented by h.*

Proof. Consider $a, b \in S$ such that $a \neq b$. Then, since h is faithful, we may assume without loss of generality that there exists $(x, y) \in h(a) \backslash h(b)$. Then $([x], [y]) \in \hat{h}(a)$. Suppose by way of contradiction that $([x], [y]) \in \hat{h}(b)$. Then there exists $(w, z) \in h(b)$ with $1'$ acting as the universal relation on $\{x, w\}$ and on $\{y, z\}$. That is, $(x, w) \in h(1')$ and $(z, y) \in h(1')$. Since h preserves composition we have that $(x, y) \in h(1'b1')$ and so $(x, y) \in h(b)$. But this violates the assumptions on (x, y). So $\hat{h}$ is faithful.

Now we turn our attention to composition under $\hat{h}$. Let $([x],[y]) \in \hat{h}(a)$ and $([y],[z]) \in \hat{h}(b)$. Without loss of generality, assume $(x,y) \in h(a)$ and $(y,z) \in h(b)$, since as before we can always compose elements with $1'$ to move around within equivalence classes. Then, as h preserves composition, $(x,z) \in h(ab)$ and so $([x],[z]) \in \hat{h}(ab)$. Similarly we have that $([x],[z]) \in \hat{h}(ab) \implies ([x],[z]) \in \hat{h}(a)\hat{h}(b)$. So $\hat{h}$ also preserves composition.

We note that $\hat{h}$ only contracts binary relations in $h(\mathcal{S})$. Hence, Boolean operations and constants 0, 1 and $1'$ are preserved in $\hat{h}$, assuming they were correctly represented by h. In particular, if $1'$ is represented correctly then $(x,y) \in h(1') \iff x = y$, and so $h(1') = \hat{h}(1')$. □

It is by this quotient that we will ensure that the elements of the Boolean monoid $\mathbf{M}(\mathcal{A})$ that are in `i` are represented as injective functions. Recall from Lemma 2.2.3 that an element $a \in$ `i` if and only if $(a0') \cdot a = 0 = (0'a) \cdot a$.

Lemma 2.3.2. *Suppose that the Boolean monoid* $\mathbf{M}(\mathcal{A})$ *is representable in a signature containing* $\{;,+,\cdot\}$ *in such a way that* 1 *is represented as an equivalence relation. Then there exists a representation in the same signature with the property that if* $a \in M(A)$ *is such that* $(a0') \cdot a = 0 = (0'a) \cdot a$*, then* a *is represented as an injective function.*

Proof. Let $h \colon \mathbf{M}(\mathcal{A}) \to \wp(X \times X)$ be such a representation of $\mathbf{M}(\mathcal{A})$ onto some base set X and consider $a \in M(A)$ such that $(a0') \cdot a = 0 = (0'a) \cdot a$. By applying Lemma 2.3.1 we may work under the assumption that $h = \hat{h}$, and note that this preserves the property that 1 is represented as an equivalence relation.

Suppose there exists $x, y, z \in X$ such that $(y,x) \in h(a)$ and $(y,z) \in h(a)$, a situation illustrated in Figure 2.3. We will show that $x = z$. As 1 is acting as the universal relation on $\{x,y,z\}$ and $01 = 10 = 0$, if $h(0)$ relates any two (potentially equal) elements of $\{x,y,z\}$ then it must act as the universal relation on all three. Since $0 \leqslant 1'$, this would imply that $1'$ is also acting as a universal relation, a situation we have precluded unless $x = z$. So assume otherwise, that is, assume that $h(0)$ is not relating any two elements of $\{x,y,z\}$.

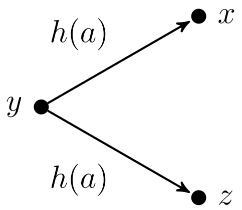

Figure 2.3: An element a not represented as a function under h

We note also that $(x,z) \notin h(0')$ as if this were the case then we would have $(y,z) \in h(a0')$. But $a \cdot (a0') = 0$, giving $(y,z) \in h(0)$. Similarly, we have that $(z,x) \notin h(0')$.

As $0' + 1' = 1$ we therefore have (x, z) and (z, x) in $h(1')$. Hence (x, x) and (z, z) are in $1'$. As $\hat{h} = h$ it follows that $x = z$ as required. That is, a is a function under h. By symmetry we also have that a is injective under h. $\square$

The requirement that $(0'a) \cdot a = 0 = (a0') \cdot a$ simply ensures that a is disjoint from $a0'$ and also from $0'a$. We can restate this with operations in $\{;, \leqslant, -\}$ such that $(0'a) \cdot a = 0$ if and only if $a \leqslant -(0'a)$, and similarly $(a0') \cdot a$ if and only if $a \leqslant -(a0')$. This allows us to replicate the previous result in the signature of ordered complemented semigroups.

Lemma 2.3.3. *Suppose that the Boolean monoid* $\mathbf{M}(\mathcal{A})$ *is representable in a signature containing* $\{;, \leqslant, -\}$ *in such a way that* 1 *is represented as an equivalence relation. Then there exists a representation in the same signature with the property that if* $a \in M(A)$ *is such that* $a \leqslant -(a0')$ *and* $a \leqslant -(0'a)$ *then* a *is represented as an injective function.*

Proof. Let $h \colon \mathbf{M}(\mathcal{A}) \to \wp(X \times X)$ be a such a representation of $\mathbf{M}(\mathcal{A})$ onto some base set X and consider $a \in M(A)$ such that $a \leqslant -(a0')$ and $a \leqslant -(0'a)$. Again we work under the assumption that $h = \hat{h}$. Since 0 is the unique element with the property that $0 \leqslant -0$, we have that $h(0) \subseteq h(1) \backslash h(0)$, and so 0 is represented correctly as the empty set.

We take x, y, z as in Figure 2.3 with $(y, x) \in h(a)$ and $(y, z) \in h(a)$. As $a \leqslant -(a0')$, we cannot have $(x, z) \in h(0')$, since we could compose to get $(y, z) \in h(a0')$. Similarly, $(z, x) \notin h(0')$. Because $1'$ and $0'$ are complementary with respect to 1, we have that (x, z) and (z, x) are in $h(1')$. We compose to realise $1'$ acting as a universal relation on $\{x, z\}$, a situation we have precluded unless $x = z$. Hence, a is represented as a function under h. By symmetry we also have that a is injective under h. $\square$

Hence, the `i` relation, as given in Definition 2.2.2, can be recovered in the case of signatures containing $\{;, +, \cdot\}$ or $\{;, \leqslant, -\}$, as long as 1 is to be represented as an equivalence relation. To complete the $\{;, \mathtt{i}, \ulcorner, \llcorner\!\lrcorner\}$-embedding required by Theorem 2.2.5, we must also check that domain and range equivalence are respected in the representation of a Boolean monoid as either a lattice-ordered semigroup or an ordered complemented semigroup.

Lemma 2.3.4. *Suppose that the Boolean monoid* $\mathbf{M}(\mathcal{A})$ *is representable in a signature containing* $\{;, +, \cdot\}$ *or in a signature containing* $\{;, \leqslant, -\}$*, and in either case suppose that* 1 *is represented as an equivalence relation. Then one can define domain and range equivalence of the elements in* $\mathbf{M}(\mathcal{A})$ *in such a way that they are respected by the representation.*

Proof. In either case, take $x \sqcap y$ if $x1 = y1$, and $x \sqcup y$ if $1x = 1y$. $\square$

This establishes that (2) $\implies$ (4) and (3) $\implies$ (4) in Theorem 2.2.5, completing the proof. Subject to the assumption that 1 is represented as an equivalence relation, we have undecidability of representability and finite representability of finite algebras in either signature. If we restrict our attention to representations over a finite base set then we can remove this assumption. The proofs are largely the same for lattice-ordered semigroups and ordered complemented semigroups, and both involve Lemma 2.3.5.

Lemma 2.3.5. *Let h be a representation of a Boolean monoid $\mathbf{M}$ onto a finite base set X respecting composition and order. Then for every nonzero idempotent $f \in M$ there exists an element of X fixed by $h(f)$ but not by $h(0)$.*

Proof. By faithfulness there exists $x, y \in X$ such that $(x, y) \in h(f)$ and $(x, y) \notin h(0)$, or such that $(x, y) \notin h(f)$ and $(x, y) \in h(0)$. Since 0 is the bottom element, we conclude that the latter is not possible and assume that $(x, y) \in h(f)$. Since f is idempotent we must witness an element $z \in X$ such that $(x, z) \in h(f)$ and $(z, y) \in h(f)$. We must continue to witness this for every pair in $h(f)$. But the representation is finite, so we must eventually witness a loop $(x_a, x_a) \in h(f)$. If $(x_a, x_a) \in h(0)$ also, then we could compose to get $(x, y) \in h(0)$, violating our initial assumption. Hence, f but not 0 fixes x_a in the representation. $\square$

We first remove the assumption that 1 be represented as an equivalence relation in finite representations of $\mathbf{M}(\mathcal{A})$ as a lattice-ordered semigroup, although in actuality only composition and meet are required for the proof. Recall that in the Boolean monoid signature we defined $\mathrm{D}(a) = (a1) \cdot 1'$ and that $\mathrm{D}(a)$ is idempotent in $\mathbf{M}(\mathcal{A})$.

Lemma 2.3.6. *Let h be a representation of the Boolean monoid $\mathbf{M}(\mathcal{A})$ onto a finite base set X respecting the operations in $\{;, \cdot\}$. Then there exists a representation h° in the same signature but representing the top element 1 as an equivalence relation. Furthermore, if h respects the operations in $\{+, -, 1', 0\}$ then so too does h°.*

Proof. For a binary relation r define the symmetric interior

$$r^\circ := \{(x, y) \mid (x, y) \in r \text{ and } (y, x) \in r\}.$$

If r is reflexive and transitive then one can view r° as the largest equivalence relation contained in r. Define $h^\circ \colon M(A) \to \wp(X \times X)$ as $h^\circ \colon a \mapsto h(a) \cap h(1)^\circ$.

Since we are only omitting non-loops in the representation, we have that h° preserves any operation in $\{+, -, 1', 0\}$, assuming that h does. For composition, consider

$(x, y) \in h^\circ(ab)$ for some $a, b \in M(A)$. Then since h respects composition there exists $z \in X$ such that $(x, z) \in h(a)$ and $(z, y) \in h(b)$. As (x, y) is in the image of h° we have that $(y, x) \in h(1)^\circ$. So $(y, z) \in h(1a)$ and, as 1 is the top element, $(y, z) \in h(1)$. Similarly, $(z, x) \in h(1)$. We conclude that $(x, z) \in h^\circ(a)$ and $(z, x) \in h^\circ(b)$. By similar composition with 1 we have that if $(x, z) \in h^\circ(a)$ and $(z, y) \in h^\circ(b)$ then $(x, y) \in h^\circ(ab)$, and so composition is respected by h°.

Now we must prove that h° is faithful. Let $a, b \in M(A)$ be distinct and assume without loss of generality that $b \nleq a$, so that $b \cdot (-a) \neq 0$. Note that we are only considering $-a$ as an element of $M(A)$, and do not require complementation to be represented in any way. As $\mathbf{M}(\mathcal{A})$ is normal we have that $\mathrm{D}(b \cdot (-a))(b \cdot (-a)) = (b \cdot (-a))$, and so $\mathrm{D}(b \cdot (-a)) \neq 0$. We established in Lemma 2.3.5 that nonzero idempotents under h fix points in X that are not fixed by 0. As such, $h^\circ(\mathrm{D}(b \cdot (-a))) \neq h^\circ(0)$. But clearly $h^\circ(\mathrm{D}(a \cdot (-a))) = h^\circ(0)$. As such $h^\circ(a) \neq h^\circ(b)$, and so h° is faithful.

Since h respects composition and order, h° represents 1 as transitive and symmetric, and hence reflexive on a subset of X. Since h° is faithful, this subset is nonempty. Hence, h° represents 1 as an equivalence relation over a nonempty subset of X. □

If we have a representation of $\mathbf{M}(\mathcal{A})$ as a lattice-ordered semigroup, then we can take the symmetric interior and then the quotient used in Lemma 2.3.1 to obtain a similar representation preserving the `i` relation and representing 1 as an equivalence relation. This allows us to remove the requirement in Lemma 2.3.2 that the top element of $\mathbf{M}(\mathcal{A})$ be represented as an equivalence relation, if the representation is to be taken over a finite set. The following lemma permits us to do the same in the case of Lemma 2.3.3, which deals with ordered complemented semigroups.

Lemma 2.3.7. *Let h be a representation of the Boolean monoid $\mathbf{M}(\mathcal{A})$ onto a finite base set X respecting the operations in $\{;, \leqslant, -\}$. Then there exists a representation h° in the same signature but representing the top element 1 as an equivalence relation. Furthermore, if h respects the operations in $\{+, \cdot, 1', 0\}$ then so too does h°.*

Proof. The proof is largely the same as for Lemma 2.3.6, though we must recover faithfulness with a different approach. Recall that 0 is forced to be represented as the empty set since $0 \leqslant -0$, and so any representation preserving $\{;, \leqslant -\}$ also trivially preserves 0. Again we define $h^\circ \colon \mathbf{M}(\mathcal{A}) \to \wp(X \times X)$ as $h^\circ \colon a \mapsto h(a) \cap h(T)^\circ$ and take distinct $a, b \in M(A)$ with the assumption that $b \nleq a$. Then there exists a nonzero c such that $c \leqslant b$ and $c \leqslant -a$. Hence we can distinguish between a and b if we witness a nonempty $h^\circ(c)$.

Since c is nonzero, we use Lemma 2.3.5 to conclude that $\mathrm{D}(c)$ fixes a point $x \in X$ under h. Now we note that, since 1 has maximum domain and range and composition on the right cannot restrict domain, $D(c) \leqslant D(c)T = cT$. We must witness this composition as in Figure 2.4 and so $h^{\circ}(D(c)) \leqslant h^{\circ}(cT) = h^{\circ}(c)h^{\circ}(T)$. That is, $h^{\circ}(c)$ is nonempty. □

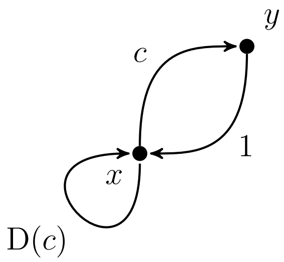

Figure 2.4: Witnessing the composition $D(c) \leqslant cT$

We noted before that the definition of complementation requires care in the absence of a top element. We used here relative complementation which mimics the definition of complementation when 1 is present: that if x is related to y by an element of the algebra, then for all relations a we have that (x, y) belongs to just one of $\{-a, a\}$. Under this weaker definition and without 1 acting as the universal relation we could have, for example, a situation as in Figure 2.3 such that no element relates x to z or z to x. If this occurs, we cannot take the complement to reason that $1'$ acts as the universal relation on these points, as we did in Lemma 2.3.3. Under the weaker definition of relative complementation, these proofs require that an element already relates these two points.

Alternatively, we can represent complementation as universal complementation in which the complement is taken with respect to $\wp(X \times X)$, where X is the base set of the representation. That is, in the absence of a top element we can take complements with respect to a universal relation. Under this interpretation it will turn out that if 1 does exist then it must act as an equivalence relation in any representation. We thank Marcel Jackson for the following observation.

Lemma 2.3.8. *Let $\mathcal{S}$ be a complemented semigroup of binary relations with complement taken with respect to a universal relation. If there exists an idempotent f such that $f(-f) = -f = (-f)f$ and $-f$ is also idempotent, then f is the universal relation.*

Proof. Suppose f relates x to y and, for contradiction, f does not relate y to x. Then $y(-f)x$ and, by assumption, $x(-f)x$. Then composing we get that $x(-f)y$, a contradiction. Hence, f is a symmetric, and so a reflexive binary relation.

We now know that f is an equivalence relation on its domain. Suppose that the domain of f is not full, and so we have that $x(-f)y$ and $y(-f)x$. Since $-f$ is idempotent we compose to get $x(-f)x$. If x is in the domain of f then we have a contradiction. If not, take y to be in the domain of f to reach a similar contradiction. Hence, f is an equivalence relation with full domain. □

Hence, any representation of $\mathbf{M}(\mathcal{A})$ respecting the operations in $\{;, \leqslant, -\}$ in which $-$ is represented with respect to a universal relation will always represent 1 as that universal relation. This extends the undecidability of representability of finite algebras in this signature to include infinite representations.

It would be interesting to see if we can do the same for infinite representability of lattice-ordered semigroups.

Problem 2.3.9. Finite representability is undecidable for lattice-ordered semigroups. Can the same be said of representability in general?

Another problem to consider is semigroups with either form of complementation but no order. As far as we can determine, this problem remains unexplored in the literature, and is mentioned by Schein [81].

Problem 2.3.10. Is representability or finite representability decidable in the signature $\{;, -\}$, with either relative or universal complementation? Are representable algebras in this signature finitely axiomatisable?

Chapter 3

Disjoint representations of semigroups

3.1 Finite disjoint representations

Recall from Chapter 1 our discussion of Cayley's Theorem (Theorem 1.1.1), which states that every group is isomorphic to a group of permutations. Actually, we represent a group as a group of permutations over itself. In particular, $a \mapsto (x, y)$ if $x ; a = y$ or, as an alternative representation, $x = a ; y$. Call either of these a *Cayley representation* of the group. As such, we have the following corollary to Cayley's Theorem.

Corollary 3.1.1. *Every finite group is isomorphic to a group of permutations over a finite set.*

A $\{;, {}^{\smile}, 1'\}$-reduct of a relation algebra is not the same as a group, since we are unable to recover the group axiom $a ; a^{\smile} = 1'$ from the relation algebra axioms. This is not an issue for a $\{;\}$- or $\{;, 1'\}$-reduct, which is a semigroup or monoid, respectively. An analogue of Cayley's Theorem exists for semigroups (or monoids), which are represented as transformations. The only adjustment is that a semigroup $\mathcal{S}$ is represented over $\mathcal{S}^{1'} = \mathcal{S} \cup \{1'\}$, the monoid into which $\mathcal{S}$ embeds.

Theorem 3.1.2 (Cayley's Theorem for semigroups)**.** *Every (finite) semigroup is isomorphic to a (finite) semigroup of transformations.*

Groups enjoy cancellativity. Take the Cayley representation h of a group such that $(x, y) \in h(a)$ whenever $x ; a = y$ and suppose $(x, y) \in h(a)$ and $(x, y) \in h(b)$. Then one can take inverses to deduce that $a = b$. Hence in a representation of a group, no edge is labelled by two distinct elements[1]. We call a representation of an algebra of relations in which every edge is labelled by at most one element *disjoint*. In semigroups, composition is not generally cancellative, and so we are left with the question of when a semigroup has a disjoint representation. This notion is formalised in Definition 3.1.3.

Definition 3.1.3. A semigroup has a *disjoint representation* if it is isomorphic to a semigroup of disjoint transformations. The representation is *finite* if the semigroup of disjoint transformations is over a finite set.

For our purposes, we can rephrase this not as a special kind of representation of a a semigroup but rather as a representation of a special kind of $\{;, \cdot, 0\}$-reduct of the relation algebra signature.

Lemma 3.1.4. *For every (finite) semigroup $\mathcal{S}$ there exists a pair of algebras $\{\mathcal{A}_1, \mathcal{A}_2\}$, each with signature $\{;, \cdot, 0\}$, such that $\mathcal{S}$ is disjointly representable (over a finite set) if and only if at least one of $\{\mathcal{A}_1, \mathcal{A}_2\}$ is (finitely) representable.*

Proof. We define $A_1 = S \cup \{0\}$ with 0 acting as $a \,;\, 0 = 0 = 0 \,;\, a$ for all $a \in A_1$. If S already has a zero then that element no longer acts as a zero in $\mathcal{A}_1$. If S has no zero element then we let $A_2 = A_1$. Otherwise we define $A_2 = S$ with the intention that it will inherit a zero element from $\mathcal{S}$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ inherit the composition of $\mathcal{S}$ and both are equipped with a meet operation $\cdot$ such that, for all $a, b \in A_1$ or $a, b \in A_2$,

$$a \cdot b = \begin{cases} a & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

Suppose S does not have a 0 element. Then a representation h of $\mathcal{S}$ cannot represent an element as the empty set. Hence a (finite) representation of $\mathcal{S}$ is easily interpreted as a (finite) representation of $\mathcal{A}_1$ with 0 represented as the empty set. If h is disjoint then it will also respect the semilattice operation of $\mathcal{A}_1$, which forbids two elements being placed on the same edge. Hence a (finite) disjoint representation of $\mathcal{S}$ gives a (finite) and disjoint representation of $\mathcal{A}_1$. Conversely, a (finite) and disjoint representation of $\mathcal{A}_1$ respects composition and forbids two elements being represented on the same edge, and so gives a disjoint representation of $\mathcal{S}$.

If S does have a 0 element then we need to consider how this might be represented in a representation of $\mathcal{S}$. Let h be a representation of $\mathcal{S}$. If $h(0) = \varnothing$ then h gives a representation of $\mathcal{A}_2$ using the argument above, with the representation of $\mathcal{A}_2$ finite if h is finite. If $h(0) \neq \varnothing$ then h gives a representation of $\mathcal{A}_1$. $\square$

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[1] In fact, every edge is labelled by *exactly* one element.

We call a semilattice with this meet operation a *flat semilattice*. It is a consequence of this lemma that when working with disjoint representations of semigroups we can assume the existence of a 0 element.

It is a result of Bredikhin and Schein [12] that every semilattice ordered semigroup (signature $\{;, \cdot\}$) can be represented as an inclusion-ordered semigroup of binary relations respecting greatest lower bound (signature $\{\circ, \cap\}$). Bredikhin later extended this to include the representation of 0 [11] (see also Andreka and Mikulas [5]). As such, every semigroup has a disjoint representation.

One can also prove this using the game theoretic techniques of Hirsch and Hodkinson, an approach we will shortly use. These techniques were introduced to prove undecidability of representability of finite relation algebras [35, 36]. The game theoretic approach is to build up the representation piece by piece. Apart from an initial move, the moves of the game are exactly the composition and witness moves introduced in Chapter 1.

We will be adjusting the game theoretic approach as used by Hirsch to prove the nonfinite axiomatisability of the class of representable ordered monoids [33]. The definitions that follow are adapted from this paper. While ordered monoids have signature $\{;, \leqslant, 1'\}$ we can define the rules of a game so as not to be concerned about the presence or representability of an identity element and so that 0 is represented as the empty set. We can then define a strategy for witness moves in such a way that the resulting representation is disjoint. First we define the 'board' on which the games are played.

Definition 3.1.5. Let $\mathcal{S}$ be an ordered semigroup with 0. A *prenetwork* (D, N) over $\mathcal{S}$ consists of a set of nodes D and a map $N \colon D \times D \to \wp(\mathcal{S})$. Furthermore, (D, N) is called a *network* over $\mathcal{S}$ if for all $x, y, z \in D$, $a \in N(x, y)$, $b \in N(y, z)$ there exists $c \in N(x, z)$ with $c \leqslant a ; b$.

We may also use N to refer to the set of nodes. That is, $x \in N$ refers to a node of the prenetwork N and $N(x, y)$ is the label between nodes x and y. We say that a prenetwork M is a *subnetwork* of prenetwork N, written $M \subseteq N$, if the nodes of M are a subset of the nodes of N and for all $x, y \in M$ and all $a \in M(x, y)$ there is $a^- \in N(x, y)$ with $a^- \leqslant a$. We say that M is an induced subnetwork of N, written $M \leqslant N$, if the nodes of M are a subset of the nodes of N and for all $x, y \in M$ we have $M(x, y) = N(x, y)$.

We can also take the union of prenetworks. If $\{N_\lambda \colon \lambda \in \Lambda\}$ is a set of prenetworks then $N = \bigcup_{\lambda \in \Lambda} N_\lambda$ is the prenetwork with nodes the union of the nodes of all N_λ and

$$N(x,y) = \bigcup_{\lambda\colon x,y \in N_\lambda} N_\lambda(x,y).$$

The game is played with two players, $\forall$ and $\exists$. $\forall$ will ask for certain states that should exist if the ordered semigroup $\mathcal{S}$ is representable, and $\exists$ will update the board with this information.

Let $0 \leqslant n \leqslant \omega$ and let $\mathcal{S}$ be an ordered semigroup with 0. A play of the game $G_n(\mathcal{S})$ has n rounds and consists of a sequence of n pre-networks $N_0, N_1, \dots, N_n$. In round 0 the initial move is played. $\forall$ chooses $a_0, a_1 \in S$ such that $a_1 \not\leqslant a_0$. The response by $\exists$ to this initial move is a network N_0, consisting of nodes x_1, x_2 with $N_0(x_1, x_2) = a_1$ and with no other edges. Note that the condition that $a_1 \not\leqslant a_0$ precludes the choice of 0 for a_1.

Suppose the game has been played to the prenetwork N_{i-1}. Then N_i is constructed with either a composition or witness move.

Composition move : $\forall$ chooses $x, y, z \in N_{i-1}$, $a \in N_{i-1}(x,y)$ and $b \in N_{i-1}(y,z)$. $\exists$ responds with $N_i \supseteq N_{i-1}$, a copy of N_{i-1}, but with $a \mathbin{;} b \in N_i(x,z)$.

Witness move : $\forall$ picks $x, y \in N_{i-1}$ and any a, b such that $c \leqslant a \mathbin{;} b$ and $c \in N_{i-1}(x,y)$. Unlike with composition moves, $\exists$ has some choice in how they may respond. They must choose a node z, either from N_{i-1} or a new node, and respond with $N_i \supseteq N_{i-1}$, a copy of N_{i-1}, but with $a \in N_i(x,z)$ and $b \in N_i(z,y)$ (these may already exist). The option for $\exists$ to choose a new node is how the network grows.

$\forall$ wins the game if there is $i < n$ and $a^- \leqslant a_0$ with $a^- \in N_i(x_1, x_2)$. If a 0 is introduced into the network at any point and the game is played sufficiently long then this winning condition would be met, as 0 will propagate through to $N(x,y)$ by composition moves. Thus, if 0 is not represented properly then $\forall$ will win.

Strictly speaking, $\exists$ cannot win the game; they can only not lose. A *strategy* for $\exists$ determines a unique move for any given witness move and any N_{i-1}. This strategy is a *winning strategy* for $\exists$ if it never leads to a winning state for $\forall$.

Theorem 3.1.6 (Variant of [33, Prop. 4.2]). *Let $\mathcal{S}$ be a countable ordered semigroup with 0. Then $\mathcal{S}$ is representable if and only if $\exists$ has a winning strategy in $G_\omega(\mathcal{S})$.*

The proof is not presented here, since the one given in [33] is sufficiently close. We do need to note how a representation is generated from a winning strategy for $\exists$. Let

$N_0 \subseteq N_1 \subseteq \ldots$ be a play of a game starting from an initial move on a_0, a_1 and for some sequence of choices made by ∀, and in which ∃ uses their winning strategy. Let $N^*_{[a_0,a_1]} = \bigcup_{i\in\omega} N_i$. Let

$$N = \bigcup_{a_1 \nleq a_0} N^*_{[a_0,a_1]}.$$

By the proof given in [33], each of $N^*_{[a_0,a_1]}$ is a network and so too is N. The representation is given by the map

$$h(\rho) = \{(x, y) \colon \exists \rho^- \leqslant \rho, \rho^- \in N(x, y)\}.$$

Let's return to the topic of disjoint semigroups, which we think of as ordered semigroups with 0, with order given by a flat meet-semilattice. By Theorem 3.1.2, every such semigroup has a representation, and so there exists a winning strategy for ∃ for every such algebra. By equipping ∃ with a suitable strategy, however, we can ensure that the resulting representation is disjoint.

Theorem 3.1.7. *Every semigroup has a disjoint representation.*

Proof. Recall that in the initial move of the game, ∀ chooses $a_0, a_1 \in S$ such that $a_1 \nleq a_0$. The response to this initial move by ∃ is a network N_0, consisting of nodes x_1, x_2 with $N_0(x_1, x_2) = a_1$. ∀ wins if, at some stage i, ∃ introduces $a^- \in N_i(x_1, x_2)$ with $a^- \leqslant a_0$. We are dealing with unordered semigroups, so this condition needs to be reconsidered. We consider all semigroups as being equipped with an antichain order. The initial move is the same, with $N_0(x_1, x_2) = a_1$. But now ∀ wins if, at any stage i, ∃ introduces $a^- \in N_i(x_1, x_2)$ with $a^- \neq a_1$. The choice of element a_0 is irrelevant here, and is disregarded.

∃ has no choice when responding to composition moves, so we need only specify their response to witness moves. Regardless of the choice of nodes or semigroup elements made by ∀, we equip ∃ with a strategy of *always* adding a new node for witness moves.

First we show that the strategy of always adding new nodes is a winning strategy. It suffices to check that for all $i < n$ there is no $a^- \neq a_1$ with $a^- \in N_i(x_1, x_2)$. Equivalently, we show that the only element ∃ can introduce along (x_1, x_2) is a_1. Suppose that we have two 'chains' of directed edges from x_1 to x_2. That is, we have a chain of nodes $x_1k_1k_2 \ldots k_Kx_2$ and $x_1l_1l_2 \ldots l_Lx_2$ with two compositions $N_i(x_1, k_1)\,;\,N_i(k_1, k_2);\ldots;N_i(k_K, x_2)$ and $N_i(x_1, l_1);N_i(l_1, l_2);\ldots;N_i(l_L, x_2)$. These edges may have been introduced by witness or composition moves. If an edge was introduced by a composition move then we can replace it with the two edges used for that composition move. As such, as we can assume that each of these chains was constructed by a sequence of witness moves above a_1, each one building on the last. By associativity of composition, $N_i(x_1, k_1);N_i(k_1, k_2);\ldots;N_i(k_K, x_2)$ and $N_i(x_1, l_1);N_i(l_1, l_2);\ldots;N_i(l_L, x_2)$ both equal a_1. So the only element that can be introduced along $N_i(x_1, x_2)$ is a_1.

The same argument can be applied to any edge in a chain above $N_i(x_1, x_2)$. This tells us that a composition move cannot introduce a semigroup element along an edge on which there is already a different semigroup element. Hence, the resulting representation is disjoint. □

There is an interesting difference here between groups and semigroups. Every finite group has a finite disjoint representation; its Cayley representation. While an analogue of Cayley's Theorem exists for finite semigroups, the resulting Cayley representation is finite but not necessarily disjoint. Theorem 3.1.7 shows us that we can construct a disjoint representation of any semigroup, but over a countably infinite set. Can we construct a representation that entertains both properties—finiteness and disjointness—for any finite semigroup, as we can for finite groups? We will answer this question in the negative.

We offer an example here of a finite semigroup (signature $\{;\}$) that is, by Theorem 3.1.7, disjointly representable, but not finitely and disjointly representable. Our particular example includes elements $1'$ and 0, though in this specific case we do not demand that they be represented correctly. Even when not playing a game with ∀ and ∃, we will use the language of witness and composition moves, as they capture the notion of representing composition correctly.

Theorem 3.1.8. *There exists a finite, representable* $\{;, \cdot\}$*-algebra with no finite representation.*

Proof. Let $\mathcal{A}$ be the algebra with Cayley table defined as in Figure 3.1. Define $\cdot$ to make $\mathcal{A}$ a flat semilattice.

;	0	$1'$	a
0	0	0	0
$1'$	0	$1'$	a
a	0	a	a

Figure 3.1: Cayley table of a finite semigroup permitting no finite disjoint representation

To see that $\mathcal{A}$ is representable, observe that we can represent it as we would the point algebra over $\mathbb{Q}$ (Example 1.1.7). Specifically, let $h\colon A \to \wp(\mathbb{Q} \times \mathbb{Q})$ be defined by $h(0) = \varnothing$, $h(1') = \mathrm{id}_{\mathbb{Q}}$ and $h(a) = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \colon x < y\}$. Since $(\mathbb{Q}, <)$ is irreflexive, transitive and dense, h is a representation of $\mathcal{A}$ over the domain $\mathbb{Q}$. Now we prove that $\mathcal{A}$ can have no representation over a finite set.

Assume for contradiction that there exists a representation $h\colon \mathcal{A} \to \wp(X \times X)$, where X is a finite set. There must exist an edge (x, y) in $h(a)\backslash h(0)$, or else h is not a faithful representation. Since a is idempotent, a witness move on $(x, y) \in h(a)$ tells us that there exists $z \in X$ such that $(x, y), (x, z), (z, y) \in h(a)$. We can continue with witness moves on each of these edges, giving us a set of vertices $z_1, z_2, \ldots, z_n \in X$ such that for all $i \neq j$ and $i, j \leqslant n$ we have $(z_i, z_j) \in h(a)$ and $(z_1, z_n) \notin h(0)$ (relabelling $z_1 = x$ and $z_n = y$). But X is finite, so if $n > |X|$, the points $z_1, z_2, \ldots, z_n$ are not all distinct and we can find $i, j \leq n$ such that $z_i = z_j$ and $(z_i, z_j) \in h(a)$. We cannot have $(z_i, z_j) \in h(0)$ (else $(z_1, z_n) \in h(a; 0; a) = h(0)$). Thus there is a node, which we can assume without loss of generality to be x, such that $(x, x) \in h(a)\backslash h(0)$.

Since $1' ; a = a$ we can consider $y \in X$ such that $(x, y) \in h(1')$ and $(y, x) \in h(a)$. Then we have $(x, x) \circ (x, y) \in h(a\, ;\, 1')$, that is, $(x, y) \in h(a)$. This situation is illustrated in Figure 3.2. But $a \cdot 1' = 0$, and this composes with a to give $(x, x) \in h(0)$, a contradiction. So a finite representation cannot exist. ☐

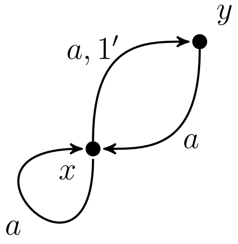

Figure 3.2: Relations on an idempotent in a finite representation

This result implies a lack of the finite representability for finite algebras (FRFA) property for reducts of the relation algebra signature, including composition and meet. This is because we can take the above example and easily include the other operations in the relation algebra signature.

Corollary 3.1.9. *Let* $\{;, \cdot\} \subseteq \tau \subseteq \{;, +, \cdot, \leq, \breve{}, -, 1', 0, 1\}$. *There are finite, representable τ-algebras with no finite representations.*

Proof. Consider the point algebra as in Example 1.1.7. This is in the full relation algebra signature. The reduct of the point algebra with only the operations in $\{;, \cdot\}$ contains a subalgebra isomorphic to the algebra used in the proof of Theorem 3.1.8, with $<$ acting as a. The point algebra is representable on $\mathbb{Q}$ with the usual dense linear order, and so every reduct of the point algebra is also representable. But every reduct of the point algebra with a signature containing $\{;, \cdot\}$ is not representable over a finite set, since such a representation would contain a finite representation of the algebra used in the proof of Theorem 3.1.8. ☐

We now consider the class $\mathscr{D}$ of semigroups with a finite disjoint representation. Initially we might ask if $\mathscr{D}$ is a *variety*, which is a class of algebras closed under taking direct products, subalgebras, and homomorphic images [2]. Since an infinite semigroup cannot have a finite representation, we know that such a class must consist only of finite semigroups. This excludes the possibility that $\mathscr{D}$ is a variety, since an infinite direct product of nonempty finite algebras is infinite.

So what if we restrict our attention to finite direct products? This is exactly what a *pseudovariety* is—a class of algebras closed under *finite direct products*, subalgebras, and homomorphic images. As it turns out, $\mathscr{D}$ is not even a pseudovariety. In fact, as the next result shows, we can move outside the class with just a single direct product.

Theorem 3.1.10. *The class of finite semigroups that are disjointly representable over a finite set is not closed under finite direct products.*

Proof. Consider the semigroup $\mathcal{S}$ consisting only of an idempotent f and a 0 element. This can be represented as a single node with a loop labelled f, and this representation is finite and disjoint. Consider now the semigroup $\mathcal{S} \times \mathcal{S}$, with Cayley table as below:

;	$(0,0)$	$(f,0)$	$(0,f)$	(f,f)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(f,0)$	$(0,0)$	$(f,0)$	$(0,0)$	$(f,0)$
$(0,f)$	$(0,0)$	$(0,0)$	$(0,f)$	$(0,f)$
(f,f)	$(0,0)$	$(f,0)$	$(0,f)$	(f,f)

Relabel $0 := (0,0)$, $a = (0,f)$ and $1' := (f,f)$. These form a subsemigroup of $S \times S$, with Cayley table as below:

;	0	a	$1'$
0	0	0	0
a	0	a	a
$1'$	0	a	$1'$

By Theorem 3.1.8, this is not finitely and disjointly representable. Hence, $\mathcal{S} \times \mathcal{S}$ is not finitely and disjointly representable. □

So $\mathscr{D}$ is not a pseudovariety. We will shortly see that $\mathscr{D}$ is not even closed under taking homomorphic images. However, $\mathscr{D}$ is closed under taking subalgebras. To see this, suppose that a finite semigroup $\mathcal{S}$ has a finite disjoint representation

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[2] A variety can also be considered as a class defined by identities. The equivalence of these two definitions is not trivial, and is the subject of Birkhoff's Theorem. See [13, Chapter 2, Theorem 11.9].

$h : \mathcal{S} \to \wp(X \times X)$. Suppose that $\mathcal{R}$ is a subsemigroup of $\mathcal{S}$ with $R \subseteq S$. Then one can simply take the restriction $h|_R$ as a finite disjoint representation of $\mathcal{R}$.

We will now explore a pair of strong necessary conditions for a finite semigroup to have a finite disjoint representation. Note that from now on we will omit the symbol ; for composition and write ab as shorthand for $a \,;\, b$.

Lemma 3.1.11. *Let $\mathcal{S}$ be a finite semigroup that is disjointly representable over a finite set. If there are distinct elements $b, c, e \in S$ such that e is idempotent and $0 \neq bce = bc$, then $c = ce$.*

Proof. Let $\mathcal{S}$ be a semigroup as above with distinct elements $b, c, e \in S$ such that e is idempotent and $0 \neq bce = bc$. Let $h : \mathcal{S} \to \wp(X \times X)$ be a disjoint representation of such a semigroup over finite X. Suppose that $bc = e$. Since e is idempotent and X is finite then h must represent $bc = e$ as a loop on a node somewhere, say x, as per the proof of Theorem 3.1.8. From x we witness $b \,;\, c = bc$ using another node y, and then compose to see a cb-labelled loop on y. This situation is drawn in Figure 3.3. We then compose again to get $(y, x) \in h(cbc)$. Since h is a disjoint representation we have that $c = cbc$ and so $c = ce$, as desired.

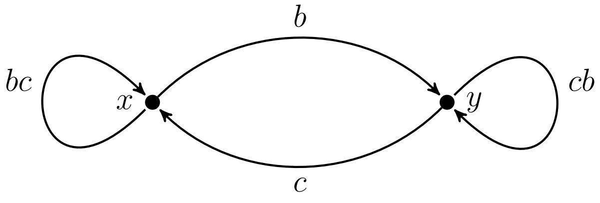


Figure 3.3: Consequence of $bc = e$

Now suppose that $bc \neq e$. For an edge labelled $bce = bc$ we must witness the composition $(bc)e = bce$. Suppose we do this by creating a new point and so a new edge labelled $bce = ce$. Since X is finite, if we repeat this step we must eventually have a loop involved in the composition. If the loop holds $bce = bc$, as in Figure 3.4, then from that node there must be an edge holding e. We compose to get an edge with $bce = bc$ and e. Since we are assuming that $bc \neq e$ and our representation h is disjoint, this is a contradiction, and so the loop must hold e instead, as in Figure 3.5.

If we also consider the witness move $b \,;\, c = bc$ it follows that in $h(S)$ we must witness the situation in Figure 3.6. Composing with this loop gives an edge holding both c and ce. Since h represents S disjointly, we conclude that $c = ce$. $\square$

It will prove useful to consider the dual of this property.

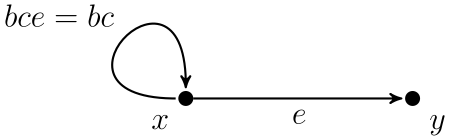


Figure 3.4: Witnessing $bce = bc$ on a loop

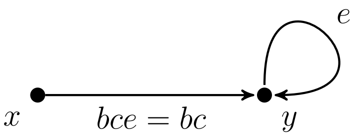


Figure 3.5: Witnessing e on a loop

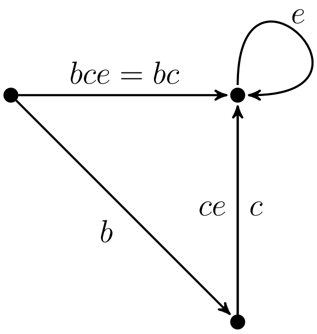


Figure 3.6: A finite representation of a semigroup with $bce = bc$

Corollary 3.1.12. *Let $\mathcal{S}$ be a finite semigroup that is disjointly representable over a finite set. If there are distinct elements $b, c, e \in S$ with e idempotent, then*

$$0 \neq ecb = cb \implies ec = c.$$

Proof. The proof is similar to that of Lemma 3.1.11. □

Corollary 3.1.13. *There exists a finitely disjointly representable semigroup with a quotient that is not finitely and disjointly representable. As a result, the class of finitely disjointly representable semigroups is not closed under homomorphisms.*

Proof. Consider the semigroup with finite disjoint representation in Figure 3.7. The only nonzero compositions here are $b\,;c = bc$, $c\,;e = ce$, $(b\,;c)\,;e = b\,;(c\,;e) = bce$, and an idempotent e. Consider the equivalence relation with $bce \sim bc$ and with everything else equivalent only to itself. In order to check that this is a congruence, it suffices to observe that right-composition by e preserves $\sim$. Then e is idempotent, $0 \neq bce = bc$, and $c \neq ce$. By Lemma 3.1.11, the quotient of this semigroup by $\sim$ is not finitely disjointly representable. □

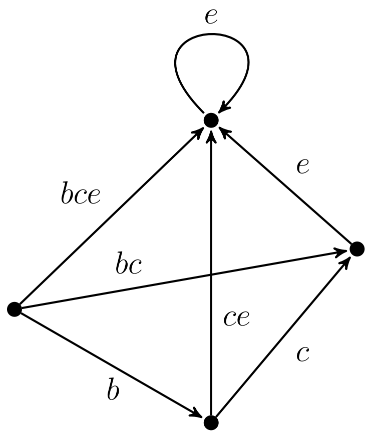


Figure 3.7: A finite disjoint representation of a semigroup

3.2 Green's relations

Before we explore the consequences of these conditions, we will need to introduce some fundamentals of semigroup theory. Of key importance are the relations of Green [29]. Much of the material used in defining these relations and their properties is from Chapters 2 and 3 of [41].

Let a and b be elements of a semigroup $\mathcal{S}$. Define $\mathcal{S}^{1'}$ to be the monoid into which $\mathcal{S}$ embeds. Then define quasiorders $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{J}$ on $\mathcal{S}$ as follows:

$$\begin{aligned} a \leqslant_{\mathcal{L}} b &\iff a \in \mathcal{S}^{1'} b, \\ a \leqslant_{\mathcal{R}} b &\iff a \in b\mathcal{S}^{1'}, \\ a \leqslant_{\mathcal{J}} b &\iff a \in \mathcal{S}^{1'} b \mathcal{S}^{1'}. \end{aligned}$$

Now define the equivalence relation $\mathcal{L}$ as $a\mathcal{L}b$ if $a \leqslant_{\mathcal{L}} b$ and $b \leqslant_{\mathcal{L}} a$, and similarly for $\mathcal{R}$ and $\mathcal{J}$. That is, $a\mathcal{L}b$ if a and b generate the same left ideals, $a\mathcal{R}b$ if they generate the same right ideal, and $a\mathcal{J}b$ if they generate the same ideal. We include the following for completeness.

Lemma 3.2.1. *$\mathcal{L}$ and $\mathcal{R}$ commute.*

Proof. Let $a, b \in S$ such that $a(\mathcal{L} \circ \mathcal{R})b$. We wish to prove that $a(\mathcal{R} \circ \mathcal{L})b$. Now there exists $c \in S$ such that $a\mathcal{L}c\mathcal{R}b$. So there exist $x, y, u, v \in S$ such that

$$xa = c, \quad cu = b, \quad yc = a, \quad bv = c.$$

So $au = ycu$. Hence,

$$auv = yvub = ybv = yc = a.$$

Similarly, $yb = ycu$, and

$$xyb = xycu = xau = cu = b.$$

Now $a\mathcal{R}(ycu)$ and $(ycu)\mathcal{L}b$, and so $a(\mathcal{R} \circ \mathcal{L})b$. The reverse inclusion is shown in a similar fashion. $\square$

With this in mind we can define a new relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and, because of Lemma 3.2.1 $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$. We then have that $\mathcal{L}, \mathcal{R}, \mathcal{D} \subseteq \mathcal{J}$. The following lemma, however, tells us that in a finite semigroup $\mathcal{D} = \mathcal{J}$. Since the focus of our investigation is on finite disjoint representations, this is of particular interest.

Lemma 3.2.2. *Let $\mathcal{S}$ be a* periodic semigroup*; that is, a semigroup in which all elements are of finite order. Then $\mathcal{D} = \mathcal{J}$.*

Proof. Suppose $a, b \in S$ and $a\mathcal{J}b$. Then there exists $x, y, u, v \in S$ such that

$$xay = b, \quad ubv = a.$$

Then

$$\begin{aligned} a &= (ux)a(yv) = (ux)^2 a(yv)^2 = (ux)^3 a(yv)^3 = \dots \\ b &= (xu)b(yv) = (xu)^2 b(vy)^2 = (xu)^3 b(vy)^3 = \dots \end{aligned}$$

Since $\mathcal{A}$ is periodic there is an idempotent power ω_1 such that $(ux)^{\omega_1}(ux)^{\omega_1} = (ux)^{\omega_1}$. As such,

$$\begin{aligned} a &= (ux)^{\omega_1} a(yv)^{\omega_1} \\ &= (ux)^{\omega_1}(ux)^{\omega_1} a(yv)^{\omega_1} \\ &= (ux)^{\omega_1} a \\ &= (ux)^{\omega_1 - 1} uxa \end{aligned}$$

Define $c := xa$. Then $a\mathcal{L}c$. Now consider ω_2 such that $(vy)^{\omega_2}(vy)^{\omega_2} = (vy)^{\omega_2}$. Following a similar argument to the above, we get that $c = bvy(vy)^{\omega_2 - 1}$. Hence, $c\mathcal{R}b$ and so $a(\mathcal{L} \circ \mathcal{R})b$, that is, $a\mathcal{D}b$. $\square$

Since only finite semigroups can have finite disjoint representations, and all finite semigroups are periodic, in this chapter we can safely refer to all $\mathcal{D}$-classes as $\mathcal{J}$-classes. The end result is that we can partition any finite semigroup into a finite number of $\mathcal{J}$-classes, which in turn we can partition into either $\mathcal{L}$-classes or $\mathcal{R}$-classes.

So far we have introduced four different relations on the elements of a semigroup: $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$, and identified $\mathcal{J}$ with $\mathcal{D}$ for finite semigroups. There is a pleasing graphical representation of these relations. We draw each $\mathcal{J}$-class of a semigroup as a rectangle, divided into a grid. If two elements are in the same row then they are $\mathcal{L}$-related, and if they are in the same column then they are $\mathcal{R}$-related. That is, each cell of the grid is an intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class. Recall that by Lemma 3.2.1, $\mathcal{L}$ and $\mathcal{R}$ commute, and so a grid is a valid tool for visualising the $\mathcal{L}$ and $\mathcal{R}$ relations within a given $\mathcal{J}$-class.

Furthermore, the $\mathcal{J}$-classes are ordered by the $\leqslant_{\mathcal{J}}$ relation. An example is given in Figure 3.8. Note that the 0 element, if it exists, is in its own $\mathcal{J}$-class, and this is the unique bottom class with respect to $\mathcal{J}$.

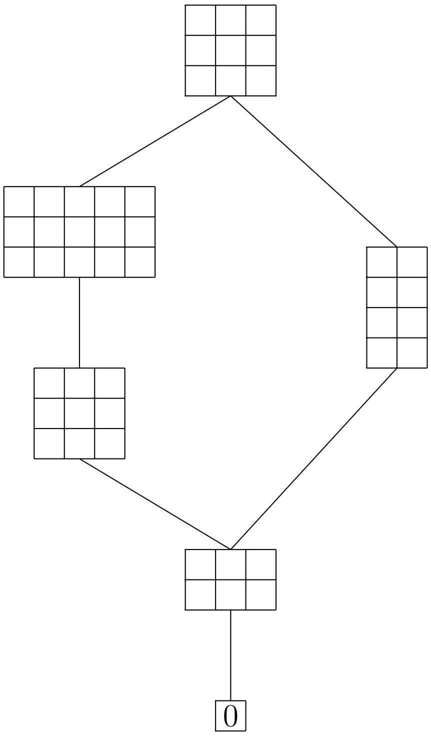


Figure 3.8: An example of a finite semigroup partitioned and ordered by its $\mathcal{J}$-classes, with $\mathcal{L}$- and $\mathcal{R}$-classes shown by grids

An element $a \in \mathcal{S}$ is called *regular* if there exists $x \in S$ such that $a = axa$. A semigroup is regular if all of its elements are. With the following lemma, we can also discuss regular $\mathcal{J}$-classes of finite semigroups. Denote by $\mathcal{J}_a$ the $\mathcal{J}$-class containing a, and similarly for $\mathcal{L}_a$, $\mathcal{R}_a$ and $\mathcal{J}_a$. The following is a standard lemma that can be found in any textbook on semigroups, but we include a proof here for completeness.

Lemma 3.2.3. *If a is a regular element of a semigroup, then so too is every element of $\mathcal{J}_a$.*

Proof. Since a is regular there exists $x \in S$ such that $a = axa$. Suppose $a\mathcal{L}b$ for some $b \in \mathcal{S}$. Then $b = ua$ for some $u \in S$. Now $b = uaxa = bxa$. As $a\mathcal{L}b$, we have that $a = vb$ for some $v \in S$. Hence, $b = bxa = b(xv)b$ and so every element in $\mathcal{L}_a$ is regular. By similar reasoning, every element in $\mathcal{R}_a$ is regular. □

We require one final notion before we can discuss Green's relations in the context of finite disjoint representations.

Lemma 3.2.4. *In a regular $\mathcal{J}$-class of a finite semigroup, each $\mathcal{L}$- and $\mathcal{R}$-class contains an idempotent.*

Proof. If $a = axa$, then xa is an idempotent of $\mathcal{L}_a$ and ax is an idempotent of $\mathcal{R}_a$. □

We can use the results of Lemma 3.1.11 and Corollary 3.1.12 to derive information about the $\mathcal{J}$-classes of a finitely and disjointly representable semigroup. In particular, we learn the following restriction on where a regular $\mathcal{J}$-class can occur in the $\leqslant_{\mathcal{J}}$ order.

Theorem 3.2.5. *In a finitely and disjointly representable semigroup, a non-zero regular $\mathcal{J}$-class is maximal with respect to $\leqslant_{\mathcal{J}}$.*

Proof. Let $\mathcal{S}$ be such a representable semigroup with $e, x \in S$ such that $0 \neq e = ee$ and $e \leqslant_{\mathcal{J}} x$. That is, there exists $a, b \in S$ such that $e = axb$. Then

$$axbe = ee = e = axb.$$

So we may assume without loss of generality that $be = b$, or derive the equality from Lemma 3.1.11. Then $b\mathcal{J}e$ and, in particular, $xb\mathcal{J}e$. Recall that in a regular $\mathcal{J}$-class, every $\mathcal{L}$- and $\mathcal{R}$-class contains at least one idempotent. So there exists $f \in S$ such that

$$e\mathcal{L}(xb)\mathcal{R}f = ff.$$

Then there exist $z \in S$ such that $fz = xb$. So $fxb = xb$. From Corollary 3.1.12, $x = fx$. That is, $x \leqslant_{\mathcal{J}} f\mathcal{J}e$, and so $x \leqslant_{\mathcal{J}} e$. Combined with the initial assumption that $e \leqslant_{\mathcal{J}} x$, we have that $e\mathcal{J}x$, and so the regular $\mathcal{J}$-class containing e is maximal with respect to $\leqslant_{\mathcal{J}}$. □

The case in which a finite semigroup contains a single non-zero $\mathcal{J}$-class which is also regular corresponds to the class of finite *completely [0-]simple*[3] semigroups. These

---

[3] The proper definition of a completely [0-]simple semigroup is not considered here. For the finite case it suffices to consider them as semigroups with a single non-zero $\mathcal{J}$-class, which is regular. The [0-] prefix indicates that a zero element may or may not be present.

semigroups correspond exactly to those that have a special type of representation. A semigroup is completely [0-]simple if and only if has a disjoint transitive representation [79]. By transitive, we mean that every edge in the representation is labelled by *at least* one element of the semigroup. This is different from a disjoint representation in which we label every edge by *at most* one element of the semigroup.

The matter of finite disjoint representability is a gap in the literature. The class of finite semigroups permitting a transitive, disjoint and finite representation is exactly the completely [0-]simple semigroups, which have a very specific and simple $\mathscr{J}$-class structure. Every finite semigroup permits a representation which is transitive and finite but not necessarily disjoint [63]. Thus the finite disjoint representability of finite semigroups is the missing case. We now know from Theorem 3.2.5 that if such a representation of a semigroup exists, then a non-zero regular $\mathscr{J}$-class of that semigroup must be maximal with respect to $\leqslant_{\mathscr{J}}$.

3.3 Rees quotients and pure direct products

In Theorem 3.1.10 we proved that the class $\mathscr{D}$ of finite semigroups permitting a finite disjoint representation is not closed under direct products. In Corollary 3.1.13 we proved that $\mathscr{D}$ is not closed under taking quotients either. We will now discuss two concepts which are similar to direct products and quotients and under which $\mathscr{D}$ is closed. We begin with a special kind of quotient introduced by Rees [72].

We define a *Rees quotient* of a semigroup $\mathcal{S}$ to be the quotient of $\mathcal{S}$ by an ideal. That is, a Rees quotient of $\mathcal{S}$ is obtained by factoring $\mathcal{S}$ by a congruence whose one nontrivial block is an ideal. We note that if $\mathcal{S}$ is a semigroup with 0, then taking a Rees quotient has the same effect as identifying an ideal with 0. Importantly for our purposes, the class $\mathscr{D}$ is closed under taking Rees quotients.

Lemma 3.3.1. *Let $\mathcal{S}$ be a finitely and disjointly representable semigroup and suppose I is an ideal of $\mathcal{S}$. Then the Rees quotient $\mathcal{S}/I$ is also finitely and disjointly representable.*

Proof. Let $\phi : \mathcal{S} \to \wp(X \times X)$ be a finite, disjoint representation, representing 0 as the empty set. Define a new representation ψ such that

$$\begin{aligned} \psi :\quad S &\to \wp((X \times S^{1'}\backslash I) \times (X \times S^{1'}\backslash I)) \\ s &\to \{((x,a),(y,as))\colon (x,y) \in \phi(s) \text{ and } a, as \in S^{1'}\backslash I\}. \end{aligned}$$

We note that if $\mathcal{S}$ contains a 0 element, then $0 \in I$ and so ψ would represent 0 as the empty set.

To verify that composition moves are represented correctly, consider the composition ab in $\psi(S/I)$. That is, consider the situation as in Figure 3.9. Since $tab \notin I$, we can be sure that $(ab) \notin I$. Then, since $(x, y) \in \phi(a)$, $(y, z) \in \phi(b)$ and $(x, z) \in \phi(ab)$, we have that $((x, t), (z, tab)) \in \psi(ab)$ as in Figure 3.10. That is, compositions are respected by ψ.

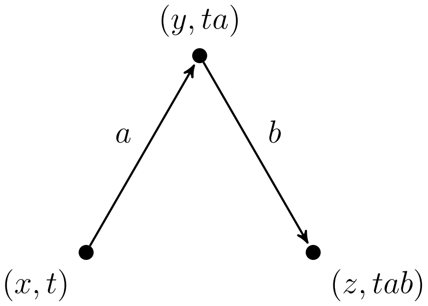


Figure 3.9: A composition in $\psi(S)$

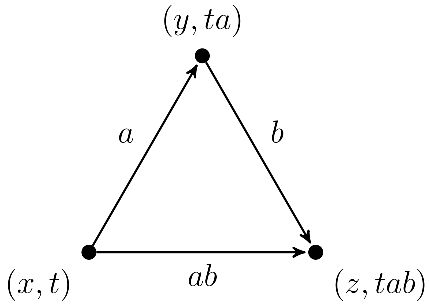


Figure 3.10: The completed composition

Now suppose $a, b, c \in S \backslash I$ such that $ab = c$ and we witness $((x, t), (z, tc)) \in \psi(c)$. First we note that there exists $y \in X$ such that $(x, y) \in \phi(a)$ and $(y, z) \in \phi(b)$, since ϕ is a representation of S. Since $tc \notin I$ and $tc = tab$ it follows that $ta \notin I$. Thus $((x, t), (y, ta)) \in \psi(a)$. Then $((y, ta), (z, tab)) \in \psi(b)$ also. Hence are are able to witness $ab = c$ in $\psi(S \backslash I)$.

Next we note that ψ represents $S \backslash I$ disjointly. This is because $((x, t), (y, ta)) \in \psi(a)$ and $((x, t), (y, ta)) \in \psi(b)$. So $(x, y) \in \phi(a)$ and $(x, y) \in \phi(b)$ which implies $a = b$. Hence, ψ is a finite and disjoint representation of $S \backslash I$.

Finally we observe that ψ represents $S \backslash I$ faithfully. By disjointness, this just requires that each $a \in S \backslash I$ labels some edge. Let $(x, y) \in \phi(a)$. Then $((x, 1'), (y, a)) \in \psi(a)$, as required. $\square$

There exists something of a weak converse to this lemma.

Lemma 3.3.2. *Let I_1, I_2 be ideals of a semigroup $\mathcal{S}$ with 0 such that $I_1 \cap I_2 = \{0\}$. If $\mathcal{S}/I_1$ and $\mathcal{S}/I_2$ are finitely and disjointly representable, then so too is S.*

Proof. Let ϕ_1 and ϕ_2 be finite disjoint representations as follows:

$$\begin{aligned} \phi_1 : \quad S/I_1 &\to \wp(X \times X) \\ \phi_2 : \quad S/I_2 &\to \wp(Y \times Y). \end{aligned}$$

Define a function ϕ such that

$$\begin{aligned} \phi: \quad S &\to \wp((X \cup Y) \times (X \cup Y)) \\ s &\to \phi_1(s) \sqcup \phi_2(s) \end{aligned}$$

where $\sqcup$ is the disjoint union. That is to say, the image of ϕ is the disjoint union of the images of ϕ_1 and ϕ_2. Since each of these two components is itself a finite disjoint representation of its respective Rees quotient, composition is preserved in each, and therefore in the image of ϕ. Furthermore, a disjoint union of representations, each of which is itself disjoint, must also be disjoint, since no elements are introduced to an edge coming from either $\phi_1(S/I_1)$ or $\phi_2(S/I_2)$.

Furthermore, for every element in the image of S under ϕ, we can witness a composition above it in either ϕ_1 or ϕ_2, or potentially both. Suppose $(x, z) \in \phi(c)$ with $ab = c$. Since $I_1 \cap I_2 = \{0\}$ the element e cannot appear in both I_1 and I_2. Suppose without loss of generality that $c \notin I_2$. Then c will appear in $\phi_2(S/I_2)$. Since $c \leqslant_{\mathcal{J}} a$ and $c \leqslant_{\mathcal{J}} b$ we know that $a, b \notin I_2$, so none of these elements will be collapsed in S/I_2. As ϕ_2 is a representation there exists $y \in Y$ such that $(x, y) \in \phi_2(a)$ and $(y, z) \in \phi_2(b)$. This node y will also appear in $\phi(S)$, and so the composition $ab = c$ is witnessed. Hence, witness moves are preserved by ϕ. □

Rees quotients can be used to define a special kind of direct product we call a *pure direct product*.

Definition 3.3.3. Let S and T be semigroups, each with a 0. The *pure direct product* of semigroups S and T, denoted $S \otimes T$, is the Rees quotient of the direct product $S \times T$ by the ideal consisting of all elements with at least one coordinate equal to 0.

We say that a congruence θ *saturates* 0 if $x \; \theta \; 0 \implies x = 0$. That is, the congruence class $[0]$ is just the singleton $\{0\}$.

Lemma 3.3.4. *Let θ_1 and θ_2 be two 0-saturating congruences on a semigroup S and such that $\theta_1 \wedge \theta_2 = \{(x, x) : x \in S\}$, the diagonal congruence. Then S embeds into the pure direct product $S/\theta_1 \otimes S/\theta_2$.*

Proof. Define the map h as follows:

$$\begin{aligned} h: \quad S &\to \quad S/\theta_1 \otimes S/\theta_2 \\ s &\mapsto \begin{cases} 0 & \text{if } s = 0 \\ (s/\theta_1, s/\theta_2) & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose $h(s) = (x, y) \in S/\theta_1 \otimes S/\theta_2$. If $x = 0$ then the θ_1 class containing s would be $\{0\}$, as θ_1 is 0-saturating. Similarly, if $y = 0$ then the θ_2 class containing s would be $\{0\}$. Hence, the map h is well-defined.

Since both θ_1 and θ_2 saturate 0, if either x or y is 0 then $s = 0$.

Suppose $h(x) = (x/\theta_1, x/\theta_2)$ and $h(y) = (y/\theta_1, y/\theta_2)$ are elements of $S/\theta_1 \otimes S/\theta_2$ with each coordinate non-zero. Suppose furthermore that $(x/\theta_1, x/\theta_2) = (y/\theta_1, y/\theta_2)$. So θ_1 and θ_2 agree on these coordinates. But $\theta_1 \wedge \theta_2 = \{(x, x) : x \in S\}$, and so $x = y$. Hence, h is injective.

Finally we check that h is a homomorphism. If $s_1 = 0$ then $h(s_1) = 0$ and if $s_2 = 0$ then $h(s_2) = 0$. In either case it is trivially true that $h(s_1)h(s_2) = h(s_1 s_2)$, so suppose that neither s_1 or s_2 is 0.

Then

$$
\begin{aligned}
h(s_1)h(s_2) &= (s_1/\theta_1, s_1/\theta_2)\,(s_2/\theta_1, s_2/\theta_2) \\
&= ((s_1/\theta_1)(s_2/\theta_1), (s_1/\theta_2)(/s_2/\theta_2)) \\
&= ((s_1 s_2)/\theta_1, (s_1 s_2)/\theta_2)\,.
\end{aligned}
$$

If $s_1 s_2 = 0$ then this last line is equal to $h(0, 0) = 0$, and so $h(s_1, s_2) = h(s_1 s_2)$. If $s_1 s_2 \neq 0$ then this last line is equal to $h(s_1 s_2)$. Hence, h is a homomorphism and so an embedding. □

Theorem 3.3.5. *Let θ_1 and θ_2 be two congruences on a semigroup S saturating* 0 *and such that $\theta_1 \wedge \theta_2 = \{(x, x) : x \in S\}$, the diagonal congruence. If S/θ_1 and S/θ_2 are finitely and disjointly representable with* 0 *represented as the empty relation, then so is S.*

Proof. Let $\phi_1 : S/\theta_1 \to \wp(X \times X)$ and $\phi_2 : S/\theta_2 \to \wp(Y \times Y)$ be such representations. Define a map $\phi : S/\theta_1 \otimes S/\theta_2 \to \wp((X \times Y) \times (X \times Y))$ such that $((x_1, y_1), (x_2, y_2)) \in \phi(a, b)$ if and only if $(x_1, x_2) \in \phi_1(a)$ and $(y_1, y_2) \in \phi_2(b)$.

Composition is pointwise and, since ϕ_1 and ϕ_2 respect composition, so too does ϕ. We also note that 0 cannot appear as a coordinate in the image of $S/\theta_1 \otimes S/\theta_2$ under ϕ. Suppose it did appear; that is, suppose without loss of generality that $((x_1, y_1), (x_2, y_2)) \in \phi(0, b)$ for some $b \in S$. Then $(x_1, x_2) \in \phi_1(0)$. But ϕ_1 represents 0 as the empty relation, so this cannot be.

Now all that remains is to verify that witness moves are preserved by ϕ. Suppose $((x_1, z_1), (x_2, z_2)) \in \phi(c_1, c_2)$ and $(a_1, a_2)(b_1, b_2) = (c_1, c_2)$. Since witness moves are preserved in ϕ_1 and ϕ_2 there exists $y_1 \in X$ and $y_2 \in Y$ such that

$$\begin{aligned} (x_1, y_1) &\in \phi_1(a_1) \qquad & (x_2, y_2) &\in \phi_2(b_1) \\ (y_1, z_1) &\in \phi_1(a_2) \qquad & (y_2, z_2) &\in \phi_2(b_2). \end{aligned}$$

So we can find the composition shown in Figure 3.11 in the image of $S/\theta_1 \otimes S/\theta_2$ under ϕ. Hence, witness moves are correctly represented by ϕ.

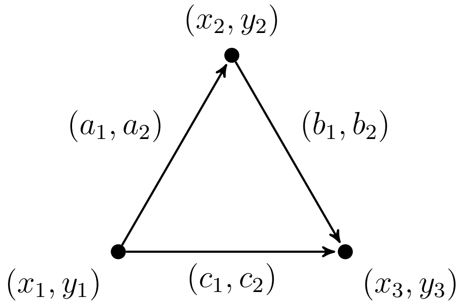


Figure 3.11: Witnessing a composition in the representation ϕ

Finally we note that ϕ represents $S/\theta_1 \otimes S/\theta_2$ disjointly, for if it did not we could project down to either coordinate to obtain a non-disjoint representation with ϕ_1 or ϕ_2. Now we take $\phi \circ h : S \to \wp((X \times Y) \times (X \times Y))$, with h as defined in Lemma 3.3.4, as our representation. $\square$

While $\mathscr{D}$ is not closed under taking (finite) direct products or quotients, it is closed under taking (finite) pure direct products and Rees quotients. Furthermore, we have from Theorem 3.2.5 that every regular $\mathcal{J}$-class in a semigroup in $\mathscr{D}$ must be maximal with respect to $\leqslant_{\mathcal{J}}$.

As we will prove shortly, the end result is that to consider finite disjoint representability of semigroups, we need only consider the semigroups with a particularly restrictive $\mathcal{J}$-structure. Specifically, we need only consider semigroups in which all nonmaximal $\mathcal{J}$ classes are not regular and in which there is only one $\mathcal{J}$-class above the $\mathcal{J}$-class containing 0.

The following lemma will set us up to use induction to prove this.

Lemma 3.3.6. *Let $\mathcal{S}$ be a finite semigroup with 0, and suppose that $\mathcal{S}$ has minimal nonzero $\mathcal{J}$ classes $U_1, U_2, \ldots, U_n$. Denote by $I_1, I_2, \ldots I_n$ the ideals $U_1 \cup \{0\}, U_2 \cup \{0\}, \ldots, U_n \cup \{0\}$. For each subset $\varnothing \neq A \subset \{I_1, I_2, \ldots, I_k\}$ let $\bar{A}$ be the ideal of all elements dividing (below with respect to the $\mathcal{J}$ order) the nonzero elements of $\bigcup_{I \in A} I$ and not dividing the nonzero elements of $\bigcup_{I \notin A} I$. Then the minimal nonzero $\mathcal{J}$-classes of $\mathcal{S}/\bar{A}$ are those in the ideals in $\{I_1, \ldots, I_k\} \backslash A$.*

Before we get to the proof, let us take a moment to visualise this lemma. Consider a semigroup with minimal nonzero $\mathcal{J}$-classes U_1, U_2, U_3, U_4. When we union each of these with 0 we get the ideals I_1, I_2, I_3, I_4. An example of such a semigroup $\mathcal{S}$ is given in Figure 3.12, and this particular example includes additional $\mathcal{J}$-classes V_1 and V_2, and with $A = \{I_3, I_4\}$.

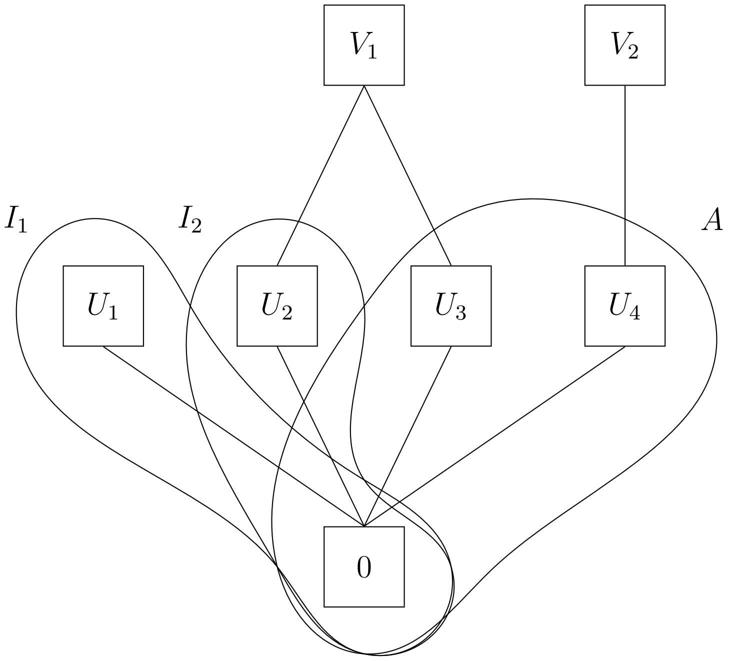


Figure 3.12: Minimal nonzero $\mathcal{J}$-classes and associated ideals for a semigroup $\mathcal{S}$

If we quotient $\mathcal{S}$ by the ideal generated by the union of the ideals in A then we identify U_3 and U_4 with 0. The $\mathcal{J}$-class V_2 'moves down' the $\mathcal{J}$-ordering. The minimal nonzero $\mathcal{J}$-classes of $\mathcal{S}/A$ are then U_1, U_2 and V_2.

Now we extend A to $\bar{A}$. In this example every element of V_2 is above an element of U_4, but not above any nonzero element in $I_1 \cup I_2$. Hence, $V_2 \subseteq \bar{A}$. This is not the case for V_1 for which every element is above an element of I_2. Hence, $V_1 \nsubseteq \bar{A}$. If we quotient $\mathcal{S}$ by $\bar{A}$ then U_3, U_4 and V_2 are identified with zero, and we are left with two remaining minimal nonzero $\mathcal{J}$-classes: U_1 and U_2. This result is shown in Figure 3.13.

In other words, this lemma allows us to quotient in such a way that we can get rid of minimal nonzero $\mathcal{J}$-classes without introducing new ones.

Proof. When we quotient by an ideal, we identify every element in that ideal with 0. Since either no element of a $\mathcal{J}$-class or every element of a $\mathcal{J}$-class is in an ideal, this has the effect of identifying entire $\mathcal{J}$-classes with 0. A nonzero $\mathcal{J}$-class $U \notin \{U_1, \ldots, U_k\}$ that is minimal in $S/\bar{A}$ can only arise if it is above a nonzero element of $\bar{A}$ and not above any nonzero element of $\{I_1, \ldots, I_k\} \backslash A$. That is, U would be a non-minimal $\mathcal{J}$-class in $\mathcal{S}$ that 'moves down' the $\mathcal{J}$-ordering when we identify $\mathcal{J}$ class below it

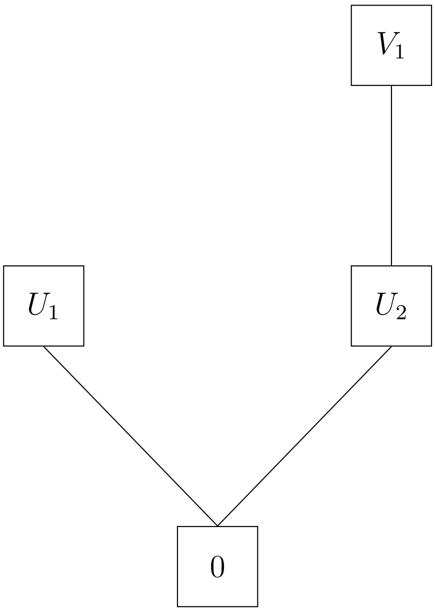


Figure 3.13: Minimal nonzero $\mathcal{J}$-classes in $\mathcal{S}/\bar{A}$

with 0. Yet we have defined $\bar{A}$ such that no $\mathcal{J}$-class can be above it without being above a nonzero element in $\{I_1, \ldots, I_k\}\backslash A$. □

As a consequence, the number of minimal nonzero $\mathcal{J}$-classes in $\mathcal{S}/\bar{A}$ is $k - |A|$. We also note that if $\mathcal{S}/A$ is representable, then so too is $\mathcal{S}/\bar{A}$ by Lemma 3.3.1. This allows us to prove the following theorem.

Theorem 3.3.7. *Let $\mathcal{S}$ be a finite semigroup with 0, and suppose that $\mathcal{S}$ has minimal nonzero $\mathcal{J}$ classes $U_1, U_2, \ldots, U_n$ with $n \geqslant 2$. Denote by $I_1, I_2, \ldots I_n$ the ideals $U_1 \cup \{0\}, U_2 \cup \{0\}, \ldots, U_n \cup \{0\}$ and let $\mathcal{I} = I_1 \cup I_2 \cup \cdots \cup I_n$. Then $\mathcal{S}$ is finitely and disjointly representable if and only if each of the Rees quotients $\mathcal{S}/\overline{(\mathcal{I}\backslash U_1)}$, $\mathcal{S}/\overline{(\mathcal{I}\backslash U_2)}$, ..., $\mathcal{S}/\overline{(\mathcal{I}\backslash U_n)}$ is finitely and disjointly representable.*

Proof. Each of the $\overline{(\mathcal{I}\backslash U_n)}$ is an ideal of $\mathcal{S}$. If S is finitely and disjointly representable then by Lemma 3.3.1 so too is each of $\mathcal{S}/\overline{(\mathcal{I}\backslash U_1)}$, $\mathcal{S}/\overline{(\mathcal{I}\backslash U_2)}$, ..., $\mathcal{S}/\overline{(\mathcal{I}\backslash U_n)}$.

We prove the converse by induction. We note that the base case of $n = 2$ is a direct application of Lemma 3.3.2. Suppose the statement is true for semigroups with k minimal nonzero $\mathcal{J}$-classes. That is, suppose that for all semigroups $\mathcal{S}$ with minimal nonzero $\mathcal{J}$-classes $U_1, U_2, \ldots, U_k$, if each of $\mathcal{S}/\overline{(\mathcal{I}\backslash U_1)}$, $\mathcal{S}/\overline{(\mathcal{I}\backslash U_2)}$, ..., $\mathcal{S}/\overline{(\mathcal{I}\backslash U_n)}$ is representable then so too is $\mathcal{S}$.

Now consider a semigroup with minimal nonzero $\mathcal{J}$-classes $U_1, U_2, \ldots, U_k, U_{k+1}$ and suppose that each of $\mathcal{S}/\overline{(\mathcal{I}\backslash U_1)}$, $\mathcal{S}/\overline{(\mathcal{I}\backslash U_2)}$, ..., $\mathcal{S}/\overline{(\mathcal{I}\backslash U_k)}$, $\mathcal{S}/\overline{(\mathcal{I}\backslash U_{k+1})}$ is finitely and disjointly representable. Then by Lemma 3.3.6, $S/\overline{I_{k+1}}$ has k minimal ideals. By the inductive assumption, $S/\overline{I_{k+1}}$ is finitely and disjointly representable. So now we have that $\overline{I_{k+1}}$ and $\overline{(\mathcal{I}\backslash U_{k+1})}$ are ideals of $\mathcal{S}$ intersecting at $\{0\}$. By Lemma 3.3.2, $\mathcal{S}$ is finitely and disjointly representable. $\square$

This restrictive $\mathscr{J}$-structure outlines a potential proof approach to defining a characterisation of $\mathscr{D}$. While we were unable to find such a representation, we conjecture that the conditions in Lemma 3.1.11 and its dual, Corollary 3.1.12, are not only necessary but sufficient for the finite disjoint representability of a semigroup.

Conjecture 3.3.8. *Let $\mathcal{S}$ be a finite semigroup. Then $\mathcal{S}$ is finitely disjointly representable if and only if for all distinct elements $b, c, e \in S$ with e idempotent, $0 \neq ecb = cb \implies ec = c$ and $0 \neq ecb = cb \implies ec = c$.*

3.4 Representations as functions

A special kind of representation is one in which every element of the algebra acts as a function[4]. That is, a representation h over a domain X is a representation as functions if for all $x, y, z \in X$ and for all elements a of the algebra, $(x, y) \in h(a)$ and $(x, z) \in h(a)$ implies $y = z$. This is analogous to the requirement that for a function to be well-formed, an element in the domain must be mapped to a *unique* element in the codomain.

We finish this chapter with a brief exploration of representations of meet-ordered semigroups in which the meet semilattice is flat. First we note that the more general case has been covered by Garvac′kiĭ [27].

Theorem 3.4.1 ([27])**.** *Let $\mathcal{A}$ be an algebra in the signature $(;, \cdot)$. Then $\mathcal{A}$ is representable as an algebra of functions if and only if the following axioms hold:*

1. $((a \cdot b)c) \cdot (ac) = (a \cdot b)c$
2. $a(b \cdot c) = (ab) \cdot (ac)$
3. $((u \cdot v)a) \cdot ((v \cdot w)b) = ((u \cdot w)a) \cdot ((v \cdot w)b)$

These conditions are much simpler for the case in which $\cdot$ is flat. Schein [82] was the first to prove the following using difference semigroups with methods not covered here. We offer another proof.

---

[4] What we call a 'function' here might be referred to as a 'partial function' in other texts.

Theorem 3.4.2. *Let $\mathcal{S}$ be an algebra in the signature $(;, \cdot, 0)$ such that $\cdot$ is the meet operation of a flat semilattice. That is, $a \cdot b = a$ if $a = b$ and 0 otherwise. Then $\mathcal{S}$ is representable as an algebra of functions if and only if*

$$0 \neq xa = xb \implies a = b.$$

Proof. For such an algebra $\mathcal{S}$ we construct a representation similar to that of Cayley's theorem (Theorem 3.1.2). Specifically, with $\mathcal{S}^{1'}$ the monoid into which the underlying semigroup of $\mathcal{S}$ embeds, we have that for all $x, y \in \mathcal{S}^{1'}$ and $a \in S$, $(x, y) \in \varphi(s)$ if and only if $xa = y$.

In the representation we then delete the point corresponding to 0, and all edges to and from it. This also has the effect of forcing 0 to be represented as the empty set.

First we prove that φ is a representation of $\mathcal{S}$. Composition moves are respected as if $(x, y) \in \varphi(a)$ and $(y, z) \in \varphi(b)$ then $y = xa$ and $z = yb$. Hence, $z = x(ab)$ and so $(x, z) \in \varphi(ab)$. Witness moves are also respected: if $(x, z) \in \varphi(sr)$ then $xs \neq 0$ and so $(x, xs) \in \varphi(s)$ and $(xs, z) \in \varphi(r)$. The representation is faithful since we included $1'$ as a point in the domain, and so $(1', a) \in \varphi(s)$ if and only if $a = s$.

Now we check that the representation is disjoint, that is, it respects $\cdot$. Suppose $(x, y) \in \varphi(a)$ and $(x, y) \in \varphi(b)$ with $a \neq b$. Then $y = xa$ and $y = xb$. Since we removed the vertex 0 from the representation, we have that $0 \neq xa = xb$. By assumption, $a = b$ and so the representation is disjoint.

Finally we check that the representation is that of an algebra of functions. Suppose $(x, y) \in \varphi(a)$ and $(x, z) \in \varphi(a)$. Then $y = xa$ and $z = xa$ and so $y = z$.

For the other direction, suppose that h is a representation of $\mathcal{S}$ as an algebra of functions. Suppose we have that $0 \neq xa = xb$. That is, in the representation given by h we have the situation as drawn in Figure 3.14.

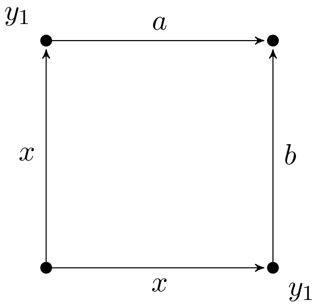


Figure 3.14: The situation $xa = xb$ in a representation of $\mathcal{S}$ as an algebra of functions

Then, since h represents every element as a function, we have that $y_1 = y_2$. Hence, $xa = xb$. Since the representation is disjoint, $a = b$. $\square$

Chapter 4

Qualitative representations

The situation with relation algebras is rather dismal. Representability cannot be characterised by a finite number of elementary axioms. In fact, there isn't even any possible algorithm that can decide, for every finite relation algebra, whether or not a representation exists. In Chapters 2 and 3 we weakened the signature or placed additional restrictions on the algebra. In this chapter, we will look instead at weakening the notion of a *representation.*

First let us fix our terminology. Let's call the concept of *representation* that we introduced in Chapter 1 a *relation algebra representation*. This is to distinguish it from the weaker notions of representation that we are about to introduce.

The first such notion is a *weak representation*[1], which has been extensively discussed in the literature. In weakening the notion of representation, we also need to weaken the abstract setting as well. What a relation algebra is to a relation algebra representation, a *nonassociative algebra* is to a weak representation. Every relation algebra is a nonassociative algebra, but the converse is not true. Every relation algebra representation is a weak representation but here too, the converse is not true.

When we talk about representations of a relation algebra we usually refer to an isomorphism onto a proper relation algebra, as per Definition 1.1.5. We would expect the same when talking about weak representations of nonassociative algebras, but this is not the case. In fact, nonisomorphic nonassociative algebras can share weak representations.

This is not the case for *qualitative representations*. Qualitative representations tend to be implicit in works on weak representations, e.g. [51], but the first clear definition is given by Hirsch et al. [38]. We will define all of these concepts properly, but

---

[1] We do not discuss the concept of a weak representation as discussed by Jónsson and Tarski [46].

intuitively, a qualitative representation is stronger than a weak representation but weaker than a relation algebra representation. In a relation algebra representation, every composition must be seen wherever it can be seen; that is, composition must be represented through both composition moves and witness moves. In a weak representation, we require only composition moves. In a qualitative representation, we require composition moves, and that every composition appear at least *once* in the representation. That is, if $c \leqslant a \,;\, b$, then in a qualitative representation we should see a triangle representing this composition *somewhere*. However, we do not need to witness $a \,;\, b$ above every c.

Weak representations have a plethora of practical applications through the use of *constraint satisfaction*. Researchers have used nonassociative algebras and their weak representations to move a robot past a human in a narrow corridor [30], identify the leaders of a flock of birds [59], and even to play Angry Birds [92]. A constraint may be something like "move the robot through the corridor" or "do not collide with a human" expressed in relational terms. A *satisfaction* of these constraints is a weak representation of the underlying nonassociative algebra in which the conjunction of these constraints is true.

This chapter is primarily concerned with qualitative representations. In Section 4.4 we survey the constraint satisfaction properties of nonassociative algebras on at most three atoms, in which the desired representation is to be qualitative. In Section 4.5 we survey the qualitative representability of all algebras on at most four atoms, with the results given in Appendix A. Where a qualitative representation exists, we give a small example as a digraph or matrix.

4.1 Weak representations

In 1983, James Allen [1] introduced a novel representation of temporal relations now known as *Allen's Interval Algebra*. Rather than associating each fact with a specific date and time, Allen's method allowed "significant imprecision", in that the temporal knowledge contained within the model is strictly relative—we can claim that one event occurred before another, even though we might not know the exact relation between the two events.

There are advantages to such a system. In particular, we abandon the concept of scale, allowing us to talk about the millennia between archaeological periods and the nanoseconds between microprocessor operations similarly. Allen's interval algebra suits us whenever we care only about the *temporal relations* between events.

The algebra itself is defined on binary relations between intervals, with 13 atoms as shown in Table 4.1, which is a copy of Table 1.10. Every relation has a converse with an intuitive interpretation; for example, if an interval a occurs *before* an interval b, then the converse of this relation is that the interval b occurs *after* the inverval a. Note that the converse of equality is equality.

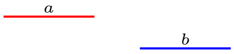
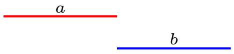
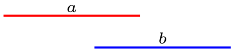
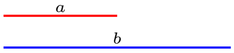
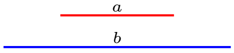
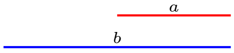
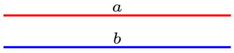
Example	Relation	Converse
	a before b	b after a
	a meets b	b is met by a
	a overlaps with b	b is overlapped by a
	a starts b	b is started by a
	a during b	b contains a
	a finishes b	b is finished by a
	a equals b	b equals a

Table 4.1: The Atoms of Allen's interval algebra

As mentioned in Chapter 1, these are the atoms of a relation algebra [58, Chapter 6]. We can build more binary relations from these atoms by equipping the algebra with the relation algebra operations. For example, we can *compose* one relation with another. If an interval a occurs before an interval b, and b starts an interval c, then we conclude that a occurs before c. We may also equip our algebra with *union* and *intersection*, to be interpreted as *or* and *and* respectively.

This example shows that algebras of relations can be used to express *qualitative reasoning*—a kind of reasoning that seeks to "represent continuous properties of the world by discrete systems of symbols" [17]. In particular, we consider temporal and spatial relations as two major families of qualitative reasoning. We'll make this definition more precise shortly, but before we do, let us consider the prototypical example of spatial reasoning: *RCC8*.

Taking its name from its authors Randell, Cui and Cohn [71], along with its number of atoms[2], RCC8 discusses the ways in which regions of space can interact. Its atoms are *equality* (EQ), *disconnected* (DC), *externally connected* (EC), *partially overlaps* (PO), *tangential proper part* and its negation (TPP and NTPP), and converse or inverses of these last two (TPPI and NTPPI). These relations, along with examples, are described in Figure 4.2.

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[2] RCC8 is also referred to as the *region connective calculus* [75, pp. 43].

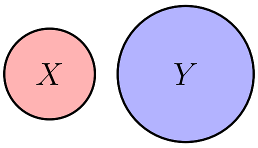
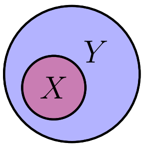
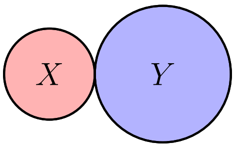
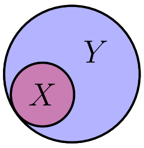
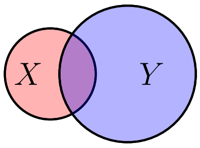
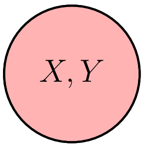
Example	Relation	Example	Relation
	$X\,\mathrm{DC}\,Y$		$X\,\mathrm{NTPP}\,Y$ $Y\,\mathrm{NTPPI}\,X$
	$X\,\mathrm{EC}\,Y$		$X\,\mathrm{TPP}\,Y$ $Y\,\mathrm{TPPI}\,X$
	$X\,\mathrm{PO}\,Y$		$X\,\mathrm{EQ}\,Y$

Table 4.2: The Atoms of RCC8

As we did with Allen's interval algebra, we can equip the atoms of RCC8 with the operations of a relation algebra. Certainly we can conjoin and disjoin spatial relations with intersection and union, respectively, and the atoms are designed to talk in terms of converses and negation. Indeed, RCC8 is a relation algebra [74].

The relation algebra representability of RCC falls apart, however, when we consider composition. Take as an example the spatial relations between regions X, Y and Z in Figure 4.3, in which we see that $\mathrm{EC} \leqslant \mathrm{EC}\,;\mathrm{EC}$. In Figure 4.4, we have a region Y with a hole filled entirely by X, and so $X\,\mathrm{EC}\,Y$. But there is no room for a third region Z to witness $\mathrm{EC} \leqslant \mathrm{EC}\,;\mathrm{EC}$. Therefore our concept of composition is simply too strong to accurately capture the composition of spatial regions under RCC8.

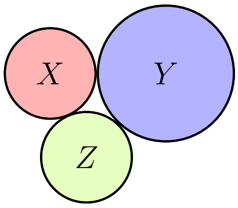


Figure 4.3: One possible composition in RCC8. We see that $\mathrm{EC} \leqslant \mathrm{EC}\,;\mathrm{EC}$.

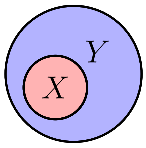


Figure 4.4: A possible situation in RCC8. Here region Y has a hole filled entirely by X, and so $X\,\mathrm{EC}\,Y$.

Thus far, we have looked at Allen's interval algebra and RCC8 as relation algebras to be assigned relation algebra representations. Yet we cannot discuss RCC8 with the terminology we have developed for relation algebras, since composition is not fully respected. Indeed, if a and b are elements of some abstract algebra of relations intended to capture the structure of RCC8, and $\varphi \to \wp(U \times U)$ is a representation of this algebra onto an algebra of binary relations between spatial regions, then instead of $\varphi(a\,;b) = \varphi(a) \circ \varphi(b)$ we see that $\varphi(a\,;b) \subseteq \varphi(a) \circ \varphi(b)$. We must define a different kind of weaker notion of representation along with a suitable abstract setting that accommodates this.

Naturally, we might ask what such an abstract algebra and corresponding notion of representation would look like. It would have to capture the aspects of relation algebras that are respected in Allen and RCC8, which are the Boolean operations, identity relation, and converse. But it would have to support a weaker notion of composition.

Such an abstract algebra would not only be limited to spatial and temporal reasoning. As an example, consider the relations of a family—*brother*, *sister*, *child*, etc.—and how the operations of a relation algebra would behave on this set of relations. We would have converses ($\mathit{child}^{\smile} = \mathit{parent}$), Boolean operations ($\mathit{brother} + \mathit{sister} = \mathit{sibling}$), and an identity relation, *self*. But composition may fail when we try to take a representation. For example, the *brother* of my *brother* is either my *brother* or *self*. If I am female, then the *brother* of my *brother* cannot be *self*. If I only have one *brother*, then *brother* of my *brother* cannot be my *brother*. Hence, in the representation, composition may only be a subset of what we would expect.

The underlying formalism we seek is explored in [49] and fully developed in [50]. In particular, we consider the weaker notion of composition as given in Chapter 11 of [50]. Recall the definition of composition of binary relations R and S over a set X:

$$R \circ S = \{(x, y) \in X \times X \colon (\exists z \in X)(x, y) \in R \text{ and } (z, y) \in S\}.$$

We consider instead *weak composition* of binary relations, defined on atoms:

$$R_i \diamond R_j = \bigcup \{R_k \colon (R_i \circ R_J) \cap R_k \neq 0\}\,.$$

As we would expect, $R \diamond S \subseteq R \circ S$ [50, Lemma 11.1]. This is exactly the composition we would use when considering representations of RCC8.

Recall that in Chapter 1 we considered the concrete world of algebras of binary relations and from that introduced the abstract world of relation algebras. If abstract relation algebras are inspired by concrete algebras of binary relations, what abstract world is inspired by algebras of binary relations with weak composition? Curiously, the foundations for the abstract world we seek were already in place before both Allen's and RCC8 entered the literature. Indeed, Maddux had already introduced the following concept as part of his work on relation algebras.

Definition 4.1.1 ([56])**.** A *nonassociative algebra*[3] is an algebra

$$\mathcal{A} = \langle A; ;, +, \cdot, -, ^{\smile}, 1', 0, 1\rangle$$

with type $\langle 2, 2, 2, 1, 1, 0, 0, 0\rangle$ such that:

- $\langle A; +, \cdot, -, 0, 1\rangle$ is a Boolean algebra,
- $x = x \mathbin{;} 1' = 1' \mathbin{;} x$ for all $x \in A$. That is, $1'$ is an identity element,
- $(x \mathbin{;} y) \cdot z = 0$ if and only if $(x^{\smile} \mathbin{;} z) \cdot y = 0$ for all $x, y, z \in A$,
- $(y \mathbin{;} x) \cdot z = 0$ if and only if $(z \mathbin{;} x^{\smile}) \cdot y = 0$ for all $x, y, z \in A$.

A nonassociative algebra is a *relation algebra*[4] if, for all $x, y, z \in A$,

$$x \mathbin{;} (y \mathbin{;} z) = (x \mathbin{;} y) \mathbin{;} z.$$

What a relation algebra representation is to a relation algebra, a weak representation is to a nonassociative algebra. Every relation algebra representation is a weak representation, although the converse does not hold. Furthermore, relation algebras are precisely the nonassociative algebras in which composition is associative.

We've glossed over the justification for using nonassociative algebras as the abstract setting for weak representations. This is best explored through the use of *partition schemes*. The intuitive idea is that, among all possible relations between two entities (intervals of time, points in space, etc.), a partition scheme is built on a finite set of "qualitative" atomic relations [50, Chapter 11]. From a partition scheme, an algebra can be derived, and this algebra is always a nonassociative algebra, but not necessarily a relation algebra. The original partition scheme defines a weak representation of the algebra. This is explored in detail in [51].

We can now formally define a weak representation of a nonassociative algebra.

Definition 4.1.2 ([51, Definition 3])**.** Let $\mathcal{A} = \langle A, ;, +, \cdot, \leqslant, -, ^{\smile}, 1', 0, 1\rangle$ be a nonassociative algebra over a set A. A *weak representation* of $\mathcal{A}$ is function $\varphi : \mathcal{A} \to \wp(U \times U)$ such that

- $\varphi : \langle A; +, \cdot, -, 0, 1\rangle \to \wp(U \times U)$,
- $\varphi(1') = \{(x, y) \in U \times U : x = y\}$,
- $\varphi(a^{\smile}) = \{(y, x) \in U \times U : (x, y) \in \varphi(a)\}$, and

---

[3] A more accurate term would be *not-necessarily-associative algebra*.

[4] One could also refer to these as *associative nonassociative algebras*. Of course, we won't do this.

- $\varphi(a\,;b) \subseteq \varphi(a) \circ \varphi(b)$.

That is to say, ϕ is a homomorphism of the Boolean reduct of $\mathcal{A}$, and preserves identity $1'$ and converse $\breve{}$, but does not fully respect composition ;. If φ is a weak representation of $\mathcal{A}$, injective, and respects composition (such that $\varphi(a\,;b) = \varphi(a) \circ \varphi(b)$), then φ is simply a representation. Clearly if $\mathcal{A}$ is a relation algebra and ϕ is a representation of $\mathcal{A}$, then ϕ is also a weak representation. This is the formalism we desired. It respects all of the properties of relation algebras and their representations that RCC8 does, but allows for a weaker concept of composition.

4.2 Constraint satisfaction

At the beginning of this chapter we mentioned that qualitative representations form the basis of a certain kind of constraint satisfaction problem. The general study of constraint satisfaction problems extends far beyond what we discuss here. We assume that all constraint networks are on binary relations, yet this does not need to hold in general. Constraint satisfaction problems are also not limited to nonassociative algebras and their representations, but could consider any relational structure.

Definition 4.2.1. A *constraint network* over a nonassociative algebra $\mathcal{A}$ is a set of variables $\{x_1, x_2, \ldots, x_n\}$ and a set of logical expressions $x_i a x_j$, where a is an element of $\mathcal{A}$. These logical expressions are referred to as *constraints*. We will often write simply $\{x_i\}$ instead of $\{x_i \colon i, j \leqslant n\}$.

A constraint network over $\mathcal{A}$ is *satisfiable* in a representation $\mathcal{M}$ if there is a map f from the variables $\{x_i\}$ to the vertices of $\mathcal{M}$ respecting the constraints. That is, for every constraint $x_i a x_j$, $(f(x_i), f(x_j)) \in a^{\mathcal{M}}$, where $a^{\mathcal{M}}$ is the assignment of a in the representation $\mathcal{M}$.

For a fixed representation $\mathcal{M}$ the problem of deciding whether or not an arbitrary constaint network is satisfiable in $\mathcal{M}$ is denoted CSP($\mathcal{M}$). When we discuss the complexity of a CSP problem, we do so with reference to the number of variables.

Recall from Chapter 1 that a relation algebra representation of an atomic relation algebra is an edge-labelled digraph in which every label is an atom of the relation algebra. The same is true for weak representations of atomic nonassociative algebras. We can therefore consider a satisfaction of a constraint network over an atomic nonassociative algebra as a *refinement* of the constraints into atoms.

Example 4.2.2 (StarVars)**.** Lee, Renz and Wolter [48] define the *StarVars* constraint language on m atoms, denoted $\mathcal{SV}_m$. The intended qualitative representation is that of oriented points in the plane. The language is defined over the domain $\mathbb{R}^2$ divided into m equal sectors centred on an oriented point. For example, $\mathcal{SV}_8$ is shown in Figure 4.5. The relation from A to B is the sector[5] in which B falls relative to A with arrow orientation relative to A.

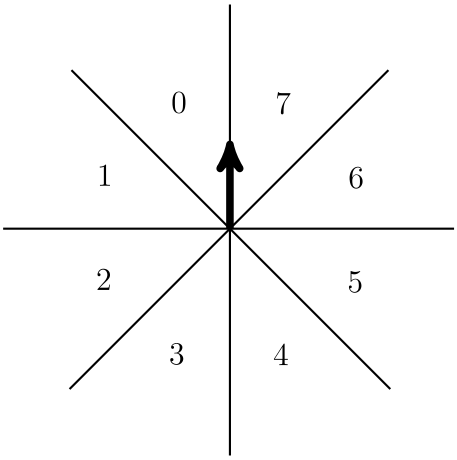


Figure 4.5: StarVars on eight variables, $\mathcal{SV}_8$

Denote by $[[n]]$ a sector labelled by n. Denote by $[[r..s]]$ the disjunction of sectors r through $s-1$. For example, $[[1..4]] = [[1]]+[[2]]+[[3]]$. A constraint satisfaction problem over $\mathcal{SV}_m$ is then a set of variables $x_1, x_2, \ldots, x_n$ with a set of constraints Ξ, with each $C \in \Xi$ a relation from the nonassociative algebra generated by $[[0]], \ldots, [[m]]$. To satisfy these constraints, one must assign to each x_i a coordinate in $\mathbb{R}^2$ and an orientation such that the relations in Ξ hold.

The problem CSP($\mathcal{SV}_m$) is the problem of deciding whether or not an arbitrary set of contraints over $\mathcal{SV}_m$ for an arbitrary finite set of points $\{x_i\}$ is satisfiable. Weak representations are the ideal setting for this problem because the only requirement is consistency; we do not need to see every witness move respected. Lee, Renz and Wolter prove that CSP($\mathcal{SV}_2$) is NP-hard and give an algorithm that solves CSP($\mathcal{SV}_m$) in NP for all m.

The authors also give a fascinating practical application of $\mathcal{SV}_8$, considering each point as a boat travelling in a particular direction. They then consider the following regulation from The International Maritime Organization (IMO).

> "When two sailing vessels are approaching one another, as to involve the risk of collision, one of them shall keep out of the way of the other as follows: (i) when each has the wind on a different side, the vessel which has the wind on the port side shall keep out of the way of the other."
> (Rule 12 i, IMO)

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[5] We ignore the lines in this example but there are several conventions one can take.

This regulation describes the disjunction of constraints $\phi(K,G,W)$ for boats K and G and wind W.

$$\phi(K,G,W) := \begin{cases} \quad K[[0..4]]W & \text{(wind on port of } K\text{)} \\ +G[[0..4]]W & \text{(wind on starboard of } G\text{)} \\ +G[[0..4]]K & (G \text{ heading towards } K) \\ +K[[0..4]]G & (K \text{ heading towards } G) \end{cases}$$

Lee, Renoz and Wolter then go on to define two relations which, according to the above regulation, force G to turn to either port or starboard.

$$\alpha(K,G,W) := \phi(K,G,W) + G[[0..4]]K$$
$$\beta(K,G,W) := \phi(K,G,W) + G[[4..0]]K$$

The IMO regulation describes a situation with two boats, but what about three? Can we find a configuration with three boats, K,G,H such that G must turn to port and starboard simultaneously? That is, can we find a satisfaction of the constraing $\alpha(K,G,W)\cdot\beta(H,G,W)$? The authors run these constraints through their algorithm and find such a situation, shown in Figure 4.6.

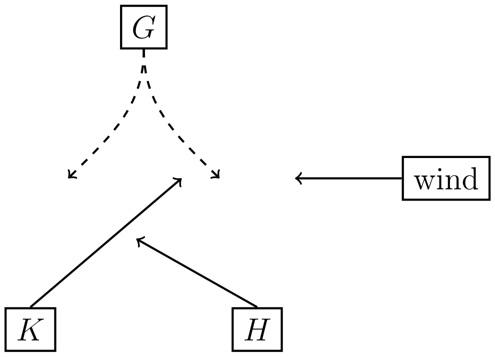


Figure 4.6: In this situation, G must turn to port and starboard simulataneously.

This is a practical problem which does not rely on the underlying associativity of a relation algebra at all. We do not need to check that witness moves are represented correctly, only composition moves. Hence, weak representations are a suitable concept for spatial constraint problems like this.

The above example considered a fixed representation of oriented points in the real plane. To study constraint satisfaction more generally, we begin with a nonassociative algebra. One can either fix a representation $\mathcal{M}$ or consider the more general problem in which a constraint network can be satisfiable in *any* representation.

Definition 4.2.3 ([19])**.** Let $\mathcal{A}$ be a nonassociative algebra. We define the following to be agnostic about the type of representation being considered. For a fixed representation $\mathcal{M}$ of $\mathcal{A}$, we denote by $\mathcal{M}$-SAT the computational problem of deciding whether or not an arbitrary instance of CSP($\mathcal{M}$) is satisfiable. The general satisfaction problem gen-SAT denotes the computational problem of deciding whether or not an arbitrary constraint network over $\mathcal{A}$ is satisfiable in *some* representation of $\mathcal{A}$.

Clearly if $\mathcal{A}$ is not representable, then gen-SAT can be decided in constant time.

The following is a consequence of a theorem of Schaefer [77], and relies on earlier work of Post [69]. Bodirsky [7] also offers a convenient summary of constraint satisfaction over templates of size 2.

Theorem 4.2.4 ([69, 77])**.** *If $\mathcal{M}$ is a representation of size 2 then $\mathcal{M}$-SAT can be solved in polynomial time.*

Definition 4.2.5. A set of constraints is *closed* if

- whenever x_iax_j and x_jbx_k are constraints there exists a constraint x_icx_k such that $c \leqslant a\,;b$,
- if x_iax_j is a constraint then $x_ja^{\smile}x_i$ is a constraint, and
- for all x_i there exists an element $e \leqslant 1'$ such that x_iex_i is a constraint.

When we consider constraint satisfaction with binary relations there are certain 'tricks' that can be performed in polynomial time and so come 'for free'. One such trick is the *propagation algorithm*, which runs in cubic time [1, 19]. For a given set of constraints Ξ, this algorithm

- adds in the constraint $x_i1'x_i$ for all x_i,
- replaces a constraint x_iax_j by $x_i(a \cdot b^{\smile})x_j$ whenever x_iax_j and x_jbx_i are constraints,
- replaces a constraint x_icx_k by $x_i(c\cdot(a;b))x_k$, or adds $x_i(a\cdot b)x_k$ if no x_icx_k exists, whenever x_iax_j and x_jbx_k are constraints, and
- can repeat the above steps until stabilised.

Because of the propagation algorithm, we can restrict our focus to closed, nonzero constraint networks. The following lemma is analogous to Proposition 5 in [19].

Lemma 4.2.6. *Let $\mathcal{M}$ be a qualitative representation of a nonassociative algebra $\mathcal{A}$. The decision problem $\mathcal{M}$-SAT reduces in cubic time to the problem of deciding whether or not a closed set of constraints is satisfiable in $\mathcal{M}$.*

One can also check to make sure that there are no *zero constraints* in linear time, simply by checking every constraint in Ξ. A constraint network with a zero relation cannot be satisfied, since 0 must always be represented as $\varnothing$. We can also collapse points related by $1'$.

Lemma 4.2.7. *We can identify points in a constraint network related by* $1'$ *in* LOGSPACE.

Proof. Consider each $1'$-labelled edge as an edge in an undirected graph. The points to be identified are those reachable from one another in this graph. Undirected graph reachability is solvable in LOGSPACE [73]. □

Definition 4.2.8. A representation $\mathcal{M}$ of an algebra is said to be *universal* if every nonzero closed network of that algebra embeds into $\mathcal{M}$.

An important consequence of the existence of a universal representation is the following analogue of [19, Lemma 6].

Lemma 4.2.9. *If a nonassociative algebra* $\mathcal{A}$ *has a universal representation* $\mathcal{M}$ *then* $\mathcal{M}$-SAT *and gen*-SAT *have at worst cubic complexity, and* $\mathcal{M}$-SAT *coincides with gen*-SAT.

4.3 Qualitative representations

In a representation of a relation algebra, if $a \,;\, b = c$ and h is a representation, then whenever $(x, y) \in h(c)$ there must exist another point z such that $(x, z) \in h(a)$ and $(z, y) \in h(b)$. To recall the terminology introduced in Chapter 1, this is just to say that all *witness moves* must be respected. This requirement is dropped when discussing weak representations. We do still demand, however, that composition moves are respected; that is, whenever $(x, z) \in h(a)$ and $(z, y) \in h(b)$, we should see $(x, y) \in h(c)$.

To use slightly different terminology, in a weak representation there are no inconsistent triangles. In a relation algebra representation there are no inconsistent triangles, and every consistent triangle appears wherever it can.

Hirsch, Jackson and Kowalski [38] introduce a new concept of a *qualitative representation*. Once again, we demand that there are no inconsistent triangles. Unlike a relation algebra representation, we do not require that every consistent triangle appears wherever it can. We simply demand that every consistent triangle appears at least *once* in the representation. That is, a qualitative representation is between a weak and a relation algebra representation.

Just as for weak representations, nonassociative algebras are the ideal abstract setting for qualitative representations. The main reason to prefer qualitative representations to weak representations is that a weak representation need not be isomorphic to the underlying nonassociative algebra. Indeed, different nonassociative algebras can have the same weak representation, and Hirsch et al. give examples.

As is the case for relation algebras and relation algebra representations, there does not exist a finite set of elementary axioms that characterise qualitative representability of nonassociative algebras.

Theorem 4.3.1 ([38, Theorem 20])**.** *The class of qualitatively representable nonassociative algebras is not finitely axiomatisable.*

Unlike relation algebras and relation algebra representations [32], qualitative representability of finite nonassociative algebras is decidable.

Theorem 4.3.2 ([38, Theorem 15])**.** *The problem of determining whether a finite atom structure has a qualitative representation is* NP*-complete.*

The algorithm given by Hirsch et al. places every consistent triangle in a partial representation and then nondeterministically 'guesses' the nonedges, checking for consistency. This algorithm extends to constraint satisfaction.

Theorem 4.3.3 ([38, Theorem 16])**.** *For a finite nonassociative algebra $\mathcal{A}$, gen-*SAT *is in* NP*.*

This is not the case for relation algebras, for which gen-SAT is undecidable [32]. For the remainder of this thesis we will focus on qualitative representations of nonassociative algebras, to the extent that we will often refer to these simply as representations. As it turns out, there is an interesting asymptotic result for nonrepresentability of nonassociative algebras.

Recall from Chapter 1 the distinction between triples and cycles, in that a cycle can contain up to six triples equivalent under Peircean transforms. We can denote a cycle by any of the triples it contains. We also speak of an algebra 'containing' a cycle, that is, the algebra satisfies the equations represented by the cycle. Before we state the next theorem, we'll relax the notation for cycles so that we do not have to use commas or brackets. As such, we can refer to a cycle (a, b, c) simply as abc.

Theorem 4.3.4. *The proportion of representable nonassociative algebras on n atoms in which every element is symmetric and with at least two identity atoms tends to* 0 *as n tends to infinity.*

Proof. Suppose for each possible cycle of a nonassociative algebra on n atoms we have selected one triple belonging to that cycle to act as a *representative*. Instead of referring to the entire set of Peircean transforms contained in a cycle, we simply refer to its representative triple. For example, in Table 4.9 we list the cycles that might appear in an algebra on four atoms, where a, b and c are symmetric while r and $r^\smile$ are not. The choice of representative is entirely arbitrary. If we want to uniquely identify a nonassociative algebra by the cycles it contains we can simply specify the representatives of those cycles, which is exactly the notational approach we take here.

We construct an algebra with $K \geqslant 2$ subidentity atoms. Let e and f be two distinct subidentity atoms, that is, $e \cdot f = 0$ and $e + f \leqslant 1'$. Since these are subidentity atoms we have that eee and fff are cycles and that $ef = 0 = fe$. We consider three symmetric (self-converse) atoms a, b, c and the situation in Figure 4.7.

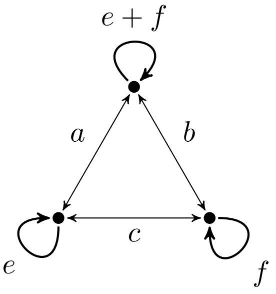


Figure 4.7: A nonrepresentable situation that must be represented

We first add cycles in order to force this configuration in any possible representation. We then create nonrepresentability by forbidding certain cycles in order to make it impossible to refine the $e + f$ loop to an atom. The cycles we require and forbid are listed in Table 4.8, where g is any subatomic identity not equal to e or f. This situation forces us to refine the loop labelled $e + f$ to be both e and f, and so is not representable.

required	forbidden
eee	afa
fff	beb
cfc	aga
abc	
cec	
aea	
bfb	

Table 4.8: Required and forbidden cycles

Suppose we have generated an algebra with $K \geqslant 2$ subidentity atoms. Suppose further that we have symmetric atoms $a_1, a_2, \ldots, a_{n-K}$, for a total of n atoms. For each cycle $a_i a_j a_k$, decide with probability $1/2$ whether or not the algebra contains $a_i a_j a_k$. We do the same for cycles involving the subidentity atoms, although for every element a_i in the algebra there must exist at least one subidentity atom, say g, such that $a_i g a_i$ is a cycle.

For atoms a_i, a_j and a_k there is a nonzero probability $\epsilon > 0$ that triples are selected such that a_i, a_j and a_k act as a, b and c above, and that the forbidden atoms are rejected. In order for the generated algebra to be representable it is necessary that every three atoms avoids this configuration. As there are $m = \lfloor n/3 \rfloor$ disjoint sets of three atoms, and the events of avoiding the configuration are independent, this gives a bound of $(1 - \epsilon)^m$ probability of the generated algebra being representable. As n goes to infinity, this probability approaches 0. □

In Section 4.5 we survey representability for all nonassociative algebras on at most four atoms. To generate the nonassociative algebras one can use cycles similar to the method used by Maddux [58]. Since the cycles involving identity are always present we focus instead on the *diversity cycles*. Each nonassociative algebra is defined by the cycles it contains, since we can always take Peircean transforms. All possible cycles for nonassociative algebras on four atoms and with atomic identity—called integral algebras—are given in Table 4.9. For each cycle we list the triples it includes as well as a 'representative' triple, which matches that used by Maddux. The possible cycles for smaller integral nonassociative algebras are easily derived from this table.

Cycles for smaller nonassociative algebras or nonassociative algebras with nonatomic identity can be derived from these tables. The code we use to generate composition tables for nonassociative algebras on four atoms is given in Appendix C.3. We actually use a brute-force method for generating algebras on fewer than three atoms, but this would be far too slow for larger algebras. Diversity cycles for all nonassociative algebras on at most four atoms are given in Appendix B.

This mirrors work done much earlier on relation algebras. Maddux notes that the effort to enumerate integral relation algebras on at most three atoms was started by Lyndon [55] and continued by McKenzie [61, pp. 38–40] and Backer [6]. A thorough treatment of representability of all relation algebras with at most eight elements (as opposed to eight atoms) is given by Andreka and Maddux[4].

Maddux lists the cycles of all relation algebras on at most five atoms, assigning them an index according to their atom structure. For example, there are 37 integral relation algebras on $1', a, r, r^{\smile}$, up to isomorphism, and so he numbers these $\#1_{37}$ through $\#37_{37}$. While we use our own indexing here, we offer Maddux's label where relevant.

In Section 4.4 we survey the constraint satisfaction properties of all nonassociative

representative	cycle
$1'1'1'$	$1'1'1'$
	$1'a1', a1'1', 1'1'a$
	$1'b1', 1'1'b, b1'1'$
	$1'1'c, c1'1', 1'c1'$
$1'aa$	$aa1', 1'aa, a1'a$
	$a1'b, ab1', b1'a, 1'ba, ba1', 1'ab$
	$1'ca, ac1', ca1', 1'ac, c1'a, a1'c$
$1'bb$	$b1'b, bb1', 1'bb$
	$1'cb, bc1', cb1', c1'b, 1'bc, b1'c$
$1'cc$	$1'cc, cc1', c1'c$
aaa	aaa
baa	aba, baa, aab
caa	aca, caa, aac
abb	abb, bab, bba
abc	$bca, abc, bac, cba, cab, acb$
acc	cca, cac, acc
bbb	bbb
cbb	cbb, bcb, bbc
bcc	cbc, bcc, ccb
ccc	ccc
raa	$aar, raa, r^\smile aa, aar^\smile, ar^\smile a, ara$
arr	$rr^\smile a, arr, r^\smile ar^\smile$
rra	$r^\smile r^\smile a, rar^\smile, arr^\smile, r^\smile ar, ar^\smile r, rra$
rar	$rar, r^\smile ra, ar^\smile r^\smile$
rrr	$rrr, rr^\smile r^\smile, r^\smile r^\smile r^\smile, r^\smile rr^\smile, rr^\smile r, r^\smile rr$
$rrr^\smile$	$rrr^\smile, r^\smile r^\smile r$

Table 4.9: Cycles for integral nonassociative algebras on four atoms

algebras on at most three atoms. This section closely follows the work of Cristani and Hirsch [19], who did the same but for relation algebras of the same size. In Section 4.4 we label our small nonassociative algebras #1 through #24 in an attempt to match the scheme used by Cristani and Hirsch. In Section 4.5—where we survey the representability of all nonassociative algebras on at most four atoms—we use a different numbering scheme #1 through #373.

In order to draw representations as small as possible, we will employ some conventions. We will consider symmetric (or self-converse) atoms a, b and c. We will also consider an atom r which is not self-converse, and so has converse $r^\smile$. We won't draw $r^\smile$ on any of the representations, since its existence can always be deduced from the placement of the atom r. Nor will we place loops on any point unless necessary since, for an algebra with an atomic identity $1'$, we can assume that there is an $1'$-labelled loop on every vertex. If the identity relation is not atomic, it will be the disjunction of atoms e_1, e_2, e_3, e_4, where these atoms exist.

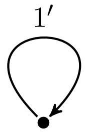
We do not require brackets or commas when denoting a cycle, for example aaa. Similar notation is used for relations between vertices. To avoid confusion, we will reserve x, y, z, u, v and w for vertices. A triple xry would then read as an r-labelled edge between x and y. We will also assume that, unless otherwise stated, all representations are square. That is, there is an edge between every two vertices. As such, we will limit our discussion to simple nonassociative algebras.

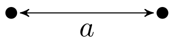
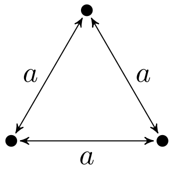
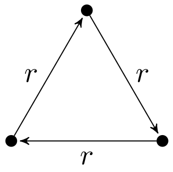
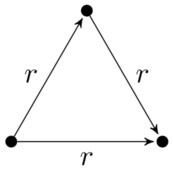
4.4 Nonassociative algebras on at most three atoms

Here we survey the nonassociative algebras on fewer than four atoms, including their constraint satisfaction properties. The numbering of these algebras matches that used in [19], where applicable. We do not discuss algebras #3 and #6–#8 in detail, as these are not simple. Algebras #19–#24 are nonassociative algebras but not relation algebras, and so are not mentioned in [19]. Recall that, unless stated otherwise, all representations here are assumed to be qualitative, rather than as relation algebras. We follow the arguments given in [19] where possible.

We begin with an overview of the qualitative representability of all nonassociative algebras on fewer than four atoms, shown in Table 4.10. Along with an atom table for the algebra we note whether or not the algebra is a relation algebra (RA). All relation algebras on fewer than four atoms are representable [4]. We also note the index of each relation algebra assigned by Maddux [58], where applicable. If the algebra is qualitatively representable, we give an example of a representation. The representations given are on the smallest number of vertices possible.

In drawing the representations, it is often convenient to omit $1'$. Unless otherwise noted, assume that all vertices in the representation are related to themselves by an $1'$-labelled loop. We can also simplify the representations by representing only one atom of a pair of converses. For example, if r is an atom and $r^{\smile}$ is its converse, we can infer from xry that $yr^{\smile}x$. Thus, it suffices to draw only an r-labelled edge.

<table>
<tr><th>atom table</th><th>RA</th><th>QRNA</th></tr>
<tr><td>#1: no atoms</td><td>yes</td><td>yes, on ∅</td></tr>
<tr><td><table><tr><td>#2</td><td>$1'$</td></tr><tr><td>$1'$</td><td>$1'$</td></tr></table></td><td>yes
1_1</td><td>
</td></tr>
</table>

atom table	RA	QRNA
$\begin{array}{c\|cc} \#3 & e_1 & e_2 \\ \hline e_1 & e_1 & 0 \\ e_2 & 0 & e_2 \end{array}$	yes	not simple: #2 × #2
$\begin{array}{c\|cc} \#4 & 1' & a \\ \hline 1' & 1' & a \\ a & a & 1' \end{array}$	yes 1_2	
$\begin{array}{c\|cc} \#5 & 1' & a \\ \hline 1' & 1' & a \\ a & a & 1 \end{array}$	yes 2_2	
$\begin{array}{c\|ccc} \#6 & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & 0 & 0 \\ e_2 & 0 & e_2 & 0 \\ e_3 & 0 & 0 & e_3 \end{array}$	yes	not simple: #2 × #2 × #2
$\begin{array}{c\|ccc} \#7 & e_1 & e_2 & a \\ \hline e_1 & e_1 & 0 & 0 \\ e_2 & 0 & e_2 & a \\ a & 0 & a & e_2 \end{array}$	yes	not simple: #2 × #4
$\begin{array}{c\|ccc} \#8 & e_1 & e_2 & a \\ \hline e_1 & e_1 & 0 & 0 \\ e_2 & 0 & e_2 & a \\ a & 0 & a & -e_1 \end{array}$	yes	not simple: #2 × #5
$\begin{array}{c\|ccc} \#9 & 1' & r & r^\smile \\ \hline 1' & 1' & r & r^\smile \\ r & r & r^\smile & 1' \\ r^\smile & r^\smile & 1' & r \end{array}$	yes 2_3	
$\begin{array}{c\|ccc} \#10 & 1' & r & r^\smile \\ \hline 1' & 1' & r & r^\smile \\ r & r & r & 1 \\ r^\smile & r^\smile & 1 & r^\smile \end{array}$	yes 1_3	

<table>
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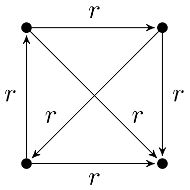
3_3</td><td> </td></tr>
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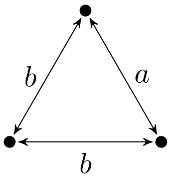
1_7</td><td> </td></tr>
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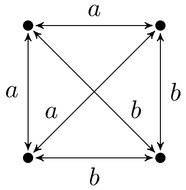
2_7</td><td> </td></tr>
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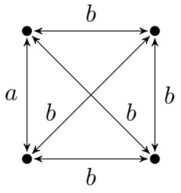
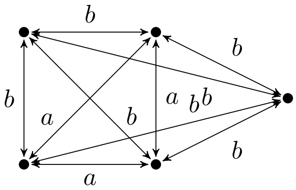
3_7</td><td> </td></tr>
<tr><td><table><tr><th>#15</th><th>$1'$</th><th>a</th><th>b</th></tr><tr><td>$1'$</td><td>$1'$</td><td>a</td><td>b</td></tr><tr><td>a</td><td>a</td><td>$-b$</td><td>b</td></tr><tr><td>b</td><td>b</td><td>b</td><td>1</td></tr></table></td><td>yes
4_7</td><td> </td></tr>
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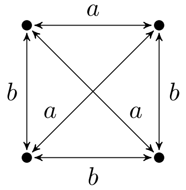
5_7</td><td> </td></tr>
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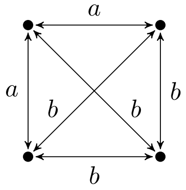
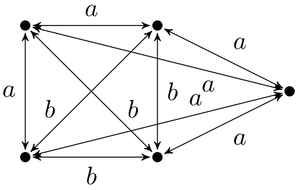
6_7</td><td> </td></tr>
<tr><td><table><tr><th>#18</th><th>$1'$</th><th>a</th><th>b</th></tr><tr><td>$1'$</td><td>$1'$</td><td>a</td><td>b</td></tr><tr><td>a</td><td>a</td><td>1</td><td>$0'$</td></tr><tr><td>b</td><td>b</td><td>$0'$</td><td>1</td></tr></table></td><td>yes
7_7</td><td> </td></tr>
</table>

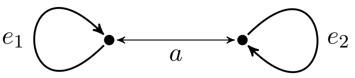
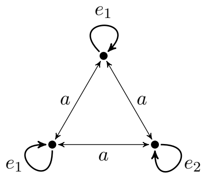
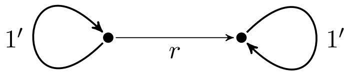
atom table	RA	QRNA
$\begin{array}{c\|ccc} \#19 & e_1 & e_2 & a \\ \hline e_1 & e_1 & 0 & a \\ e_2 & 0 & e_2 & a \\ a & a & a & 1' \end{array}$	no	
$\begin{array}{c\|ccc} \#20 & e_1 & e_2 & a \\ \hline e_1 & e_1 & 0 & a \\ e_2 & 0 & e_2 & a \\ a & a & a & 1 \end{array}$	no	
$\begin{array}{c\|ccc} \#21 & 1' & r & r^{\smile} \\ \hline 1' & 1' & r & r^{\smile} \\ r & r & 0 & 1' \\ r^{\smile} & r^{\smile} & 1' & 0 \end{array}$	no	
$\begin{array}{c\|ccc} \#22 & 1' & a & b \\ \hline 1' & 1' & a & b \\ a & a & 1' & 0 \\ b & b & 0 & 1' \end{array}$	no	no
$\begin{array}{c\|ccc} \#23 & 1' & a & b \\ \hline 1' & 1' & a & b \\ a & a & 1' & 0 \\ b & b & 0 & -a \end{array}$	no	no
$\begin{array}{c\|ccc} \#24 & 1' & a & b \\ \hline 1' & 1' & a & b \\ a & a & -b & 0 \\ b & b & 0 & -a \end{array}$	no	no

Table 4.10: Representability of nonassociative algebras on fewer than four atoms

We will now explore the representability and constraint satisfaction properties of these algebras in greater detail.

#1

This is the algebra with no atoms and $0 = 1$. It has only one representation, and it is on the empty set. Any nonempty constraint network is trivially unsatisfiable.

#2

#2	$1'$
$1'$	$1'$

This algebra contains only one atom, the identity atom. There is only one representation up to isomorphism, and it is on one point. Any nonzero constraint network is trivially satisfiable in this representation.

#3

#3	e_1	e_2
e_1	e_1	0
e_2	0	e_2

This algebra is not simple, as it is the direct product of two copies of #2.

#4

#4	$1'$	a
$1'$	$1'$	a
a	a	$1'$

We claim that this has a unique qualitative representation on 2 points, given in Figure 4.11. This is because the only non-identity element a is not idempotent; that is, $a \not\leq a ; a$. So there would be no way to label a triangle on 3 distinct points. As such, there is only one representation $\mathcal{M}$ and, by Theorem 4.2.4, $\mathcal{M}$-SAT is tractable. Therefore, Gen-SAT is also tractable. Furthermore, this representation is universal [19, Lemma 8].

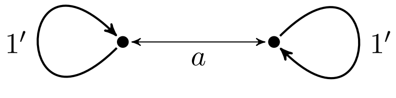


Figure 4.11: The only representation of #4, up to isomorphism

#5

#5	$1'$	a
$1'$	$1'$	a
a	a	1

The two atoms of this algebra are identity and an idempotent diversity. One can construct a representation of any size at least 3 from the complete graph on n vertices, K_n. Label every edge of K_n by a and add a $1'$-labelled loop on every vertex. This clearly respects all operations as long as $n \geqslant 3$. Hence, $\mathcal{M}$-SAT is NP-complete for finite $\mathcal{M}$. The smallest such representation is shown in Figure 4.12.

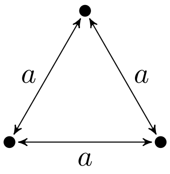


Figure 4.12: The unique smallest representation of algebra #5, up to isomorphism

A universal countably infinite representation for algebra #5 is given in [19]. We offer a different argument here. We do not need our constraint network to be closed to be representable; a constraint network only needs to be nonzero and to satisfy one other condition.

Lemma 4.4.1. *A constraint network over algebra #5 is satisfiable if and only if it is nonzero and does not contain a constraint configuration given in Figure 4.13 with* $k \geqslant 2$.

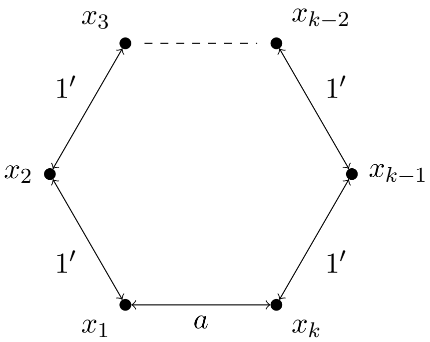


Figure 4.13: Unsatisfiable constraints in algebra #5

Proof. A network containing a 0-constraint is trivially inconsistent, so we focus on the sufficiency and necessity of the absence of the configuration in Figure 4.13 instead.

Let Ξ be a nonzero constraint network over algebra #5. Variables related by $1'$ are identified as equal, while variables related by a are identified as distinct, as a is the diversity element. Since Ξ is nonzero it will be satisfiable if and only if two distinct variables are not identified. This is exactly the situation excluded in Figure 4.13. $\square$

#6

#6	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_2	0
e_3	0	0	e_3

This algebra is not simple, as it is the direct product of three copies of #2.

#7

#7	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	0	a	e_2

This algebra is not simple, as it is the direct product of #2 and #4.

#8

#8	e_1	e_2	a
e_1	e_1	0	0
e_2	0	e_2	a
a	0	a	$-e_1$

This algebra is not simple, as it is the direct product of #2 and #5.

#9

#9	$1'$	r	$r^{\smile}$
$1'$	$1'$	r	$r^{\smile}$
r	r	$r^{\smile}$	$1'$
$r^{\smile}$	$r^{\smile}$	$1'$	r

There exists only one representation up to isomorphism, and it is on three points, illustrated in Figure 4.14. Any representation must witness $rr\breve{r}$ and so must contain this triangle. When we attempt to add a fourth point to the representation, there is no way to label edges to the new point without introducing an inconsistent triangle rrr. Hence, $\mathcal{M}$-SAT is NP-complete for all $\mathcal{M}$, and so is Gen-SAT. This is the same as the relation algebra case.

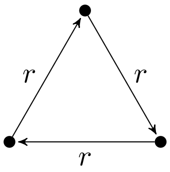


Figure 4.14: The unique representation of algebra #9, up to isomorphism

#10

#10	$1'$	r	$\breve{r}$
$1'$	$1'$	r	$\breve{r}$
r	r	r	1
$\breve{r}$	$\breve{r}$	1	$\breve{r}$

The algebra contains only one diversity cycle: rrr. In representing this as a relation algebra, one would look for dense linear orders. There exists only one countable dense linear order up to isomorphism, $(\mathbb{Q}, <)$. Unlike the relation algebra case, finite qualitative representations exist, as the density requirement can be dropped. The smallest qualitative representation is shown in Figure 4.15. Since all finite representations require at least three points, $\mathcal{M}$-SAT is NP-complete for finite $\mathcal{M}$. Like the relation algebra case, $\mathcal{M}$-SAT can be checked in $O(n^3)$ for infinite $\mathcal{M}$, and similarly for gen-SAT.

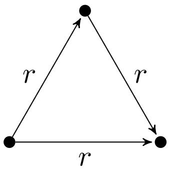


Figure 4.15: The unique smallest representation of algebra #10, up to isomorphism

#11

$\#11$	$1'$	r	$r^{\smile}$
$1'$	$1'$	r	$r^{\smile}$
r	r	$0'$	1
$r^{\smile}$	$r^{\smile}$	1	$0'$

A relation algebra representation of this algebra must be on at least seven vertices [4]. The representation on exactly seven points is unique. To explore qualitative representations we introduce the concept of a *tournament graph*.

A tournament is an orientation of a complete graph [22, Chapter 10]. That is, for every two distinct vertices x, y of a tournament, exactly one of (x, y) and (y, x) is an edge. Assume that tournaments do not have loops.

Let $\mathcal{M}$ be a representation of #11, finite or otherwise. We will construct a tournament from $\mathcal{M}$. Construct a graph G with vertices the domain of $\mathcal{M}$ such that $(x, y) \in E(G)$ if and only if xry. This is a tournament because:

- G has no loops, since xrx violates the representation of $1'$,
- $xry \implies yr^{\smile}x$ by the representation of $^{\smile}$, and so every edge is unidirectional,
- for $x \neq y$, either xry or yrx by the representation of 0.

Hence, exactly one of (x, y) and (y, x) is in $E(G)$.

This translation from representation $\mathcal{M}$ to tournament G does work in the other direction, subject to additional conditions. We can translate every directed edge of a tournament to an r relation in a representation, and add the identity relation as a loop on every vertex, but we want to be sure that we witness every composition at least once. In order for $\mathcal{M}$ to include the cycles rrr and $rrr^{\smile}$, we need to see the two graphs in Figure 4.16 as subgraphs in G.

Figure 4.16: Subgraphs needed to be witnessed in order to translate a tournament to a representation of algebra #11.

A tournament needs at least 4 vertices to contain these two graphs as subgraphs, and so the minimum size of a representation of $\mathcal{M}$ is at least 4. Such a representation does exist, and is shown in Figure 4.17. There are two representations on 4 points up to isomorphism, because one can start with the right graph in Figure 4.16, construct the left graph by adding an additional vertex, and then fill in the single remaining nonedge in one of two ways.

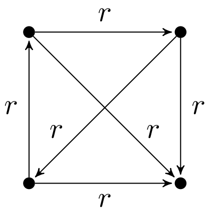


Figure 4.17: One of two smallest representations of algebra #11, up to isomorphism

One can always add new vertices to either of the graphs giving rise to a smallest representation, and so representations exist on any finite number of vertices n, as long as $n \geqslant 4$. It follows that CSP($\mathcal{M}$) is NP-complete for all finite $\mathcal{M}$.

As for infinite representations, we might be tempted to consider the class of all countable tournaments. It turns out, however, that we need consider only one special countably infinite tournament. The construction is similar to that of the Rado graph (algebra #15). A proof that such a graph exists and is unique up to isomorphism is not easily accessible in the literature, and so is included here for completeness.

This construction relies on something like a limit but for finite relational structures. This concept—the *Fraïssé limit* of a *Fraïssé class*—was introduced by Roland Fraïssé in 1954 [25] to derive the ordered set of rationals as the Fraïssé limit of the class of finite linear orderings.

We'll get to the actual definitions in a moment, but first let us consider what a Fraïssé class *means*. Let L be a signature and D an L-structure. The *age* of D is the class of all finitely generated structures that can be embedded in D. So for example, the age of $(\mathbb{Q}, <)$ is the class of all finite linear orderings. Similarly, the age of $(\mathbb{Z}, <)$ is the class of all finite linear orderings. The Fraïssé limit can be thought of as something of a 'partial converse' to the procedure of taking an age. When we take the Fraïssé limit of the class of all finite linear orderings, we get $(\mathbb{Q}, <)$, and not $(\mathbb{Z}, <)$.

We need to know what separates a Fraïssé class—which has a Fraïssé limit—from a class that does not have a Fraïssé limit. Fraïssé classes have some special properties. We will be using the concept as presented in [40].

Definition 4.4.2. Let L be a countable signature and let $\mathcal{K}$ be a non-empty finite or countable set of finitely generated L-structures. $\mathcal{K}$ is a Fraïssé class if it satisfies the following five conditions:

1. $\mathcal{K}$ is closed under isomorphism.
2. $\mathcal{K}$ is closed under taking induced substructures.

3. $\mathcal{K}$ has at most countably many members up to isomorphism.
4. $\mathcal{K}$ has the amalgamation property.
5. $\mathcal{K}$ has the joint embedding property.

One might readily accept the first three conditions, but the last two are sure to cause some confusion.

Definition 4.4.3 (Amalgamation property)**.** Let $A, B, C \in \mathcal{K}$. An *amalgam* is a tuple (A, f, B, g, C) such that $f : A \to B$ and $g : A \to C$ are embeddings. $\mathcal{K}$ has the amalgamation property if for every such amalgam there exists $D \in \mathcal{K}$ and embeddings $f' : B \to D$ and $g' : C \to D$ such that $f' \circ f = g' \circ g$. This situation is illustrated in Figure 4.18.

$$\begin{tikzcd}
B \arrow[r, dashed, "f'"] & D \\
A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, dashed, "g'"']
\end{tikzcd}$$

Figure 4.18: An amalgam with corresponding embeddings f' and g'

Definition 4.4.4 (Joint embedding property)**.** $\mathcal{K}$ has the joint embedding property if for all $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A and B are embeddable in C.

If all of these conditions are satisfied, we can use Fraïssé's Theorem to construct the sort of unique, countable structure for which we're aiming.

Theorem 4.4.5 (Fraïssé's Theorem)**.** *Let L be a countable signature and let $\mathcal{K}$ be Fraïssé class of finitely generated L-structures. Then there is an L-structure D, unique up to isomorphism, such that:*

- *D is at most countably infinite,*
- *$\mathcal{K}$ is the age of D, and*
- *D is ultrahomogenous; that is, every isomorphism between finitely generated substructures of D extends to an automorphism of D.*

We call D the Fraïssé limit *of $\mathcal{K}$.*

If $\mathcal{K}$ contains all the finite representations of a particular algebra, then any representation of that algebra embeds into the Fraïssé limit of $\mathcal{K}$. That is, we can use Theorem 4.4.5 to construct universal representations. This is exactly the approach we will now take for tournament graphs.

Lemma 4.4.6. *The class $\mathcal{K}$ of all finite tournament graphs is a Fraïssé class.*

Proof.

1. Graph isomorphisms preserve edges and non-edges, and so a tournament remains a tournament under isomorphism.
2. Let H be an induced subgraph of a finite tournament graph G, that is, $V(H) \subseteq V(G)$. For all $x, y \in V(H)$ just one of $(x, y) \in E(G)$ and $(y, x) \in E(G)$. So just one of (x, y) and $(y, x) \in H$, and so H is a tournament graph.
3. For every $n \in \mathbb{N}$ there are a finite number of tournament graphs on n vertices. Hence $\mathcal{K}$ is a countable union of finite sets, and so is countable. This assumes the axiom of countable choice for finite sets.
4. Let (A, f, B, g, C) be an amalgam of finite tournaments as in Figure 4.18. Define

   $$\begin{aligned} V_{B'} &= V(B) \backslash V(f(A)), \\ E_{B'} &= E(B) \backslash E(f(A)), \\ V_{C'} &= V(C) \backslash V(g(A)), \text{ and} \\ E_{C'} &= E(C) \backslash E(g(A)). \end{aligned}$$

   Assume without loss of generality that $V_{B'} \cap V_{C'} = \varnothing$. Define a new graph D with $V(D) = V(A) \cup V_{B'} \cup V_{C'}$ and $E(D) = E(A) \cup E_{B'} \cup E_{C'}$. Replace the nonedges, but not the nonloops, in $E(D)$ with randomly directed edges, to ensure that D is a tournament graph.

   Define the map $f' : B \to D$ by

   $$f'(x) = \begin{cases} x & \text{if } x \in V_{B'} \\ f^{-1}(x) & \text{otherwise.} \end{cases}$$

   and similarly for $g' : C \to D$. These are embeddings, and so $\mathcal{K}$ has the amalgamation property.
5. For the joint embedding property, let $A, B \in \mathcal{K}$. Take C to be a the disjoint union of A and B, with the non-edges randomly filled in. Then C is a finite tournament with A and B both embeddable in C using the identity map.

□

Thus the class $\mathcal{K}$ of all finite tournament graphs has Fraïssé limit T. By translating this infinite tournament to a representation, we construct our infinite representation

$\mathcal{M}$. As $\mathcal{K}$ is the age of T, we can represent any nonzero closed set of constraints Ξ in $\mathcal{M}$. To do this, first we assume without loss of generality that our constraint network does not contain two distinct points related by $1'$, as by Lemma 4.2.7 we can always remove these. We translate the constraints of Ξ into a *partial* tournament as before, treating $r^{\smile}$, $r + r^{\smile}$ and 1 as nonedges, and $1' + r$ and r as edges.

#12

#12	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	b
b	b	b	$-b$

This algebra has only one diversity cycle: *abb*. As such, we can construct a representation on three points as in Figure 4.19. There is also a representation on four points, also shown in Figure 4.19. Up to isomorphism, these are the only two representations of algebra #12. The representation on four points—but not the one on three—is also a representation as a relation algebra.

The reason for such limited representability is that we cannot compose $a\,;a$ on three distinct points. So whenever xay, every non-loop edge coming off x or y must be labelled by b. Starting with the representation on three points, which witnesses the *abb* cycle, we can extend this in only one way to four points. In attempting to add a fifth point, however, we are forced to witness either *bbb* or *baa*, neither of which is consistent.

Since there exist only finite representations, and on at least three points, $\mathcal{M}$-SAT is NP-complete for all $\mathcal{M}$, as is gen-SAT.

Figure 4.19: The two representations of algebra #12, up to isomorphism

#13

#13	$1'$	a	b
$1'$	$1'$	a	b
a	a	$-b$	b
b	b	b	$-b$

We begin by noting that this algebra contains cycles aaa and abb, but not bbb. This means that we can partition any representation of this algebra into two parts, each of which is a complete graph with every edge between distinct vertices labelled by an a-relation. Between each two vertices in different parts is a b-related edge. An infinite representation of this kind is shown in Figure 4.20.

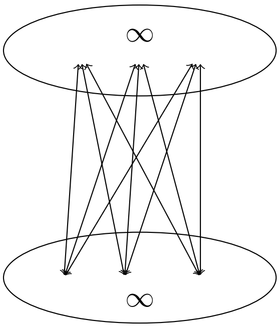


Figure 4.20: An infinite representation of algebra #13

We can look at restrictions of this representation to find other (qualitative) representations. For example, we can limit one or both a-labelled parts to a finite number of vertices. This gives us the smallest representation of algebra #13 in Figure 4.21, which is unique up to isomorphism.

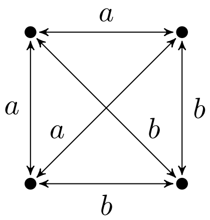


Figure 4.21: The unique smallest representation of algebra #13, up to isomorphism

The representation in Figure 4.20 is presented as a universal relation algebra representation in Lemma 15 of [19]. This is inaccurate. There is an issue with the constraint network in Figure 4.22, which is a single $a + b$-labelled edge surrounded by a chain of $1' + b$-labelled edges of length k.

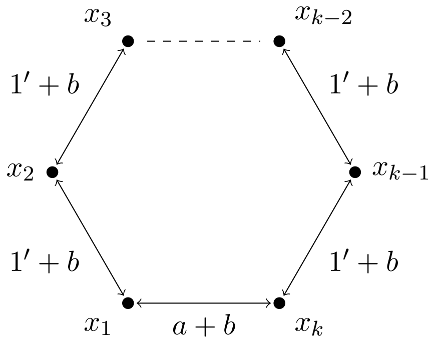


Figure 4.22: The problem configuration

In Lemma 4.4.7 we give a nonzero closed network containing this configuration (with $k = 3$) and prove that it is not satisfiable[6].

Lemma 4.4.7. *There does not exist a universal representation of algebra #13.*

Proof. It suffices to provide a nonzero closed constraint network which cannot be refined to a consistent network of atomic constraints. Such a network would then not be satisfiable in any representation.

Such a network is shown in Figure 4.23. Since bbb is not a cycle in this algebra, exactly one of (x, v) and (y, v) must refine to $1'$. Without loss of generality suppose $x1'v$. If vaz then $x1'vaz$ would lead to xaz, which is not allowed by the constraint network. So vbz and also xbz. As bbb is not a cycle in this algebra, we are forced to refine to $y1'z$. Similarly, if zaw, $y1'zaw$ would lead to yaw, also not allowed by the constraint network. So zbw and ybw. Again, as bbb is not a cycle, wav. Composing these refinements gives us $x1'vaw$, that is, xaw. This is not permitted by the constraint network, and so these constraints cannot be satisfied in any representation. A similar situation occurs if $y1'v$. □

The issue with the proof of universality given in [19] is that the configuration in Figure 4.22 is interperted with $1' + b$ as equality and $a + b$ as inequality. This causes problems because equality is transitive, and so two points which are unequal are made equal. We will now amend the proof given in [19], showing that the configuration in Figure 4.22 is the only set of constraints not representable by the original strategy.

Theorem 4.4.8. *Let Ξ be a non-zero closed set of constraints not containing the constraint system appearing in Figure 4.22. The constraints Ξ are satisfiable in the*

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[6] When I shared this problem with Prof. Robin Hirsch, one of the authors of [19], he provided such an example. The example presented here was developed subsequently.

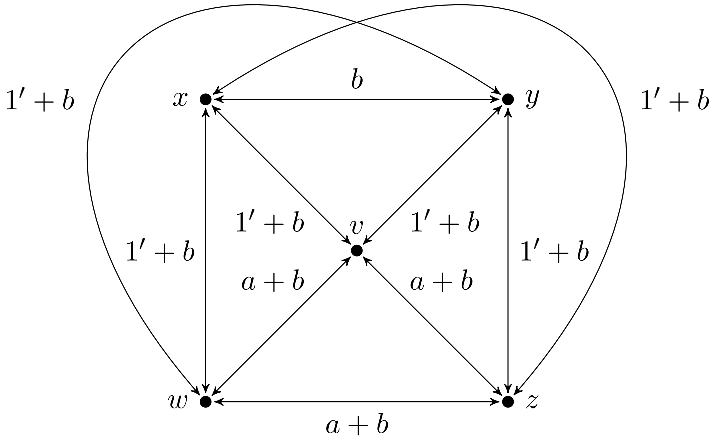

Figure 4.23: Nonzero and closed constraint network over algebra #13, but not satisfiable in any representation

following representation, illustrated in Figure 4.24.

$$\begin{aligned}
1'^{\mathcal{M}} &= \{(x,x) \colon x \in Y \cup Z\} \\
a^{\mathcal{M}} &= \{(y,y') \colon y \neq y' \in Y\} \cup \{(z,z') \colon z \neq z' \in Z\} \\
b^{\mathcal{M}} &= \{(y,z),(z,y) \colon y \in Y, z \in Z\}
\end{aligned}$$

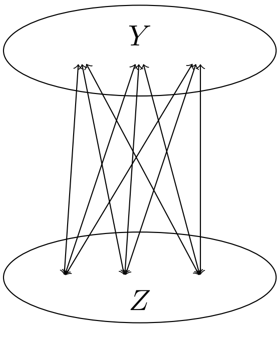

Figure 4.24: An infinite representation of algebra #13

Proof. Recall algebra #4:

#4	$1'$	$0'$
$1'$	$1'$	$0'$
$0'$	$0'$	$1'$

We construct a new constraint network Θ over algebra #4 from Ξ according to the following correspondence.

Constraint over #13 in Ξ	Constraint over #4 in Θ
0	0
$1'$	$1'$
a	$1'$
b	$0'$
$1' + a$	$1'$
$1' + b$	1
$a + b$	1
1	1

Lemma 4.4.9. *Θ is a nonzero closed set of constraints in #4.*

Proof. Since $C \neq 0$ for all $C \in \Xi$, the given mapping implies $0 \notin \Theta$. Hence, Θ is non-zero. To check that Θ is closed, we first note that all elements are symmetric and so for all $x_i A x_j$ in Θ we have that $x_j A x_i$ is in Θ, and that $A = A^{\smile}$.

We want to show that for all constraints $x_i A x_j$, $x_j B x_k$ in Θ there exists $x_i C x_k$ in Θ such that $C \leqslant A \mathbin{;} B$. Since Ξ is closed there must exist at least one constraint in Ξ that is mapped to a constraint between x_i and x_k in Θ. As a consequence if either A or B is 1 then that particular triangle is trivially closed, and so we can disregard those situations. The only possible nonclosed triangles in Θ are given in Figure 4.25.

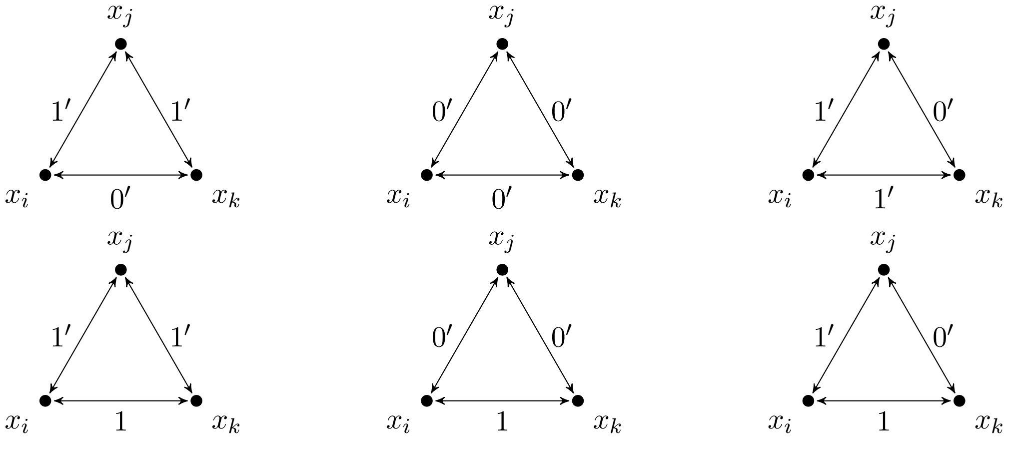

Figure 4.25: Nonclosure in Θ can only arise from a nonclosed triangle in Ξ.

In Figure 4.26 we take the 'preimage' of the situations in Figure 4.25, converting constraints in Θ to constraints in Ξ, noting that the only relation that maps to $0'_{\#4}$ is b. It suffices to consider the minimal elements in the preimage of A and B and the maximal element in the preimage of C. Each of the situations in Figure 4.26 is a nonclosed triangle over algebra #13. Since Ξ is closed we conclude that none of the nonclosed triangles in Figure 4.25 appears in Θ. $\square$

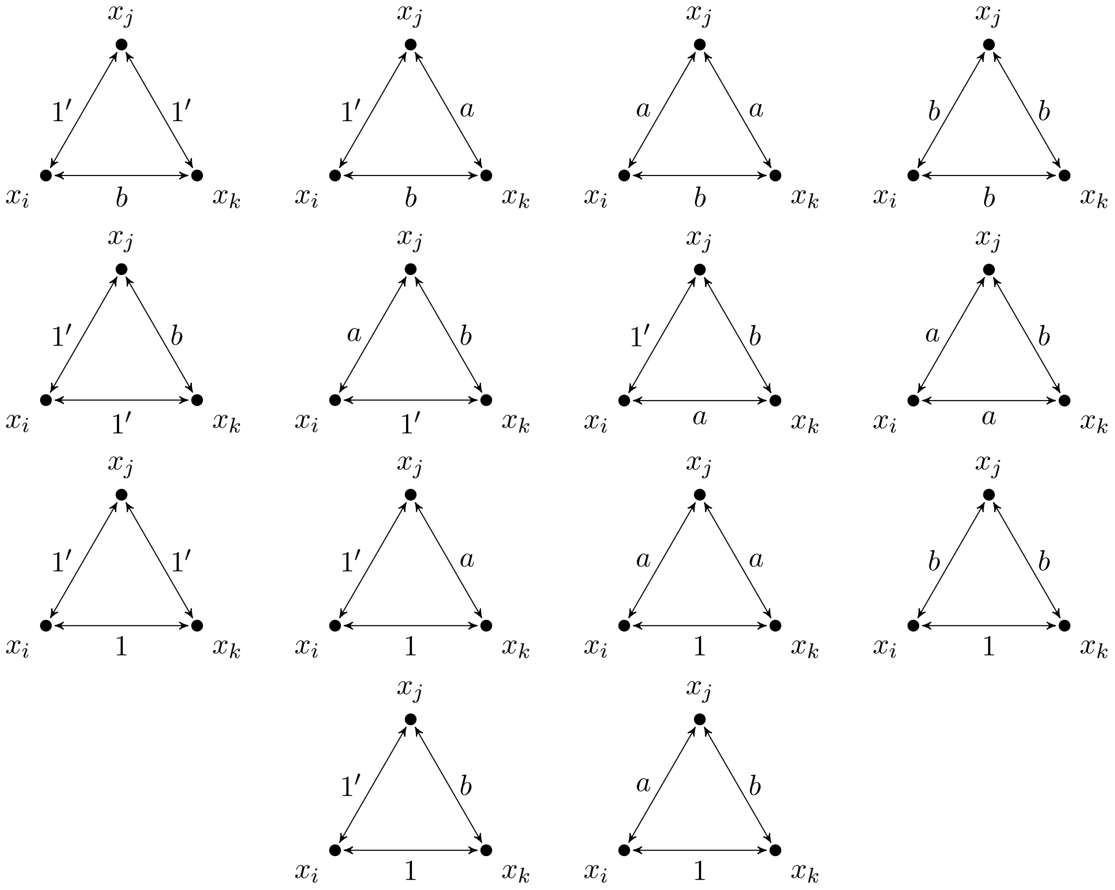

Figure 4.26: Converting the situations in Figure 4.25 into constraints in Ξ

As we noted in our earlier discussion, algebra #4 has only one representation up to isomorphism, and it is on two points. Moreover this representation is universal, so Θ is satisfiable.

Suppose that we satisfy Θ on two distinct points u, v by a map φ. We consider preimages $\varphi^{-1}(u)$ of all variables of Θ mapped to u under this representation. We construct a set of constraints Λ over algebra #5 with variables $\varphi^{-1}(u)$. The constraints in Λ are defined by $E = C \cdot (-b)$ for all $C \in \Xi$ where C is a relation between points in $\varphi^{-1}(u)$, interpreting $a_{\#13}$ as $0'_{\#5}$.

#5	$1'$	$0'$
$1'$	$1'$	$0'$
$0'$	$0'$	1

We will now prove that Λ is satisfiable over a countably infinite representation of algebra #5.

Lemma 4.4.10. *Λ is satisfiable*[7] *over a countably infinite representation of algebra #5.*

Proof. Recall that when we mapped Ξ to Θ, we mapped only $b_{\#13}$ constraints to the diversity element $0'_{\#4}$. The set of variables of Λ is the preimage of a single variable in the image of this map, so none of the relations $C \in \Xi$ used in the definition of a relation $E \in \Lambda$ will be equal to b. As such, when we negate by b we cannot introduce a zero relation, and so Λ is nonzero.

Since Λ is nonzero there is only one set of constraints that can cause it to be unsatisfiable. This is shown in Figure 4.13, interpreting a as $0'$. This configuration identifies points related by $0'$. The configuration in Figure 4.13 can only occur in Λ when the configuration in Figure 4.22 occurs in Ξ. The $a + b$- and $1' + b$-labelled edges in Ξ would become $0'$- and $1'$-labelled edges in Λ. Since we have excluded the configuration in Figure 4.22 from appearing in Ξ we can now satisfy Λ over a countably infinite set by Lemma 4.4.1. □

We can construct a set of constraints ζ similar to Λ but with variables from $\varphi^{-1}(v)$. These are also nonzero and closed. Represent $\varphi^{-1}(u)$ over an infinite set Y by variable assignment f_0 and $\varphi^{-1}(v)$ over an infinite set Z by f_1. From these sets we construct a satisfaction of Ξ given by

$$\begin{aligned} 1'^{\mathcal{M}} &= \{(x,x) \colon x \in Y \cup Z\} \\ a^{\mathcal{M}} &= \{(y,y') \colon y \neq y' \in Y\} \cup \{(z,z') \colon z \neq z' \in Z\} \\ b^{\mathcal{M}} &= \{(y,z),(z,y) \colon y \in Y, z \in Z\}. \end{aligned}$$

The variable assignment $f = f_0 \cup f_1$ is defined on all variables of Ξ and satisfies all of its constraints. □

The tractability of gen-SAT for this algebra is still an open matter, as is that of $\mathcal{M}$-SAT for infinite $\mathcal{M}$.

---

[7] The original argument claims that Λ is closed and therefore representable. The closure of Λ is not guaranteed, but fortunately is not necessary for satisfiability.

#14

#14	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	b
b	b	b	1

The smallest representation is on four points, and is unique up to isomorphism. It is shown in Figure 4.27. This representation can be built by first witnessing the cycle abb and then witnessing the cycle bbb. As $bb = 1$ we can always add an extra point to a representation and label the nonedges with b. As such, there exist finite representations of every size at least 3, and so $\mathcal{M}$-SAT is NP-complete for finite $\mathcal{M}$.

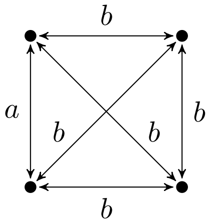

Figure 4.27: The unique smallest representation of algebra #14, up to isomorphism

Lemma 16 of [19] gives the following universal relation algebra representation of algebra #14, which can be viewed as a $2 \times \omega$ grid $M = \{n, n' \colon n \in \mathbb{N}\}$.

$$\begin{aligned}(1')^{\mathcal{M}} &= \{(n,n),(n',n') \colon n \in \mathbb{N}\}\\ (a)^{\mathcal{M}} &= \{(n,n'),(n',n) \colon n \in \mathbb{N}\}\\ (b)^{\mathcal{M}} &= (1')^{\mathcal{M}} \backslash (a)^{\mathcal{M}}\end{aligned}$$

There is a larger class of infinite qualitative representations. For example, one could start with two vertices x, y such that xay, and then add a countably infinite number of vertices, labelling all of the new edges by b and putting an $1'$-labelled loop on every vertex. This witnesses both cycles abb and bbb, but does not feature a non-identity composition of a with a and so is consistent. It is not a relation algebra because there exist b-edges above which we cannot witness the cycle abb. Note that this is a restriction of the universal representation.

More generally, let G be a simple graph containing no connected component of more than two vertices. Suppose G does contain such a component, and that there are three pairwise nonadjacent vertices. We can construct a representation $\mathcal{G}$ of algebra #14 from G with the vertices of G as the domain. We begin by translating every edge of G to an a-relation in G and every nonedge to a b-relation. Finally we add the identity relation as a loop on every vertex. Because of the conditions we imposed on G, we witness both cycles abb and bbb, without witnessing a non-identity composition of a with a. That is, $\mathcal{G}$ is a representation of algebra #14, with $\mathcal{G}$ finite if G is. We can also turn every finite representation of this algebra into a graph with the same properties.

Let $\mathcal{K}$ be the class of all finite simple graphs containing no connected component of more than two vertices. By the above we know that $\mathcal{K}$ contains, in a sense, all finite representations of algebra #14. This is a Fraïssé class, and the proof is largely similar to that used for algebra #11, except that in step 4 of the proof of Lemma 4.4.6 we do not replace the nonedges in $E(D)$ with any edges.

This gives us a Fraïssé limit D, and we can build from this a universal *qualitative* representation $\mathcal{D}$ of algebra #14.

#15

#15	$1'$	a	b
$1'$	$1'$	a	b
a	a	$-b$	b
b	b	b	1

The smallest representation is on five points, and is shown in Figure 4.28. We can construct this by first representing the bbb cycle on a triangle. From any of the edges we witness the cycle abb by adding a fourth point. Next we witness aaa from the only a-edge in the representation, and represent the identity as loops on every vertex. There is only one way up to isomorphism to label the edges at each step, and so the representation is unique up to isomorphism. There exist finite representations on any number of vertices at least five, as we can always add an extra point and label the missing edges with b. As such, $\mathcal{M}$-SAT is NP-complete for finite $\mathcal{M}$.

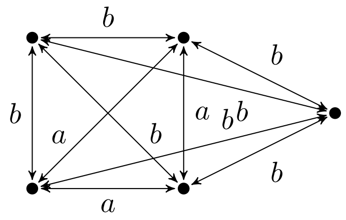

Figure 4.28: The unique smallest representation of algebra #15, up to isomorphism

Lemma 17 of [19] gives the following universal representation on a domain $\mathbb{N}\times\mathbb{N}$:

$$\begin{aligned}(1')^{\mathcal{M}} &= \{(n,n) \colon n \in \mathbb{N}\}\\ (a)^{\mathcal{M}} &= \{((m,n),(m',n) \colon m,m',n \in \mathbb{N}, m \neq m'\}\\ (b)^{\mathcal{M}} &= \{((m,n),(m',n') \colon m,m',n,n' \in \mathbb{N}, n \neq n'\}\end{aligned}$$

As for algebra #14 we can take restrictions of this algebra to obtain qualitative representations which are not relation algebra representations. For example, one could restrict the domain to all $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that $1 \leqslant m \leqslant 3$, and still witness the required cycles.

Once again we can construct a Fraïssé class $\mathcal{K}$, the class of all finite graphs G such that there does not exist vertices x, y, z such that $(x,y),(y,z) \in E(G)$ and $(x,z) \notin E(G)$. The argument is carried out similarly to that used for algebra #14.

#16

#16	$1'$	a	b
$1'$	$1'$	a	b
a	a	$-a$	$0'$
b	b	$0'$	$-b$

This algebra forbids the cycles *aaa* and *bbb*. It is called *the pentagon algebra*, because if one considers a and b to be two distinct colours then an edge-colouring of the complete graph K_5 without monochromatic triangles will give a relation algebra representation of this algebra.

In fact, there exist only two qualitative representations of algebra #16 up to isomorphism, on four and five points. They are shown in Figure 4.30. As such, $\mathcal{M}$-SAT is NP-complete for all $\mathcal{M}$, as is gen-SAT. The representation on four points is not a relation algebra representation since along the bottom b-edge we cannot witness $b \leqslant a \mathbin{;} a$.

To develop the representation on four points, we witness our only two cycles, *abb* and *aab*, disjointly on two triangles as in Figure 4.29. No matter how we try to label the nonedges between y, u, z and v, we are forced to witness a situation isomorphic to that in which $y = u$ and $z = v$. Hence we assume this without loss of generality.

From here there is only one way to label the remaining nonedge. This gives a representation on four points which is unique up to isomorphism. If we add a fifth vertex

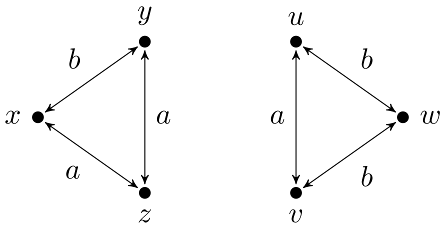

Figure 4.29: Witnessing the cycles of algebra #16

then there is only one way to label the nonedges, leading to the representation on five vertices which is also unique up to isomorphism.

These are the only edge-labellings of undirected graphs on four or five vertices that are consistent with algebra #16. If we attempt to add a sixth vertex to the representation of size five, there is no way to label the new edges without introducing either an *aaa* or *bbb* cycle.

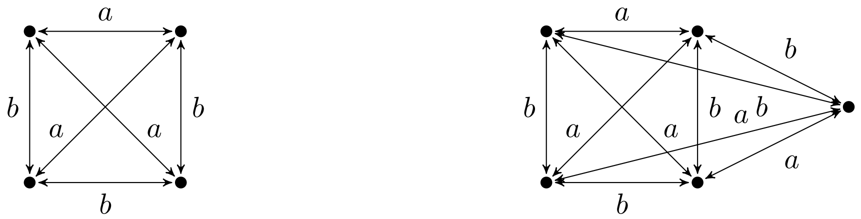

Figure 4.30: The unique representations of algebra #16, up to isomorphism

#17

#17	$1'$	a	b
$1'$	$1'$	a	b
a	a	$-a$	$0'$
b	b	$0'$	1

The smallest qualitative representation is on 4 points. As $bb = 1$, this can conceivably be enlarged to any $4 < n < \infty$, and $\mathcal{M}$-SAT is NP-complete for all finite $\mathcal{M}$. Finite relation algebra representations also exist [19], but not all qualitative representations are relation algebra representations, for example, Figure 4.31.

Lemma 14 of [19] gives a universal relation algebra representation $\mathcal{N}$ based on N, the infinite triangle-free graph. This graph can be constructed as a Fraïssé limit of the class of finite triangle-free graphs, similar to the construction of the infinite tournament for algebra #11. As such, it is a universal representation of algebra #17.

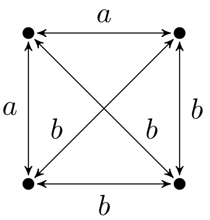

Figure 4.31: A qualitative representation of algebra #17 on 4 points

The lack of triangles ensures that we never see the composition $a \leqslant a \,;\, a$.

$$\begin{aligned}(1')^{\mathcal{N}} &= \{(x,x) \colon x \in V(N)\}\\ (a)^{\mathcal{N}} &= \{(x,y) \colon x \neq y \text{ and } (x,y) \in E(N)\}\\ (b)^{\mathcal{N}} &= \{(x,y) \colon x \neq y \text{ and } (x,y) \notin E(N)\}\end{aligned}$$

#18

#18	$1'$	a	b
$1'$	$1'$	a	b
a	a	1	$0'$
b	b	$0'$	1

The smallest qualitative representation is on 5 points. As $bb = 1$, this can conceivably be enlarged to any $5 < n < \infty$, and $\mathcal{M}$-SAT is NP-complete for all finite $\mathcal{M}$. Finite relation algebra representations also exist [19], but not all qualitative representations are relation algebra representations, for example, Figure 4.32.

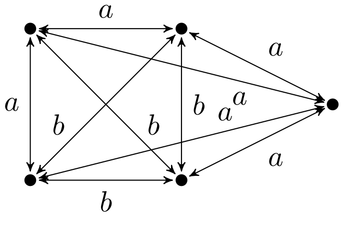

Figure 4.32: A qualitative representation of algebra #18 on 5 points

Lemma 14 of [19] gives a universal relation algebra representation $\mathcal{R}$ based on R, the infinite random graph, or simply *the random graph*. The random graph, also known as the *Rado graph*, can be constructed in multiple ways, but the random construction is the most obvious (see [15]). Alternatively, one can use a Fraïssé limit similar to the construction of the infinite tournament for algebra #11. As such, R can be used to construct a universal representation for algebra #18:

$$\begin{aligned}\mathcal{R}(1') &= \{(x, x) \colon x \in V(R)\} \\ \mathcal{R}(a) &= \{(x, y) \colon x \neq y \text{ and } (x, y) \in E(R)\} \\ \mathcal{R}(b) &= \{(x, y) \colon x \neq y \text{ and } (x, y) \notin E(R)\}\end{aligned}$$

#19

#19	e_1	e_2	a
e_1	e_1	0	a
e_2	0	e_2	a
a	a	a	$1'$

This algebra is not associative as $(a ; e_1) ; e_2 \neq a ; (e_1 ; e_2)$. There exists only one representation up to isomorphism, and it is on two points. This representation is shown in Figure 4.33. As a is the only nonidentity element and $a ; a = 1'$, we are unable to add a third point to the representation. By Theoerem 4.2.4, $\mathcal{M}$-SAT is tractable for all $\mathcal{M}$, as is gen-SAT.

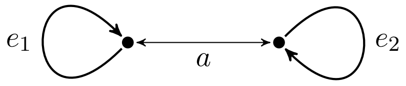

Figure 4.33: The only representation of #19, up to isomorphism

#20

#20	e_1	e_2	a
e_1	e_1	0	a
e_2	0	e_2	a
a	a	a	1

This algebra is also not associative. First we note that $a = -1'$ and hence must be the diversity relation. In order to witness the aaa cycle we must see at least three points in the representation. A representation on three points is shown in Figure 4.34. As such, $\mathcal{M}$-SAT is NP-complete for all finite $\mathcal{M}$.

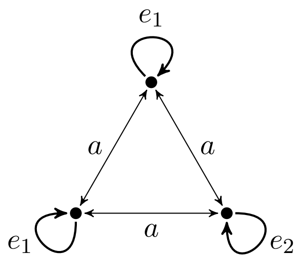

Figure 4.34: One of the smallest representations of algebra #20

For all $n \geqslant 3$ we can construct $2^n - 2$ unique (up to isomorphism) representations of #20 on n points. Our construction relies on the fact that any two distinct points in a representation of #20 must be related by a. Begin by taking a complete graph K_n and label every edge by a. For every vertex we add a loop which we can label by either e_1 or e_2. This respects composition because $e_1 ; a = e_2 ; a = a = a ; e_2 = a ; e_1$. This gives us 2^n weak representations, but in order to witness all compositions we require at least one e_1-loop and one e_2-loop.
This construction can be extended to a countably infinite representation, which arises from the unique countably infinite complete graph.

We can also construct a universal representation by considering the cartesian product of two copies of the countably infinite complete graph, $K_\omega \times K_\omega$. For one copy we place an e_1-labelled loop on every vertex, and for the other an e_2-labelled loop on every vertex. A nonzero closed constraint network can then be represented using an inductive method similar to that used for algebra #5. When mapping a new point in the representation, we place it on an e_1-labelled vertex if the constraint network demands it, otherwise an e_2-labelled vertex.

#21

#21	$1'$	r	$r^\smile$
$1'$	$1'$	r	$r^\smile$
r	r	0	$1'$
$r^\smile$	$r^\smile$	$1'$	0

This algebra is not associative as $(r ; r) ; r^\smile \neq r ; (r ; r^\smile)$. There exists only one representation up to isomorphism, and it's on two points. This representation is shown in Figure 4.33. As r and $r^\smile$ are the only nonidentity elements and $r ; r = 0 = r^\smile ; r^\smile$, we are unable to add a third point to the representation. By Theorem 4.2.4, $\mathcal{M}$-SAT is tractable for all $\mathcal{M}$, as is gen-SAT.

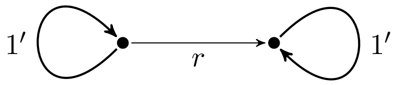

Figure 4.35: The only representation of #21, up to isomorphism

#22, #23 and #24

#22	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	0
b	b	0	$1'$

#23	$1'$	a	b
$1'$	$1'$	a	b
a	a	$1'$	0
b	b	0	$-a$

#24	$1'$	a	b
$1'$	$1'$	a	b
a	a	$-b$	0
b	b	0	$-a$

None of these algebras is associative, for in all of the three we have that $(a\,;a)\,;b \neq a;(a;b)$. Moreover, none of these algebras is representable. This is because $a;b = 0$. We are forced to witness xay along distinct points x and y, and ubv along distinct points u and v. However there is no way to label the nonedges here without introducing a composition ab, introducing a 0-labelled edge. As such, gen-SAT can be decided in constant time.

4.5 Nonassociative algebras on four atoms

In this section we discuss the 373 nonassociative algebras on 4 atoms, which were generated by the program in Section C.3. These algebras are listed in Appendix A, along with their cycles in Appendix B. We denote them by #1 through #373. This is distinct from the numbering system used in Section 4.4. Just in case this is not confusing enough, Maddux [58] numbers the relation algebras on atoms $1', a, b, c$ as 1_{37} through 37_{37}, and the relation algebras on atoms $1', a, rr^{\smile}$ as 1_{65} through 65_{65}. We will replicate his indexing by referring to the 24 algebras on 3 or fewer atoms in Section 4.4 with subscript $\leqslant 3$. For example, algebra $\#12_{\leqslant 3}$ is on 3 atoms, while #12 is on 4 atoms.

We consider symmetric atoms a, b and c. We also consider an atom r which is not symmetric, and so has converse $r^{\smile}$. We won't draw $r^{\smile}$ on any of the representations, since its existence can always be deduced from the placement of the atom r. Nor will we place loops on any point unless necessary since, for an algebra with an atomic identity $1'$, we can assume that there is an $1'$-labelled loop on every vertex. If the identity relation is not atomic, it will be the disjunction of atoms e_1, e_2, e_3, e_4, where these atoms exist. We will represent our algebras as edge-labelled digraphs wherever possible, but if the representation is too big then we will use adjacency matrices in which the ij-th entry refers to the label on the edge from vertex i to vertex j.

In moving from 3 to 4 atoms there is a large jump in difficulty of proving nonrepresentability. The next few lemmas detail a proof of the qualitative nonrepresentability of algebras #334 to #339, along with #346 and #347.

Lemma 4.5.1. *Let $\mathcal{A}$ be a simple nonassociative algebra on atoms $\{1', a, r, r^{\smile}\}$ containing rra and $rrr^{\smile}$ and not rrr. Suppose also that if $\mathcal{A}$ contains raa then it does not contain any of rar, raa or aaa. Then a qualitative representation of $\mathcal{A}$ cannot contain either of the following as subgraphs.*

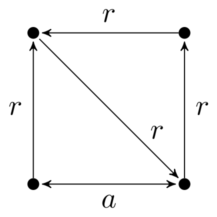

Figure 4.36: ①

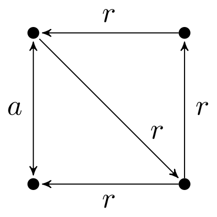

Figure 4.37: ②

Proof. ① excluded by rrr and $raa \cdot arr$. ② excluded by rrr and $raa \cdot rar$. □

Lemma 4.5.2. *Let $\mathcal{A}$ be a simple nonassociative algebra on atoms $\{1', a, r, r^{\smile}\}$ containing rra and $rrr^{\smile}$ and not rrr. Suppose also that if $\mathcal{A}$ contains raa then it does not contain any of rar, raa or aaa. Then a qualitative representation of $\mathcal{A}$ cannot contain the following as a subgraph.*

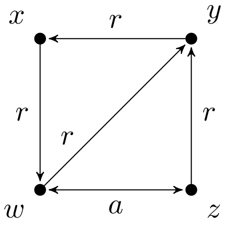

Figure 4.38: ③

Proof. We note that this witnesses arr. We exclude all possibilities to get the figure below, which also witnesses rar. We will now investigate all of the ways in which we can add a fifth distinct point, u, as in Figure 4.39. In particular, we need to do this without introducing either rrr or raa, either of which would violate the conditions of the lemma.

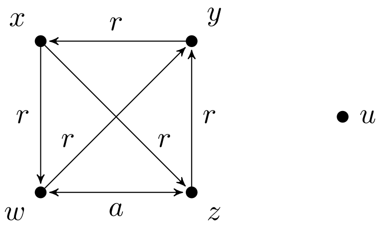

Figure 4.39: Introducing a fifth point u

yRu	zSu	problem
r	r	introduces rrr
r	$r^{\smile}$	introduces ①
r	a	introduces ②
$r^{\smile}$	r	introduces rrr
$r^{\smile}$	$r^{\smile}$	introduces rrr
a	$r^{\smile}$	introduces ①
a	a	introduces raa

This leaves us with two options: $yr^{\smile}u$ and zau as in Figure 4.40, or yau and zru as in Figure 4.41. By continuing to exclude rrr and raa we can complete the remaining nonedges without ambiguity.

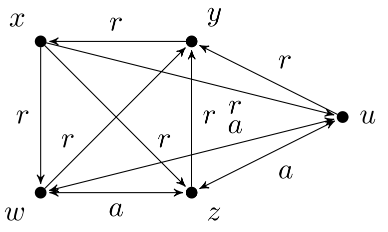

Figure 4.40: $yr^{\smile}u$ and zau

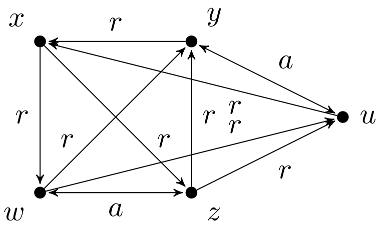

Figure 4.41: yau and zru

We are yet to witness rra. We do this by introducing a triangle vut as in Figure 4.42, although we do not assume that the new points are distinct. That is, we allow for the possibility that $1'$ can relate two points.

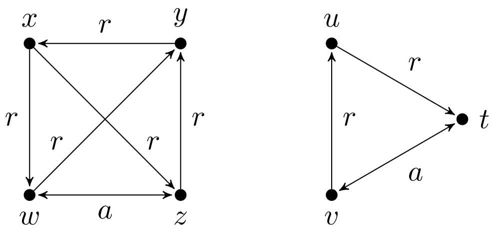

Figure 4.42: Witnessing rra

We will now consider all of the ways in which we can label the nonedges, keeping in mind that $xyzw$ can only be connected to an external point in a manner that is consistent with either Figure 4.40 or Figure 4.41. As such, we can only relate y to u by $r^{\smile}$ or a, and z to v by $1'$, r or a. We will consider and eliminate all 6 cases, ensuring that there is no consistent way to witness rra as well as the structure in Figure 4.38 without introducing either rrr or raa, violating the conditions of the lemma. One particularly useful trick here is to note that these conditions force $aa = 1' + a$.

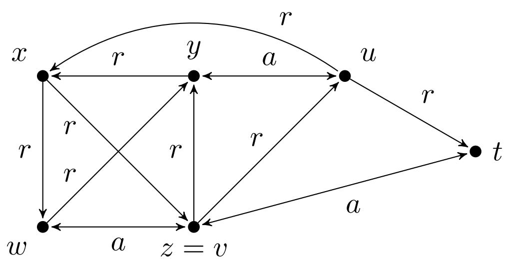

Figure 4.43: yau and $z1'v$

- $x1't$ would introduce urz, but zru.
- xrt would introduce $urxrt$ and urt, and so rrr.
- $xr^{\smile}t$ would introduce $urtrx$ and urx, and so rrr.
- xat forces xaz as $xataz$, and so raa.

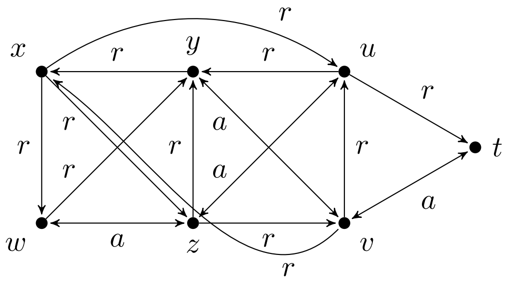

Figure 4.44: $yr^{\smile}u$ and zrv

- Introduces rrr on vxu.

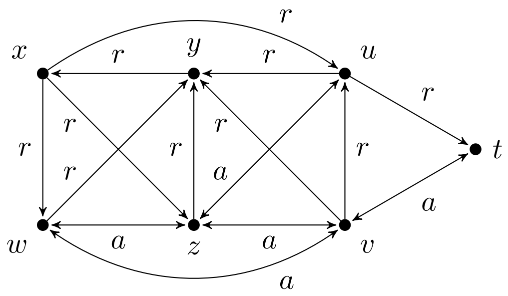

Figure 4.45: $yr^{\smile}u$ and zav

- Introduces rrr on vuy.

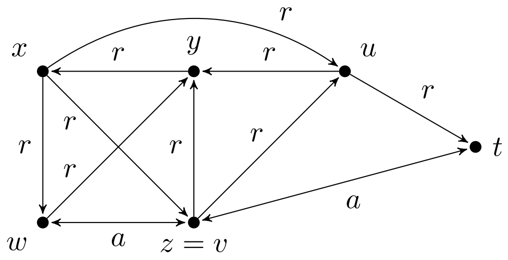

Figure 4.46: $yr^{\smile}u$ and $z1'v$

- Introduces rrr on xzv.

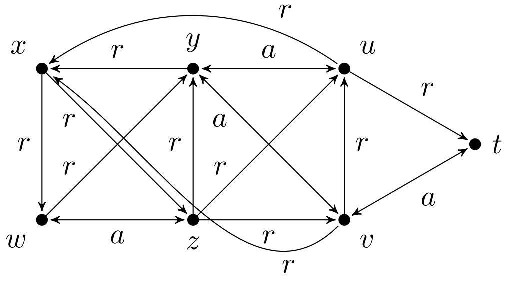

Figure 4.47: yau and zrv

- Introduces rrr on vux.

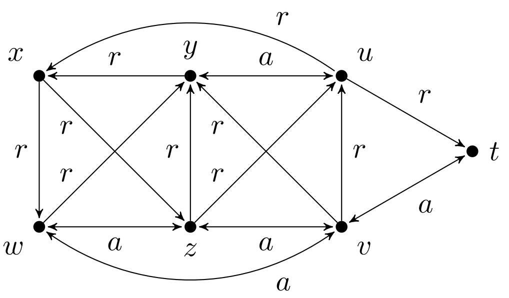

Figure 4.48: yau and zav

- $x1't$ would introduce $wavax$ and xrw, and so raa.
- xrt would introduce $urxrt$ and urt, and so rrr.
- $xr^{\smile}t$ would introduce $urtrx$ and urx, and so rrr.
- xat forces xav because $xarav$. Now we have $xavaz$ and xrz, and so raa.

□

Lemma 4.5.3. *Let $\mathcal{A}$ be a simple nonassociative algebra on atoms $\{1', a, r, r^{\smile}\}$ containing rra and $rrr^{\smile}$ and not rrr. Suppose also that if $\mathcal{A}$ contains raa then it does not contain any of rar, raa or aaa. Then $\mathcal{A}$ has no qualitative representation.*

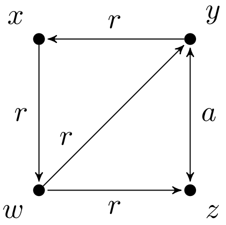

Figure 4.49: ④

Proof. We need to witness rra and $rrr^{\smile}$, that is, we need to see the structure in Figure 4.50. The proof follows by attempting to label (x, u) and (y, v) without introducing a contradiction. In particular, we cannot introduce ①, ②, ③ or ④. We do not exclude the possibility that two nodes may be related by $1'$, that is, that two nodes may be equal. We iterate through all possible combinations in Table 4.52, leaving one particularly tricky case for last.

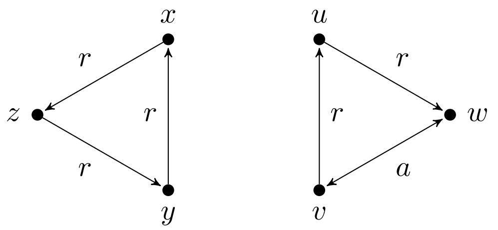

Figure 4.50: Witnessing rra and $rrr^{\smile}$

The case in which xau, yav and xav, illustrated in figure 4.51, requires some care in eliminating. First we note that we see raa on yxv. Since $x1'w$ would violate meet on (x, u), xrw would introduce arr on xuw and raw would introduce aaa on xvw, we conclude that $xr^{\smile}w$.

Finally we consider the edge (y, w). $y1'w$ would introduce ①, yrw would introduce rrr on ywx, and $yr^{\smile}w$ would introduce rrr on wyx, and yaw would introduce aaa on ywv. We have exhausted all possibilities, and conclude that there is no way to label (x, u) and (y, v) that doesn't lead to an inconsistency in the representation.

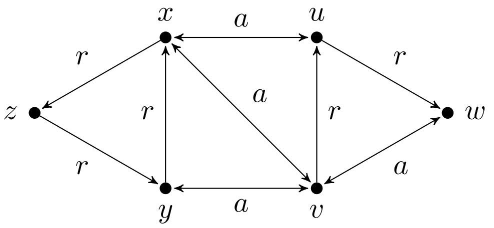

Figure 4.51: Witnessing rra and $rr\breve{r}$ with xau, yav and xav

xRu	ySv	problem
$1'$	$1'$	introduces ②.
$1'$	r	introduces rrr on yvx.
$1'$	$r^{\smile}$	introduces rrr on vyx.
$1'$	a	introduces ③.
r	$1'$	introduces rrr on yxu.
r	r	$y1'u$ would violate converse on (u, v). yru would introduce rrr on yvu. $yr^{\smile}u$ would introduce ②. yau would introduce ②.
r	$r^{\smile}$	$x1'v$ would violate converse on (y, v). xrv would introduce rrr on xvu. $xr^{\smile}v$ would introduce rrr on vyx. xav would introduce ①.
r	a	$x1'v$ would violate meet on (y, x). xrv would introduce ②. $xr^{\smile}v$ would introcue ③. xav would introduce raa on yxu, arr on vxu.
$r^{\smile}$	$1'$	introduces rrr on yux.
$r^{\smile}$	r	$y1'u$ would violate converse on (u, v). yru would introduce rrr on yvu. $yr^{\smile}u$ would introduce ②. yau would introduce ③.
$r^{\smile}$	$r^{\smile}$	$x1'v$ would violate converse on (y, v). xrv introduces ②. $xr^{\smile}v$ would introduce rrr on vux. xav would introduce ①.
$r^{\smile}$	a	$y1'u$ would violate meet on (v, u).

xRu	ySv	problem
		yru would introduce rrr on yux.
		$yr^{\smile}u$ would introduce rrr on uyx.
		yau would introduce ③.
a	$1'$	$z1'u$ violates converse on (y, z).
		zru would introduce rrr on zyu.
		$zr^{\smile}u$ would introduce ③.
		zau would introduce raa on xzu and rar on yxu.
a	r	$x1'v$ would violate meet on (x, u).
		xrv would introduce rrr on yxv.
		$xconrv$ would introduce rrr on yvx.
		xav would introduce ④.
a	$r^{\smile}$	$x1'v$ would violate converse on (x, y).
		xrv would introduce ④.
		$xr^{\smile}v$ would introduce rrr on vyx.
		xav would introduce ①.
a	a	$x1'v$ would vioalte meet on (x, y).
		xrv would introduce ②.
		$xr^{\smile}v$ would introduce ③.
		xav would introduce raa. See additional comments.

Table 4.52: Excluding possible edge-labellings in Figure 4.50

This particular method of proof is essentially a proof by exhaustion within a proof by exhaustion. It's an exhausting approach, and a single proof of this style is unlikely to cover more than a few algebras. As such, we turn to an automated method of proving nonrepresentability. Using the cycle structure of a nonassociative algebra, we can create a set of assumptions that can be run through an automated prover, Prover9 [60], using the syntax we detail below.

#146	$1'$	a	b	c
$1'$	$1'$	a	b	c
a	a	$1' + b$	$a + c$	$b + c$
b	b	$a + c$	$1' + b$	a
c	c	$b + c$	a	$1' + a$

Consider the nonassociative algebra #146, with an atom table shown above. This numbering will be explained shortly. Algebra #146 has the usual cycles involving identity, as well as bbb, baa, acc and abc. We encode these cycles as binary relations, with the assumption that the identity relates every (x, x). So for example, we wish to witness the cycle baa, so we write:

```
exists x exists y exists z (B(x,z) & A(z,y) & A(x,y)).
```

We do not wish to witness an aaa cycle, so we negate it by writing:

```
A(x,z) & A(z,y) -> -A(x,y).
```

We also include assumptions for completeness, disjointness, faithfulness and the correct representation of converse. The input for this particular algebra is given in Program 1. The output is given in Section C.4. □

The same code can be used to generate representations, if they exist. This is done through Prover9's counterpart, Mace4. Some algebras, such as #123, are too computationally difficult for Prover9. In these cases, we can rely on Mace4 to reject representations of any size up to a given number. Jackson et al. give an algorithm for deciding qualitative representability of nonassociative algebras. The resulting representation, if it exists, is on 3 times the number of cycles not involving identity. We have, at most, 10 such cycles with the algebras we are considering here, and we can assume that at least two of the resulting triangles are not disjoint, giving us a size of 29. That is, if there do not exist representations up to size 29, we can declare that the algebra is not qualitatively representable.

If a finite nonassociative algebra has n diversity cycles, one can construct an edge-labelled disconnected digraph on $3n$ vertices with each cycle as a separate triangle. This digraph will contain every consistent triangle. As a consequence of the algorithm behind Theorem 4.3.2, if this algebra is representable then it will be representable on $3n$ points.

With these tools we survey the qualitative representability of all nonassociative algebras on 4 atoms, with results given in Appendix A, and corresponding cycles in Appendix B. For each algebra we provide a possible atom table. We also note whether or not the algebra is a relation algebra; if so, we give the number used by Maddux [58], if applicable. If the algebra is qualitatively representable, we give an example of a representation. The representation is on the smallest number of vertices, but it is not necessarily the only representation up to isomorphism with that property.

If the representation is on too many points to draw in any useful manner, it is represented by a matrix with entries from the atoms of the algebra. To interpret these representations, we interpret the rows and columns of the matrix as vertices of a representation, such that the uv-th entry of the matrix is the relation from u to v.

Program 1 Prover9 input for proving nonrepresentability of algebra #146.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Generic testing of qualitative representations %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

formulas(assumptions).

x = y | A(x,y) | B(x,y) | C(x,y).     % completeness

A(x,y) -> x != y & -B(x,y) & -C(x,y). % disjointness
B(x,y) -> x != y & -A(x,y) & -C(x,y).
C(x,y) -> x != y & -B(x,y) & -A(x,y).

exists x exists y A(x,y).             % faithfulness
exists x exists y B(x,y).
exists x exists y C(x,y).

A(x,y) -> A(y,x).                     % converses
B(x,y) -> B(y,x).
C(x,y) -> C(y,x).

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                     Cycles                     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

A(x,z) & A(z,y) -> -A(x,y).
exists x exists y exists z (B(x,z) & A(z,y) & A(x,y)).
C(x,z) & A(z,y) -> -A(x,y).
A(x,z) & B(z,y) -> -B(x,y).
exists x exists y exists z (A(x,z) & B(z,y) & C(x,y)).
exists x exists y exists z (A(x,z) & C(z,y) & C(x,y)).
exists x exists y exists z (B(x,z) & B(z,y) & B(x,y)).
C(x,z) & B(z,y) -> -B(x,y).
B(x,z) & C(z,y) -> -C(x,y).
C(x,z) & C(z,y) -> -C(x,y).

end_of_list.
```

Chapter 5

Conclusion

Algebras of relations have a rich history. They were born in the late 19th century with Peirce's work on extending the algebra of Boole's logic. Much of the early work on model theory in the 20th century paid respect to algebras of relations. In the middle of the 20th century, Tarski's work on the "calculus of relations" defined relation algebras as we know them today. Using this calculus, Tarski and Givant were even able to create a language for doing set theory without variables—a language so expressive that it's equivalent equivalent a system of first-order logic with just three variables.

Tarski, along with Jónsson, asked if every relation algebra is isomorphic to a proper relation algebra. Lyndon answered this in the negative. And, in 2001, Hirsch and Hodkinson announced that the problem of deciding if a finite relation algebra is isomorphic to a proper relation algebra is undecidable.

We know that representability is easily determined for some reducts of the relation algebra signature, such as algebras with just an associative binary relation. Where in the relation algebra signature does the boundary between decidability and undecidability lie? This thesis narrows the gap, by proving undecidability of representability for lattice-ordered semigroups, as well as for complemented semigroups in which complements are to be represented with universal complementation. Furthermore, these results apply to any reduct with a signature between that of a lattice-ordered semigroup and a Boolean monoid, or between that of an ordered complemented semigroup and a Boolean monoid.

We then explored a special type of representation for finite ordered semigroups—that of a finite disjoint representation. These representations are a gap in the literature; the class of finite semigroups permitting a finite transitive disjoint representation has been classified, and every finite semigroup permits a representation which is finite and transitive, but not necessarily disjoint. We explored the $\mathcal{J}$-structure of semigroups which are finitely and disjointly representable, and in doing so provided necessary conditions for finite disjoint representability (Lemma 3.1.11 and its dual, Corollary 3.1.12). We conjectured that these conditions are also sufficient.

One can also study algebras of relations by investigating notions of representability weaker than that of relation algebra representability. Qualitatative calculi are an active area of study with many practical applications. We surveyed the qualitative representability and decidability of qualitative representability for all nonassociative algebras on at most three atoms. With the assistance of *Sage* and *Prover9Mace4*, we surveyed the representability of all 373 nonassociative algebras on four atoms, providing examples of qualitative representations where applicable.

Qualitative representability is very different to the traditional representability of relation algebras. Previous work in the area has determined that qualitative representability is decidable for finite nonassociative algebras, and that a finite qualitatively representable nonassociative algebra is representable over a finite number of vertices. Qualitative calculi can be used to guide robots, to identify the leader of a flock of birds, or even to find contradictions in maritime law. Continued work in the theoretical underpinnings of qualitative calculi will help to bring more results like these to fruition.

Appendix A

Qualitative representability of nonassociative algebras on up to four atoms

A.1 Atoms: two fragment identity and two symmetric

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$\begin{array}{c\|cccc} \#5 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & 0 \\ a & a & 0 & e_1 + a & 0 \\ b & b & 0 & 0 & e_1 \end{array}$	no	not simple: $\#2_{\leqslant 3} \times \#23_{\leqslant 3}$
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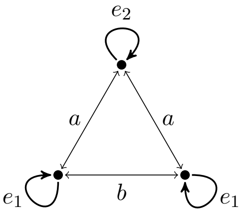
</td><td>no</td><td>
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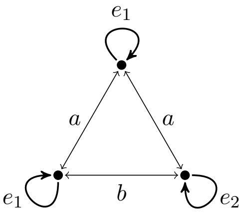
</td><td>no</td><td>
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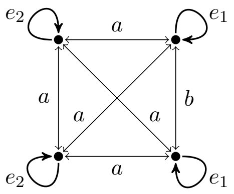
</td><td>no</td><td>
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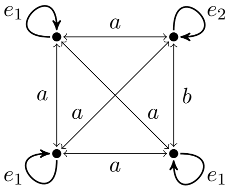
<tr><td><table><tr><th>#19</th><th>e_1</th><th>e_2</th><th>a</th><th>b</th></tr><tr><td>e_1</td><td>e_1</td><td>0</td><td>a</td><td>b</td></tr><tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>b</td></tr><tr><td>a</td><td>a</td><td>a</td><td>1</td><td>a</td></tr><tr><td>b</td><td>b</td><td>b</td><td>a</td><td>$1'$</td></tr></table></td><td>no</td><td> </td></tr>
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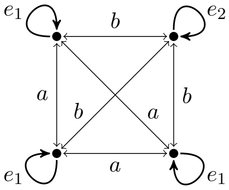
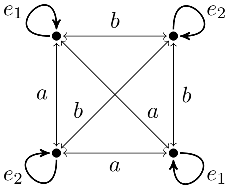
<tr><td><table><tr><th>#23</th><th>e_1</th><th>e_2</th><th>a</th><th>b</th></tr><tr><td>e_1</td><td>e_1</td><td>0</td><td>a</td><td>b</td></tr><tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>b</td></tr><tr><td>a</td><td>a</td><td>0</td><td>e_1+a</td><td>b</td></tr><tr><td>b</td><td>b</td><td>b</td><td>b</td><td>$-b$</td></tr></table></td><td>no</td><td> </td></tr>
<tr><td><table><tr><th>#24</th><th>e_1</th><th>e_2</th><th>a</th><th>b</th></tr><tr><td>e_1</td><td>e_1</td><td>0</td><td>a</td><td>b</td></tr><tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>b</td></tr><tr><td>a</td><td>a</td><td>a</td><td>$-b$</td><td>b</td></tr><tr><td>b</td><td>b</td><td>b</td><td>b</td><td>$-b$</td></tr></table></td><td>no</td><td> </td></tr>
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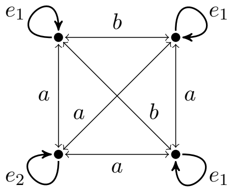
</table></td><td>no</td><td>
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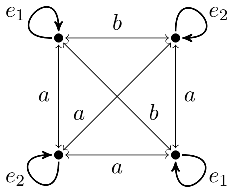
</table></td><td>no</td><td>
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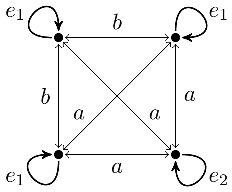
</table></td><td>no</td><td>
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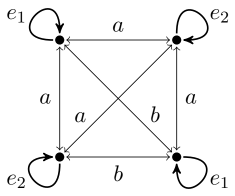
</td><td>no</td><td>
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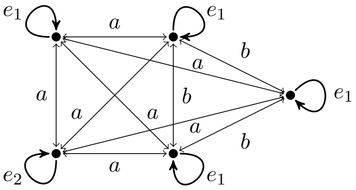
</td><td>no</td><td>
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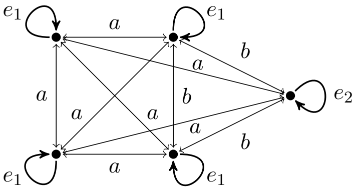
</td><td>no</td><td>
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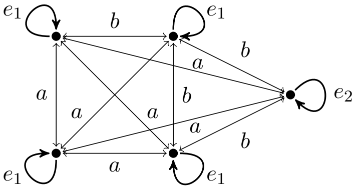
</td><td>no</td><td>
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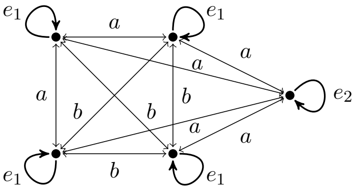
</td><td>no</td><td>
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A.2 Atoms: two fragment identity and one nonsymmetric

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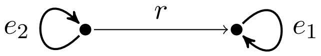
</td><td>yes</td><td>
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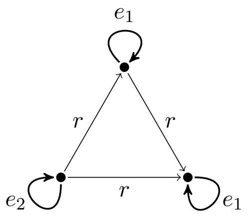
</td><td>no</td><td>
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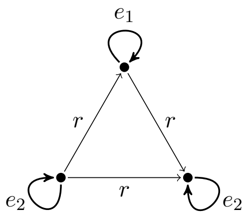
</td><td>no</td><td>
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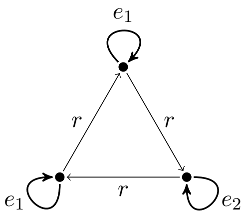
</td><td>no</td><td>
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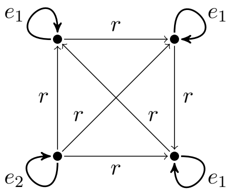
</td><td>no</td><td>
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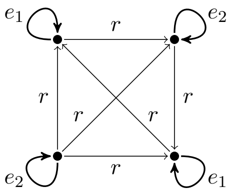
</td><td>no</td><td>
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A.3 Atoms: three fragment identity

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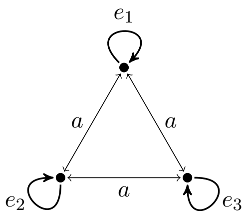
$\begin{array}{c\|cccc} \#64 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & a \\ e_3 & 0 & 0 & e_3 & 0 \\ a & a & a & 0 & e_1 + e_2 \end{array}$	no	not simple: $\#2_{\leqslant 3} \times \#19_{\leqslant 3}$
$\begin{array}{c\|cccc} \#65 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & a \\ e_3 & 0 & 0 & e_3 & a \\ a & a & a & a & 1' \end{array}$	no	no
$\begin{array}{c\|cccc} \#66 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & 0 \\ e_3 & 0 & 0 & e_3 & 0 \\ a & a & 0 & 0 & e_1 + a \end{array}$	yes	not simple: $\#2_{\leqslant 3} \times \#2_{\leqslant 3} \times \#5_{\leqslant 3}$
$\begin{array}{c\|cccc} \#67 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & a \\ e_3 & 0 & 0 & e_3 & 0 \\ a & a & a & 0 & -e_3 \end{array}$	no	not simple: $\#2_{\leqslant 3} \times \#20_{\leqslant 3}$
$\begin{array}{c\|cccc} \#68 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & a \\ e_3 & 0 & 0 & e_3 & a \\ a & a & a & a & 1 \end{array}$	no	

A.4 Atoms: four fragment identity

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A.5 Atoms: atomic identity and three symmetric

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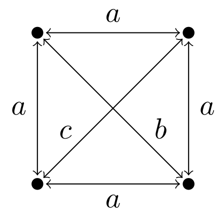
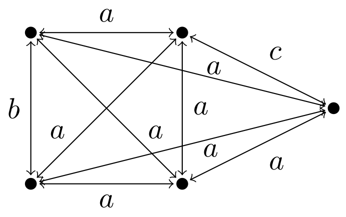
<tr><td>$\begin{array}{c|cccc} \#74 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & a & a \\ b & b & a & 1' & 0 \\ c & c & a & 0 & 1' \end{array}$</td><td>no</td><td> </td></tr>
<tr><td>$\begin{array}{c|cccc} \#75 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a \\ b & b & a & 1' & 0 \\ c & c & a & 0 & 1' \end{array}$</td><td>no</td><td> </td></tr>
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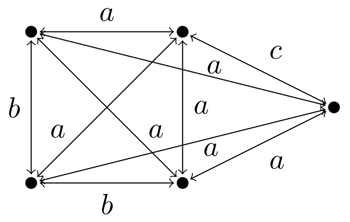
</td><td>no</td><td> </td></tr>
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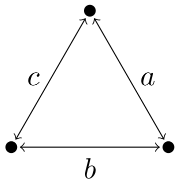
RRA</td><td> </td></tr>
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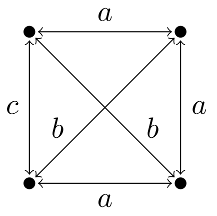
</td><td>no</td><td> </td></tr>
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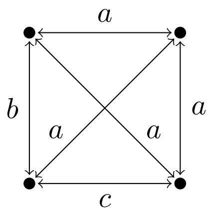
</td><td>no</td><td>
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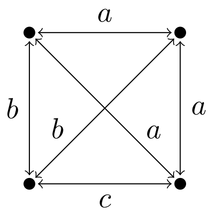
RRA</td><td>
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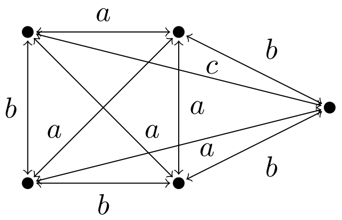
</td><td>no</td><td>
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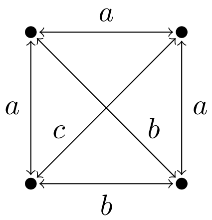
</td><td>no</td><td>
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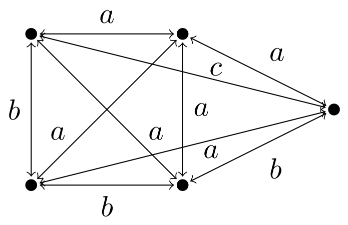
</td><td>no</td><td>
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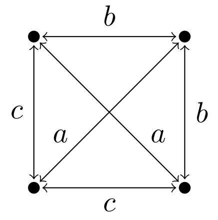
</td><td>no</td><td>
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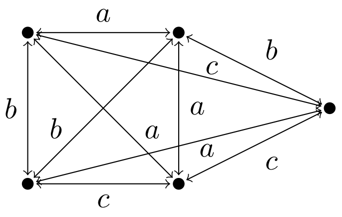
RRA</td><td>
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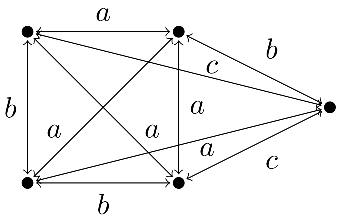
</td><td>no</td><td>
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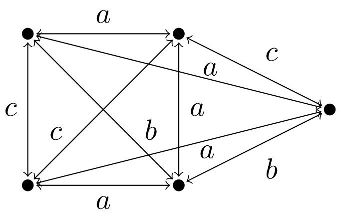
</td><td>no</td><td>
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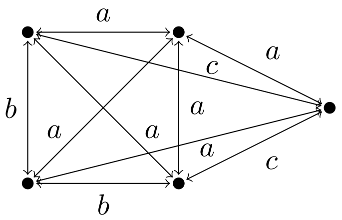
RRA</td><td>
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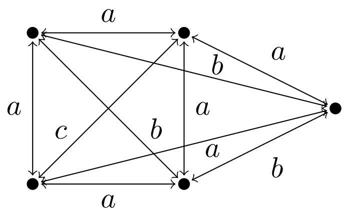
</td><td>no</td><td>
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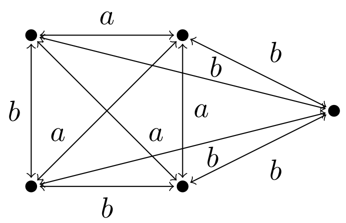
</td><td>no</td><td>
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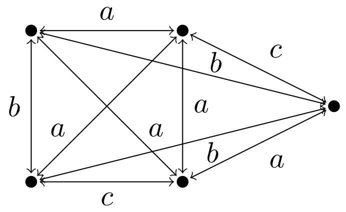
</td><td>no</td><td> </td></tr>
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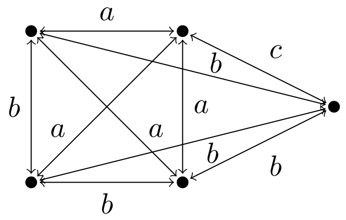
RRA</td><td> </td></tr>
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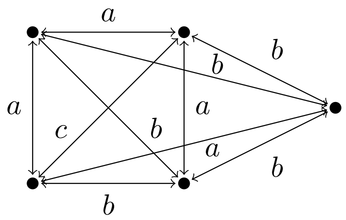
</td><td>no</td><td> </td></tr>
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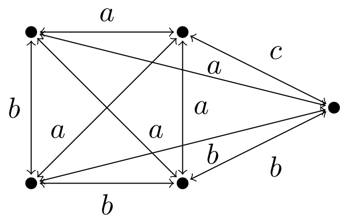
</td><td>no</td><td> </td></tr>
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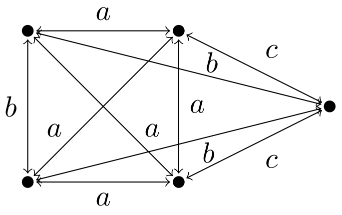
</td><td>no</td><td>
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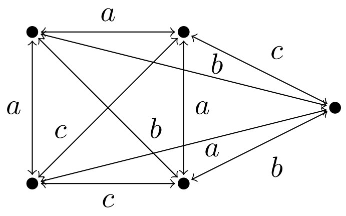
</td><td>no</td><td>
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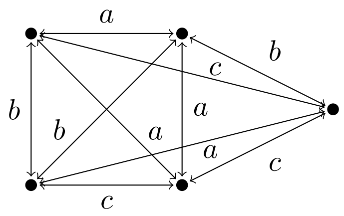
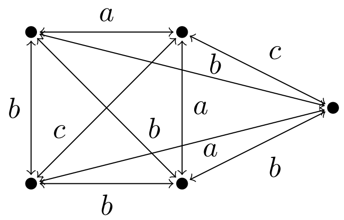
$\begin{array}{c\|cccc} \#153 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & b+c & b+c \\ b & b & b+c & -c & a \\ c & c & b+c & a & 1'+a \end{array}$	no	
$\begin{array}{c\|cccc} \#154 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+b & 0' & b+c \\ b & b & 0' & -c & a \\ c & c & b+c & a & 1'+a \end{array}$	no	
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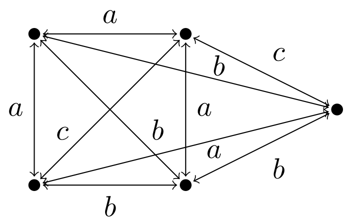
atom table	RA	QRNA
$\begin{array}{c\|cccc} \#158 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & 0' & 0' \\ b & b & 0' & -c & a \\ c & c & 0' & a & 1'+a \end{array}$	no	
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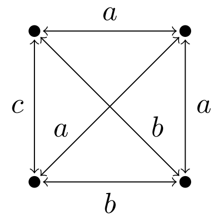
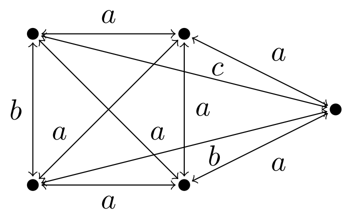
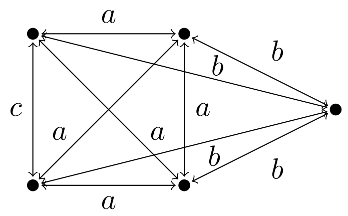
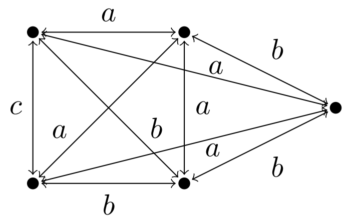
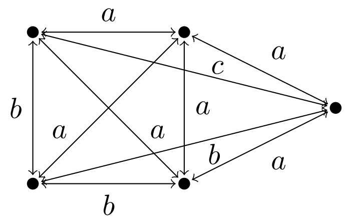
$\begin{array}{c\|cccc} \#160 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & a & a \\ b & b & a & 1'+c & b \\ c & c & a & b & 1' \end{array}$	yes 1_{65} RRA	
$\begin{array}{c\|cccc} \#161 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a \\ b & b & a & 1'+c & b \\ c & c & a & b & 1' \end{array}$	yes 5_{65} RRA	
$\begin{array}{c\|cccc} \#162 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & b & a \\ b & b & b & -b & b \\ c & c & a & b & 1' \end{array}$	yes 3_{65} RRA	
$\begin{array}{c\|cccc} \#163 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & a+b & a \\ b & b & a+b & -b & b \\ c & c & a & b & 1' \end{array}$	yes 15_{65} RRA	
$\begin{array}{c\|cccc} \#164 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+b & a \\ b & b & a+b & -b & b \\ c & c & a & b & 1' \end{array}$	yes 16_{65} RRA	

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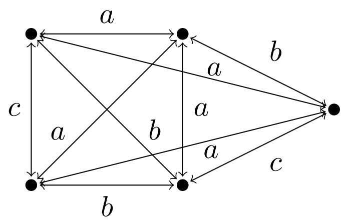

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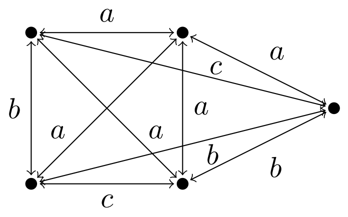

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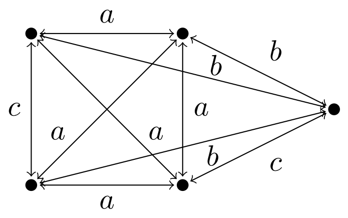

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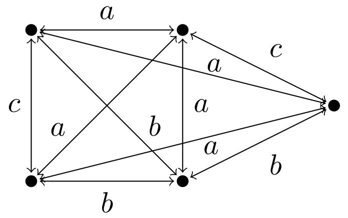

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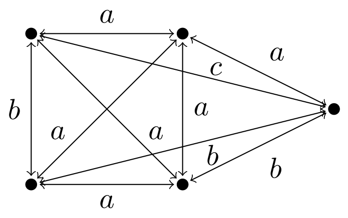

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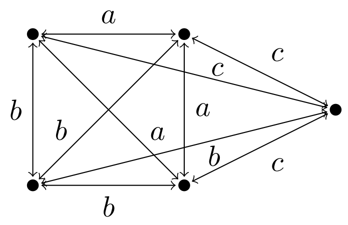
RRA</td><td>
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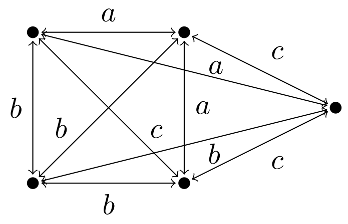
RRA</td><td>
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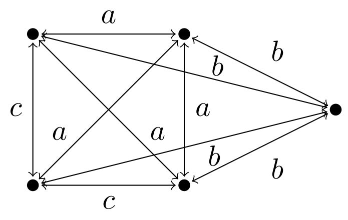
RRA</td><td>
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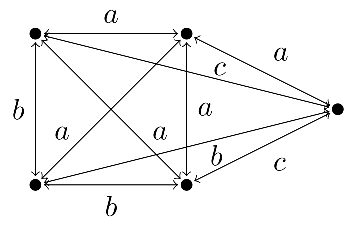

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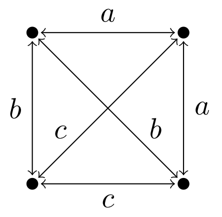

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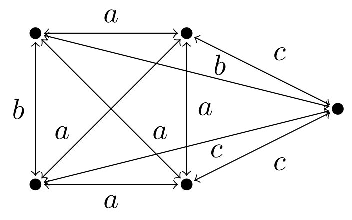

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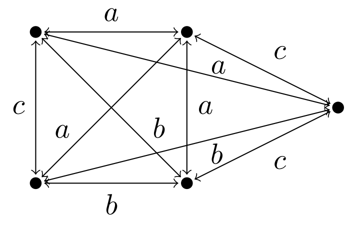

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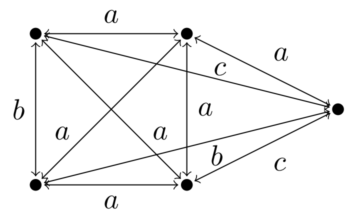

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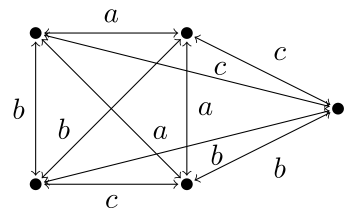

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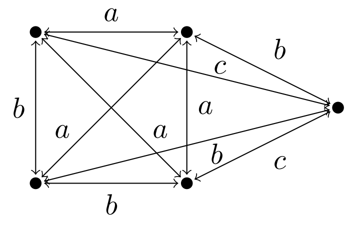

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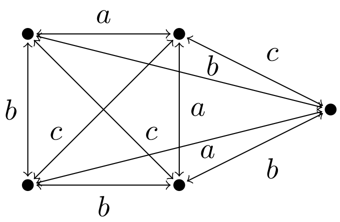

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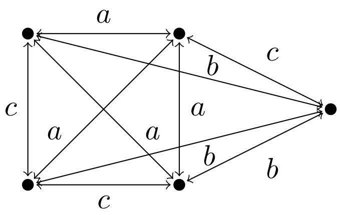

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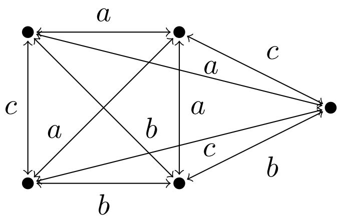

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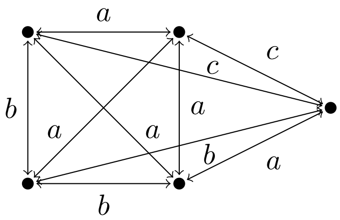

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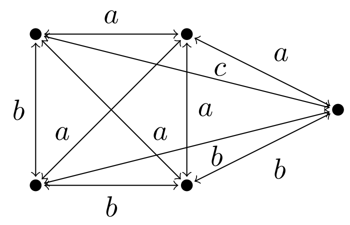
<tr><td>$\begin{array}{c|cccc} \#195 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & a+b \\ b & b & 0' & 1 & a+b \\ c & c & a+b & a+b & 1' \end{array}$</td><td>no</td><td> </td></tr>
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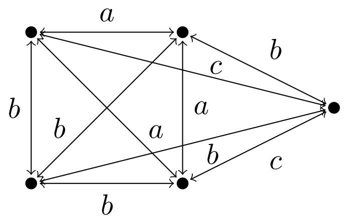
</td><td>no</td><td> </td></tr>
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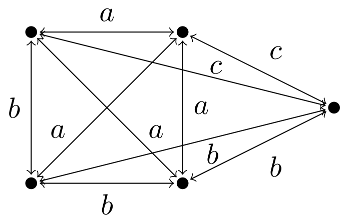
RRA</td><td> </td></tr>
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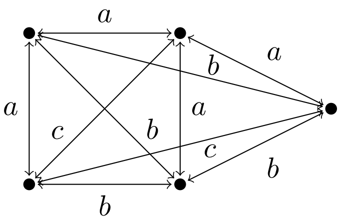
$\notin$ RRA</td><td> </td></tr>
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atom table	RA	QRNA
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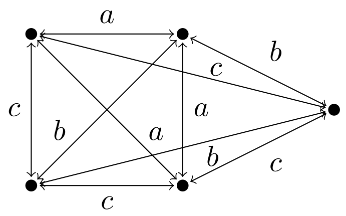
$\begin{array}{ccccc} \#227 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & c & b+c \\ b & b & c & 1'+b & a+c \\ c & c & b+c & a+c & -c \end{array}$	no	

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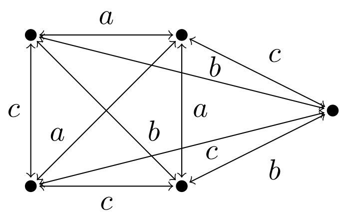
RRA</td><td> </td></tr>
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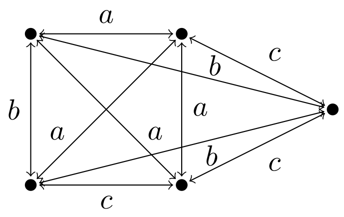
RRA</td><td> </td></tr>
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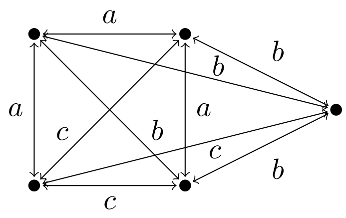
RRA</td><td> </td></tr>
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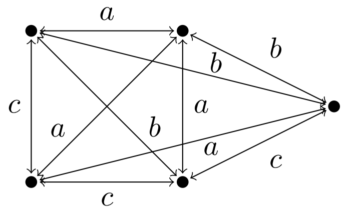
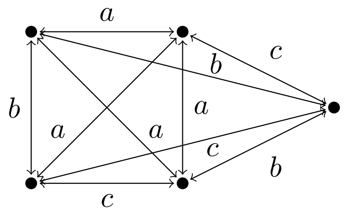
$\begin{array}{c\|cccc} \#238 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & 0' & 0' \\ b & b & 0' & -b & 0' \\ c & c & 0' & 0' & -c \end{array}$	yes 62_{65} RRA	
$\begin{array}{c\|cccc} \#239 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & 0' \\ b & b & 0' & -b & 0' \\ c & c & 0' & 0' & -c \end{array}$	yes 63_{65} RRA	
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$\begin{array}{c\|cccc} \#241 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a+c \\ b & b & a & -a & b+c \\ c & c & a+c & b+c & -c \end{array}$	no	$\begin{bmatrix} 1' & a & a & b & c & a & b \\ a & 1' & a & a & a & c & a \\ a & a & 1' & a & a & c & a \\ b & a & a & 1' & b & a & b \\ c & a & a & b & 1' & a & c \\ a & c & c & a & a & 1' & a \\ b & a & a & b & c & a & 1' \end{bmatrix}$

atom table	RA	QRNA
$\begin{array}{c c c c c} \#242 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+b & a+c \\ b & b & a+b & 1 & b+c \\ c & c & a+c & b+c & -c \end{array}$	yes 23_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & b & c & b \\ a & 1' & a & a & a & a \\ a & a & 1' & b & c & b \\ b & a & b & 1' & b & b \\ c & a & c & b & 1' & c \\ b & a & b & b & c & 1' \end{bmatrix}$
$\begin{array}{c c c c c} \#243 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & c & 0' \\ b & b & c & -a & 0' \\ c & c & 0' & 0' & -c \end{array}$	no	$\begin{bmatrix} 1' & a & a & c & a & c \\ a & 1' & a & a & c & c \\ a & a & 1' & c & c & b \\ c & a & c & 1' & b & b \\ a & c & c & b & 1' & b \\ c & c & b & b & b & 1' \end{bmatrix}$
$\begin{array}{c c c c c} \#244 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+c & 0' \\ b & b & a+c & -a & 0' \\ c & c & 0' & 0' & -c \end{array}$	yes 60_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & b & c & c \\ a & 1' & a & a & a & c \\ a & a & 1' & c & b & b \\ b & a & c & 1' & b & c \\ c & a & b & b & 1' & b \\ c & c & b & c & b & 1' \end{bmatrix}$
$\begin{array}{c c c c c} \#245 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & 0' \\ b & b & 0' & 1 & 0' \\ c & c & 0' & 0' & -c \end{array}$	yes 64_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & a \\ a & 1' & a & a & a & c \\ a & a & 1' & b & b & c \\ b & a & b & 1' & b & c \\ c & a & b & b & 1' & c \\ a & c & c & c & c & 1' \end{bmatrix}$
$\begin{array}{c c c c c} \#246 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & 0 & 0 \\ b & b & 0 & 1'+b & 0 \\ c & c & 0 & 0 & 1'+c \end{array}$	no	no
$\begin{array}{c c c c c} \#247 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -c & a & 0 \\ b & b & a & 1'+b & 0 \\ c & c & 0 & 0 & 1'+c \end{array}$	no	no
$\begin{array}{c c c c c} \#248 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a \\ b & b & a & 1'+b & 0 \\ c & c & a & 0 & 1'+c \end{array}$	no	$\begin{bmatrix} 1' & a & a & b & a & b & a \\ a & 1' & a & a & c & a & c \\ a & a & 1' & a & a & a & a \\ b & a & a & 1' & a & b & a \\ a & c & a & a & 1' & a & c \\ b & a & a & b & a & 1' & a \\ a & c & a & a & c & a & 1' \end{bmatrix}$

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29_{65}
RRA</td><td>$\begin{bmatrix} 1' & a & a & b & c & a \\ a & 1' & a & a & b & b \\ a & a & 1' & c & a & c \\ b & a & c & 1' & a & c \\ c & b & a & a & 1' & b \\ a & b & c & c & b & 1' \end{bmatrix}$</td></tr>
<tr><td>$\begin{array}{|c|cccc|} \hline \#255 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -c & 0' & b \\ b & b & 0' & -c & a \\ c & c & b & a & 1'+c \\ \hline \end{array}$</td><td>no</td><td>no</td></tr>
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<tr><td>b</td><td>b</td><td>$b+c$</td><td>$-c$</td><td>a</td></tr>
<tr><td>c</td><td>c</td><td>$a+b$</td><td>a</td><td>$1'+c$</td></tr>
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<tr><td>a</td><td>a</td><td>1</td><td>$0'$</td><td>$a+b$</td></tr>
<tr><td>b</td><td>b</td><td>$0'$</td><td>$-c$</td><td>a</td></tr>
<tr><td>c</td><td>c</td><td>$a+b$</td><td>a</td><td>$1'+c$</td></tr>
</table></td><td>yes
31_{65}
RRA</td><td>$\begin{bmatrix} 1' & a & a & b & a & b \\ a & 1' & a & a & c & c \\ a & a & 1' & b & b & a \\ b & a & b & 1' & a & b \\ a & c & b & a & 1' & c \\ b & c & a & b & c & 1' \end{bmatrix}$</td></tr>
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<tr><td>b</td><td>b</td><td>b</td><td>$-c$</td><td>0</td></tr>
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<tr><td>a</td><td>a</td><td>$-c$</td><td>$a+b$</td><td>c</td></tr>
<tr><td>b</td><td>b</td><td>$a+b$</td><td>$-c$</td><td>0</td></tr>
<tr><td>c</td><td>c</td><td>c</td><td>0</td><td>$-b$</td></tr>
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<tr><td>a</td><td>a</td><td>1</td><td>$a+b$</td><td>$a+c$</td></tr>
<tr><td>b</td><td>b</td><td>$a+b$</td><td>$-c$</td><td>0</td></tr>
<tr><td>c</td><td>c</td><td>$a+c$</td><td>0</td><td>$-b$</td></tr>
</table></td><td>no</td><td>$\begin{bmatrix} 1' & a & a & b & a & a & b & a \\ a & 1' & a & a & c & a & a & c \\ a & a & 1' & b & a & a & b & a \\ b & a & b & 1' & a & a & b & a \\ a & c & a & a & 1' & c & a & c \\ a & a & a & a & c & 1' & a & c \\ b & a & b & b & a & a & 1' & a \\ a & c & a & a & c & c & a & 1' \end{bmatrix}$</td></tr>
<tr><td><table>
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<tr><td>a</td><td>a</td><td>$1'+a$</td><td>$b+c$</td><td>$b+c$</td></tr>
<tr><td>b</td><td>b</td><td>$b+c$</td><td>$-c$</td><td>a</td></tr>
<tr><td>c</td><td>c</td><td>$b+c$</td><td>a</td><td>$-b$</td></tr>
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<tr><td><table>
<tr><th>#262</th><th>$1'$</th><th>a</th><th>b</th><th>c</th></tr>
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<tr><td>a</td><td>a</td><td>$-c$</td><td>$0'$</td><td>$b+c$</td></tr>
<tr><td>b</td><td>b</td><td>$0'$</td><td>$-c$</td><td>a</td></tr>
<tr><td>c</td><td>c</td><td>$b+c$</td><td>a</td><td>$-b$</td></tr>
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atom table	RA	QRNA
$\begin{array}{c\vert cccc} \hline \#263 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & 0' \\ b & b & 0' & -c & a \\ c & c & 0' & a & -b \\ \hline \end{array}$	yes 34_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & a \\ a & 1' & a & a & c & c \\ a & a & 1' & b & a & b \\ b & a & b & 1' & a & b \\ c & c & a & a & 1' & c \\ a & c & b & b & c & 1' \end{bmatrix}$
$\begin{array}{c\vert cccc} \hline \#264 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a \\ b & b & a & -a & b \\ c & c & a & b & 1'+c \\ \hline \end{array}$	yes 8_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & b & c \\ a & 1' & a & a & a & a & a \\ a & a & 1' & a & a & a & a \\ b & a & a & 1' & b & b & b \\ c & a & a & b & 1' & b & c \\ b & a & a & b & b & 1' & b \\ c & a & a & b & c & b & 1' \end{bmatrix}$
$\begin{array}{c\vert cccc} \hline \#265 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+b & a \\ b & b & a+b & 1 & b \\ c & c & a & b & 1'+c \\ \hline \end{array}$	yes 20_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & b & c \\ a & 1' & a & a & a & b & a \\ a & a & 1' & b & a & b & a \\ b & a & b & 1' & b & b & b \\ c & a & a & b & 1' & b & c \\ b & b & b & b & b & 1' & b \\ c & a & a & b & c & b & 1' \end{bmatrix}$
$\begin{array}{c\vert cccc} \hline \#266 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+c & a+b \\ b & b & a+c & -a & a+b \\ c & c & a+b & a+b & 1'+c \\ \hline \end{array}$	yes 36_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & b & b & b \\ a & 1' & a & a & c & a \\ a & a & 1' & c & a & c \\ b & a & c & 1' & b & c \\ b & c & a & b & 1' & b \\ b & a & c & c & b & 1' \end{bmatrix}$
$\begin{array}{c\vert cccc} \hline \#267 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & a+b \\ b & b & 0' & 1 & a+b \\ c & c & a+b & a+b & 1'+c \\ \hline \end{array}$	yes 53_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & c \\ a & 1' & a & a & a & b \\ a & a & 1' & b & b & b \\ b & a & b & 1' & b & b \\ c & a & b & b & 1' & c \\ c & b & b & b & c & 1' \end{bmatrix}$
$\begin{array}{c\vert cccc} \hline \#268 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -c & a & c \\ b & b & a & -a & b \\ c & c & c & b & -b \\ \hline \end{array}$	no	no
$\begin{array}{c\vert cccc} \hline \#269 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a & a+c \\ b & b & a & -a & b \\ c & c & a+c & b & -b \\ \hline \end{array}$	no	$\begin{bmatrix} 1' & a & a & b & c & a & b & c \\ a & 1' & a & a & a & c & a & a \\ a & a & 1' & a & a & c & a & a \\ b & a & a & 1' & b & a & b & b \\ c & a & a & b & 1' & a & b & c \\ a & c & c & a & a & 1' & a & a \\ b & a & a & b & b & a & 1' & b \\ c & a & a & b & c & a & b & 1' \end{bmatrix}$

atom table	RA	QRNA
$\begin{array}{ccccc} \#270 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & b & a+c \\ b & b & b & 1 & b \\ c & c & a+c & b & -b \end{array}$	yes 14_{65} RRA	$\begin{bmatrix} 1' & a & a & c & b & b & c \\ a & 1' & a & a & b & b & c \\ a & a & 1' & c & b & b & c \\ c & a & c & 1' & b & b & c \\ b & b & b & b & 1' & b & b \\ b & b & b & b & b & 1' & b \\ c & c & c & c & b & b & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#271 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+b & a+c \\ b & b & a+b & 1 & b \\ c & c & a+c & b & -b \end{array}$	no	$\begin{bmatrix} 1' & a & a & b & c & b & c \\ a & 1' & a & a & a & b & a \\ a & a & 1' & b & c & b & c \\ b & a & b & 1' & b & b & b \\ c & a & c & b & 1' & b & c \\ b & b & b & b & b & 1' & b \\ c & a & c & b & c & b & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#272 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -c & a+c & b+c \\ b & b & a+c & -a & a+b \\ c & c & b+c & a+b & -b \end{array}$	yes 42_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & b & b & b \\ a & 1' & a & a & c & c \\ a & a & 1' & a & c & c \\ b & a & a & 1' & b & b \\ b & c & c & b & 1' & c \\ b & c & c & b & c & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#273 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+c & 0' \\ b & b & a+c & -a & a+b \\ c & c & 0' & a+b & -b \end{array}$	yes 50_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & b & b & b \\ a & 1' & a & a & c & c \\ a & a & 1' & a & a & c \\ b & a & a & 1' & b & b \\ b & c & a & b & 1' & c \\ b & c & c & b & c & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#274 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & b+c & 0' \\ b & b & b+c & 1 & a+b \\ c & c & 0' & a+b & -b \end{array}$	yes 38_{65} $\notin$ RRA	$\begin{bmatrix} 1' & a & a & c & b & c \\ a & 1' & a & a & b & b \\ a & a & 1' & c & b & c \\ c & a & c & 1' & b & c \\ b & b & b & b & 1' & b \\ c & b & c & c & b & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#275 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & 0' \\ b & b & 0' & 1 & a+b \\ c & c & 0' & a+b & -b \end{array}$	yes 61_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & c \\ a & 1' & a & a & a & c \\ a & a & 1' & b & b & b \\ b & a & b & 1' & b & b \\ c & a & b & b & 1' & c \\ c & c & b & b & c & 1' \end{bmatrix}$
$\begin{array}{ccccc} \#276 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & a+b & a+c \\ b & b & a+b & 1 & b+c \\ c & c & a+c & b+c & 1 \end{array}$	yes 24_{65} RRA	$\begin{bmatrix} 1' & a & a & b & c & b & c \\ a & 1' & a & a & a & a & c \\ a & a & 1' & b & c & b & c \\ b & a & b & 1' & b & b & c \\ c & a & c & b & 1' & c & c \\ b & a & b & b & c & 1' & c \\ c & c & c & c & c & c & 1' \end{bmatrix}$

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<tr><td>a</td><td>a</td><td>1</td><td>$0'$</td><td>$0'$</td></tr>
<tr><td>b</td><td>b</td><td>$0'$</td><td>1</td><td>$0'$</td></tr>
<tr><td>c</td><td>c</td><td>$0'$</td><td>$0'$</td><td>1</td></tr>
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65_{65}
RRA</td><td>$$\begin{bmatrix} 1' & a & a & b & c & c \\ a & 1' & a & a & a & c \\ a & a & 1' & b & b & c \\ b & a & b & 1' & b & c \\ c & a & b & b & 1' & c \\ c & c & c & c & c & 1' \end{bmatrix}$$</td></tr>
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A.6 Atoms: atomic identity, one symmetric and one nonsymmetric

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<tr><td>$r^\smile$</td><td>$r^\smile$</td><td>0</td><td>$1'$</td><td>0</td></tr>
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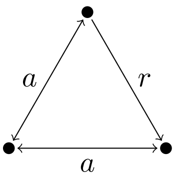
</td><td>no</td><td>
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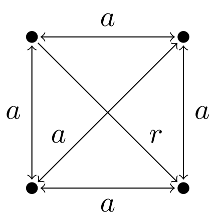
</td><td>no</td><td>
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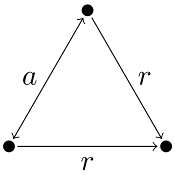
</td><td>no</td><td>
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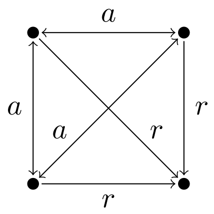
</td><td>no</td><td>
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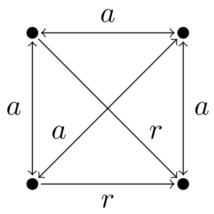
</td><td>no</td><td>
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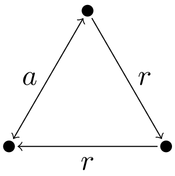
RRA</td><td>
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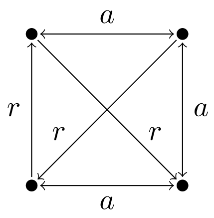
</td><td>no</td><td>
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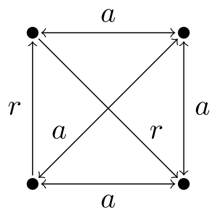
</td><td>no</td><td>
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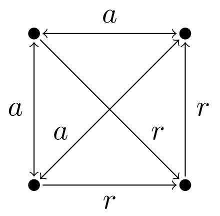
</td><td>no</td><td>
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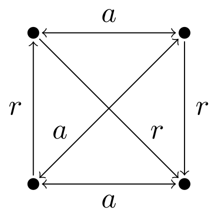
</td><td>no</td><td>
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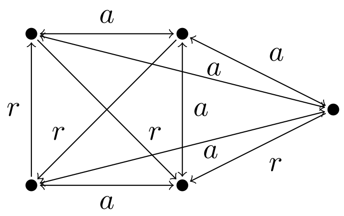
</td><td>no</td><td>
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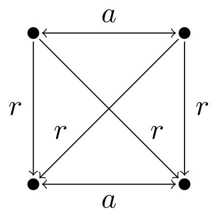
</td><td>no</td><td>
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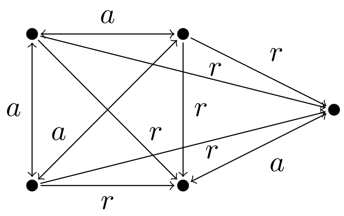
</td><td>no</td><td>
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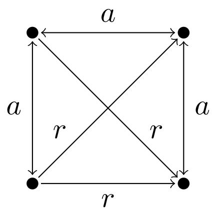
</td><td>no</td><td>
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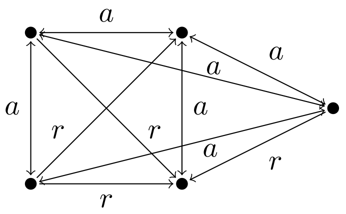
</td><td>no</td><td>
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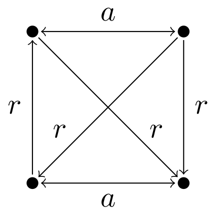
</td><td>no</td><td>
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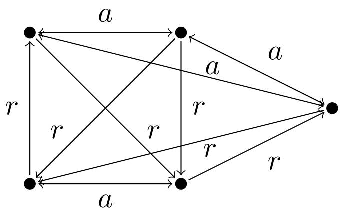
RRA</td><td>
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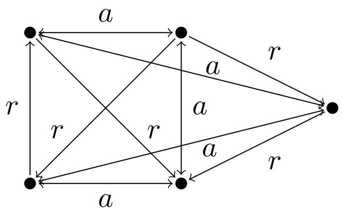
</td><td>no</td><td>
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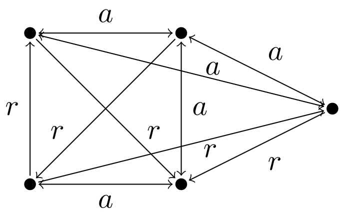
RRA</td><td>
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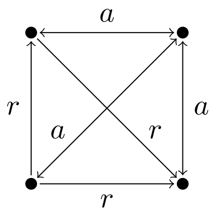
RRA</td><td> </td></tr>
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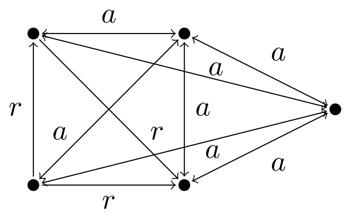
RRA</td><td> </td></tr>
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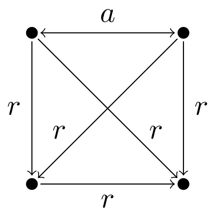
</td><td>no</td><td> </td></tr>
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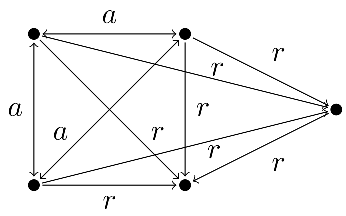
</td><td>no</td><td> </td></tr>
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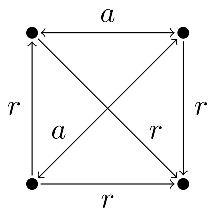
</td><td>no</td><td> </td></tr>
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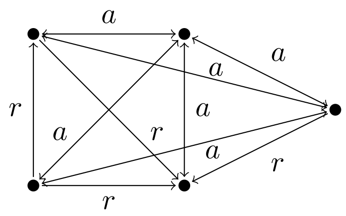
RRA</td><td> </td></tr>
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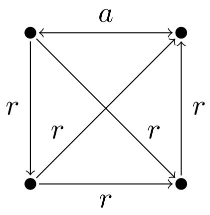
</td><td>no</td><td> </td></tr>
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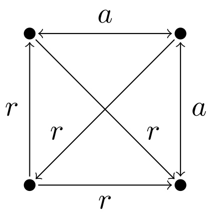
RRA</td><td> </td></tr>
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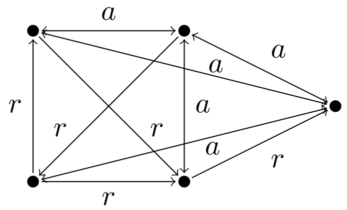
$\notin$ RRA</td><td> </td></tr>
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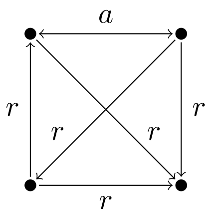
</td><td>no</td><td> </td></tr>
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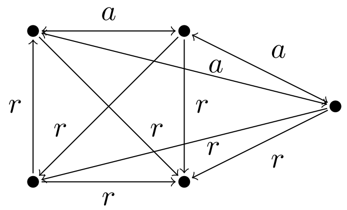
</td><td>no</td><td> </td></tr>
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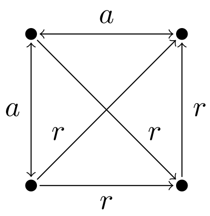
$\notin$ RRA</td><td> </td></tr>
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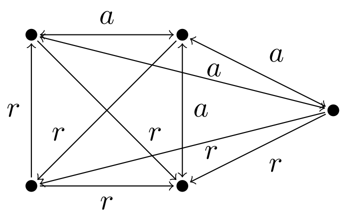
$\notin$ RRA</td><td> </td></tr>
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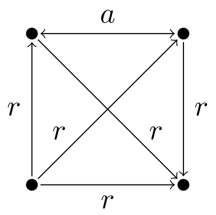
RRA</td><td> </td></tr>
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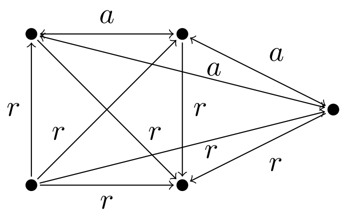
RRA</td><td> </td></tr>
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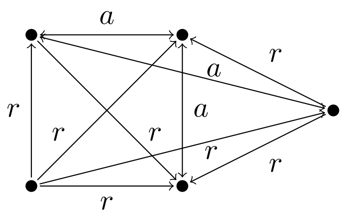
$\notin$ RRA</td><td> </td></tr>
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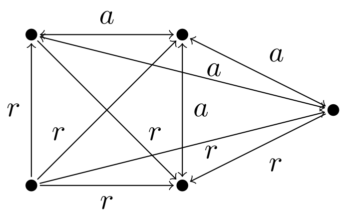
RRA</td><td> </td></tr>
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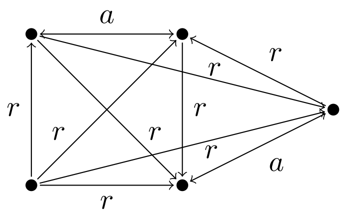
</td><td>no</td><td> </td></tr>
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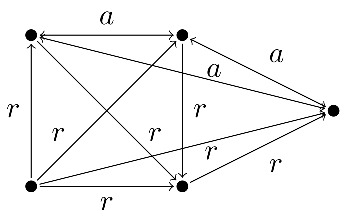
$\notin$ RRA</td><td> </td></tr>
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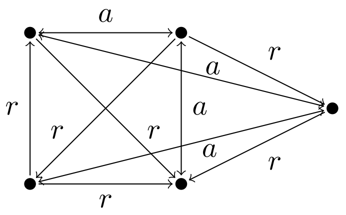
$\notin$ RRA</td><td>
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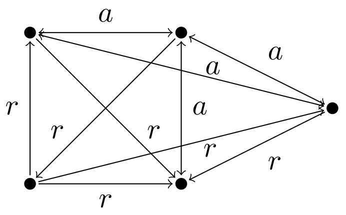
RRA</td><td>
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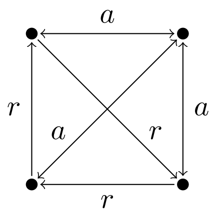
RRA</td><td>
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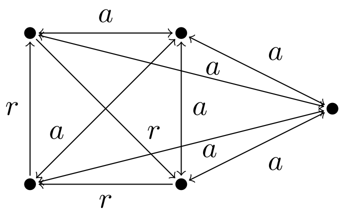
RRA</td><td>
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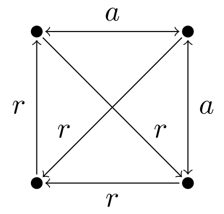
</td><td>no</td><td> </td></tr>
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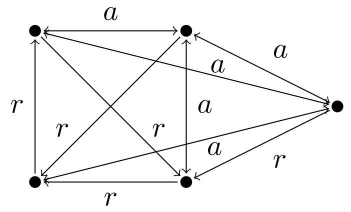
</td><td>no</td><td> </td></tr>
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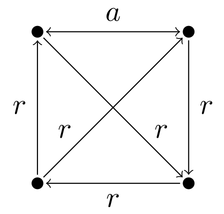
RRA</td><td> </td></tr>
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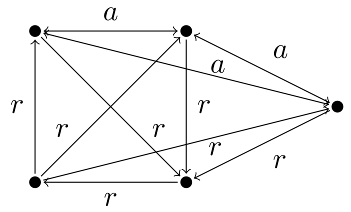
RRA</td><td> </td></tr>
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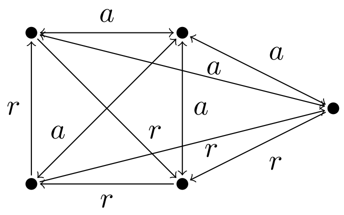
</table></td><td>no</td><td>
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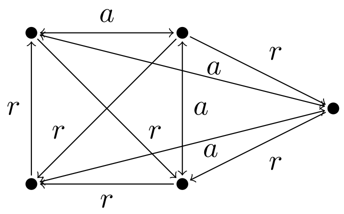
$\notin$ RRA</td><td>
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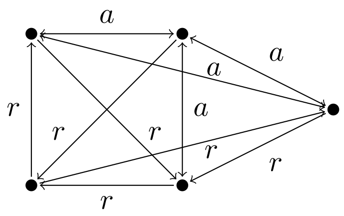
RRA</td><td>
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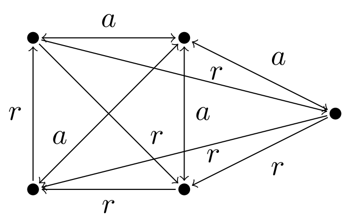

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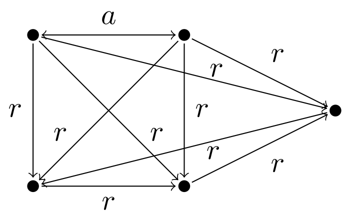

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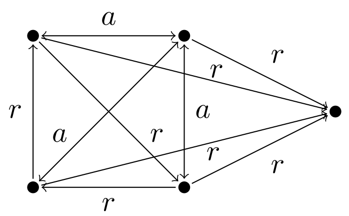

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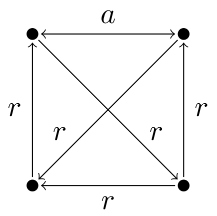

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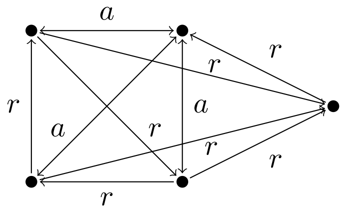
$\notin$ RRA</td><td>
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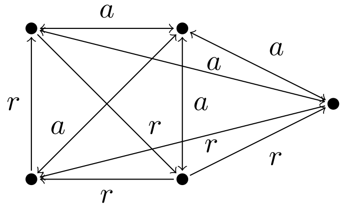
$\notin$ RRA</td><td>
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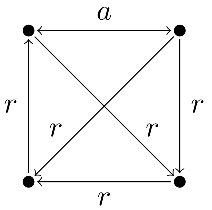
</td><td>no</td><td>
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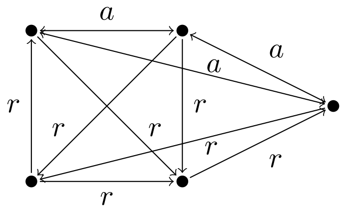
</td><td>no</td><td>
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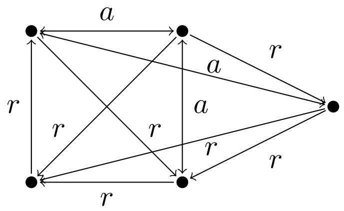
$\notin$ RRA</td><td>
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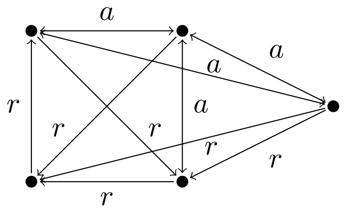
RRA</td><td>
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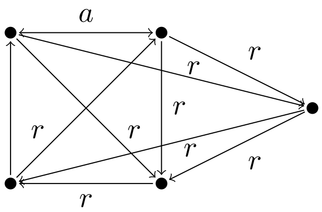
RRA</td><td> </td></tr>
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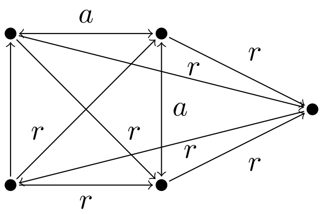
$\notin$ RRA</td><td> </td></tr>
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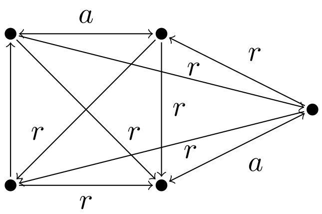
</td><td>no</td><td> </td></tr>
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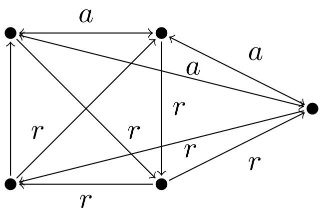
RRA</td><td> </td></tr>
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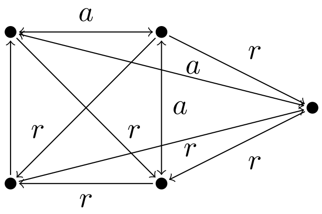
RRA</td><td> </td></tr>
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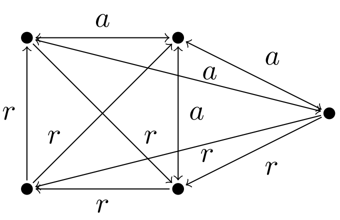
<td>
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Appendix B

Cycles of nonassociative algebras on four atoms

B.1 Atoms: two fragment identity and two symmetric

The identity is $e_1 + e_2$.

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atom table	e_1aa	e_1bb	e_2aa	e_2bb	aaa	baa	abb	bbb	RA
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$\begin{array}{c\|cccc} \#5 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & 0 \\ a & a & 0 & e_1+a & 0 \\ b & b & 0 & 0 & e_1 \end{array}$	e_1aa	e_1bb	$\ldots$	$\ldots$	aaa	$\ldots$	$\ldots$	$\ldots$	no
$\begin{array}{c\|cccc} \#6 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & 0 & b \\ e_2 & 0 & e_2 & a & 0 \\ a & 0 & a & e_2+a & 0 \\ b & b & 0 & 0 & e_1 \end{array}$	$\ldots$	e_1bb	e_2aa	$\ldots$	aaa	$\ldots$	$\ldots$	$\ldots$	yes
$\begin{array}{c\|cccc} \#7 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & 0 \\ a & a & a & -b & 0 \\ b & b & 0 & 0 & e_1 \end{array}$	e_1aa	e_1bb	e_2aa	$\ldots$	aaa	$\ldots$	$\ldots$	$\ldots$	no
$\begin{array}{c\|cccc} \#8 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & b \\ a & a & 0 & e_1+a & 0 \\ b & b & b & 0 & 1' \end{array}$	e_1aa	e_1bb	$\ldots$	e_2bb	aaa	$\ldots$	$\ldots$	$\ldots$	no
$\begin{array}{c\|cccc} \#9 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & b \\ a & a & a & -b & 0 \\ b & b & b & 0 & 1' \end{array}$	e_1aa	e_1bb	e_2aa	e_2bb	aaa	$\ldots$	$\ldots$	$\ldots$	no
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</table></td><td>e_1aa</td><td>e_1bb</td><td>e_2aa</td><td>e_2bb</td><td>aaa</td><td>baa</td><td>abb</td><td>…</td><td>no</td></tr>
<tr><td><table>
<tr><th>#34</th><th>e_1</th><th>e_2</th><th>a</th><th>b</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>a</td><td>b</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>0</td></tr>
<tr><td>a</td><td>a</td><td>0</td><td>e_1+a</td><td>0</td></tr>
<tr><td>b</td><td>b</td><td>0</td><td>0</td><td>e_1+b</td></tr>
</table></td><td>e_1aa</td><td>e_1bb</td><td>…</td><td>…</td><td>aaa</td><td>…</td><td>…</td><td>bbb</td><td>no</td></tr>
<tr><td><table>
<tr><th>#35</th><th>e_1</th><th>e_2</th><th>a</th><th>b</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>0</td><td>b</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>0</td></tr>
<tr><td>a</td><td>0</td><td>a</td><td>e_2+a</td><td>0</td></tr>
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atom table	e_1aa	e_1bb	e_2aa	e_2bb	aaa	baa	abb	bbb	RA
$\begin{array}{c\|cccc} \#36 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & 0 \\ a & a & a & -b & 0 \\ b & b & 0 & 0 & e_1+b \end{array}$	e_1aa	e_1bb	e_2aa	…	aaa	…	…	bbb	no
$\begin{array}{c\|cccc} \#37 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & b \\ a & a & a & -b & 0 \\ b & b & b & 0 & -a \end{array}$	e_1aa	e_1bb	e_2aa	e_2bb	aaa	…	…	bbb	no
$\begin{array}{c\|cccc} \#38 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & 0 \\ a & a & 0 & -e_2 & a \\ b & b & 0 & a & e_1+b \end{array}$	e_1aa	e_1bb	…	…	aaa	baa	…	bbb	yes
$\begin{array}{c\|cccc} \#39 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & 0 & b \\ e_2 & 0 & e_2 & a & 0 \\ a & 0 & a & -e_1 & a \\ b & b & 0 & a & e_1+b \end{array}$	…	e_1bb	e_2aa	…	aaa	baa	…	bbb	no
$\begin{array}{c\|cccc} \#40 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & 0 \\ a & a & a & 1 & a \\ b & b & 0 & a & e_1+b \end{array}$	e_1aa	e_1bb	e_2aa	…	aaa	baa	…	bbb	no
$\begin{array}{c\|cccc} \#41 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & b \\ a & a & 0 & -e_2 & a \\ b & b & b & a & -a \end{array}$	e_1aa	e_1bb	…	e_2bb	aaa	baa	…	bbb	no
$\begin{array}{c\|cccc} \#42 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & a & b \\ a & a & a & 1 & a \\ b & b & b & a & -a \end{array}$	e_1aa	e_1bb	e_2aa	e_2bb	aaa	baa	…	bbb	no
$\begin{array}{c\|cccc} \#43 & e_1 & e_2 & a & b \\ \hline e_1 & e_1 & 0 & a & b \\ e_2 & 0 & e_2 & 0 & 0 \\ a & a & 0 & -e_2 & a+b \\ b & b & 0 & a+b & -e_2 \end{array}$	e_1aa	e_1bb	…	…	aaa	baa	abb	bbb	yes

<table>
<tr><th>atom table</th><th>e_1aa</th><th>e_1bb</th><th>e_2aa</th><th>e_2bb</th><th>aaa</th><th>baa</th><th>abb</th><th>bbb</th><th>RA</th></tr>
<tr><td><table>
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<tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>0</td></tr>
<tr><td>a</td><td>0</td><td>a</td><td>$-e_1$</td><td>$a+b$</td></tr>
<tr><td>b</td><td>b</td><td>0</td><td>$a+b$</td><td>$-e_2$</td></tr>
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<tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>0</td></tr>
<tr><td>a</td><td>a</td><td>a</td><td>1</td><td>$a+b$</td></tr>
<tr><td>b</td><td>b</td><td>0</td><td>$a+b$</td><td>$-e_2$</td></tr>
</table></td><td>e_1aa</td><td>e_1bb</td><td>e_2aa</td><td>$\ldots$</td><td>aaa</td><td>baa</td><td>abb</td><td>bbb</td><td>no</td></tr>
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<tr><td>e_2</td><td>0</td><td>e_2</td><td>a</td><td>b</td></tr>
<tr><td>a</td><td>a</td><td>a</td><td>1</td><td>$a+b$</td></tr>
<tr><td>b</td><td>b</td><td>b</td><td>$a+b$</td><td>1</td></tr>
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</table>

B.2 Atoms: two fragment identity and one nonsymmetric

The identity is $e_1 + e_2$.

<table>
<tr><th>atom table</th><th>e_1rr</th><th>$e_1r^{\smile}r^{\smile}$</th><th>e_2rr</th><th>$e_2r^{\smile}r^{\smile}$</th><th>rrr</th><th>$rrr^{\smile}$</th><th>RA</th></tr>
<tr><td><table>
<tr><th>#47</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>0</td><td>e_1</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>0</td><td>e_1</td><td>0</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>$\ldots$</td><td>$\ldots$</td><td>$\ldots$</td><td>$\ldots$</td><td>no</td></tr>
<tr><td><table>
<tr><th>#48</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>0</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>0</td><td>e_2</td></tr>
<tr><td>$r^{\smile}$</td><td>0</td><td>$r^{\smile}$</td><td>e_1</td><td>0</td></tr>
</table></td><td>$\ldots$</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>$\ldots$</td><td>$\ldots$</td><td>$\ldots$</td><td>yes</td></tr>
<tr><td><table>
<tr><th>#49</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>0</td><td>$1'$</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>e_1</td><td>0</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>$\ldots$</td><td>$\ldots$</td><td>$\ldots$</td><td>no</td></tr>
</table>

<table>
<tr><th>atom table</th><th>e_1rr</th><th>$e_1r^{\smile}r^{\smile}$</th><th>e_2rr</th><th>$e_2r^{\smile}r^{\smile}$</th><th>rrr</th><th>$rrr^{\smile}$</th><th>RA</th></tr>
<tr><td><table>
<tr><th>#50</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>r</td><td>r</td><td>r</td><td>0</td><td>$1'$</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$1'$</td><td>0</td></tr>
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<tr><td><table>
<tr><th>#51</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
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<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>r</td><td>$-e_2$</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>0</td><td>$-e_2$</td><td>$r^{\smile}$</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>...</td><td>...</td><td>rrr</td><td>...</td><td>yes</td></tr>
<tr><td><table>
<tr><th>#52</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>0</td><td>$r^{\smile}$</td></tr>
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<tr><td>r</td><td>r</td><td>0</td><td>r</td><td>$-e_1$</td></tr>
<tr><td>$r^{\smile}$</td><td>0</td><td>$r^{\smile}$</td><td>$-e_2$</td><td>$r^{\smile}$</td></tr>
</table></td><td>...</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>...</td><td>rrr</td><td>...</td><td>no</td></tr>
<tr><td><table>
<tr><th>#53</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>r</td><td>1</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$-e_2$</td><td>$r^{\smile}$</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>...</td><td>rrr</td><td>...</td><td>no</td></tr>
<tr><td><table>
<tr><th>#54</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>r</td><td>r</td><td>r</td><td>r</td><td>1</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>1</td><td>$r^{\smile}$</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>$e_2r^{\smile}r^{\smile}$</td><td>rrr</td><td>...</td><td>no</td></tr>
<tr><td><table>
<tr><th>#55</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>$r^{\smile}$</td><td>e_1</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>0</td><td>e_1</td><td>r</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>...</td><td>...</td><td>...</td><td>$rrr^{\smile}$</td><td>yes</td></tr>
<tr><td><table>
<tr><th>#56</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
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<tr><td>r</td><td>r</td><td>0</td><td>$r^{\smile}$</td><td>e_2</td></tr>
<tr><td>$r^{\smile}$</td><td>0</td><td>$r^{\smile}$</td><td>e_1</td><td>r</td></tr>
</table></td><td>...</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>...</td><td>...</td><td>$rrr^{\smile}$</td><td>no</td></tr>
<tr><td><table>
<tr><th>#57</th><th>e_1</th><th>e_2</th><th>r</th><th>$r^{\smile}$</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>r</td><td>$r^{\smile}$</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>r</td><td>0</td></tr>
<tr><td>r</td><td>r</td><td>0</td><td>$r^{\smile}$</td><td>$1'$</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>e_1</td><td>r</td></tr>
</table></td><td>e_1rr</td><td>$e_1r^{\smile}r^{\smile}$</td><td>e_2rr</td><td>...</td><td>...</td><td>$rrr^{\smile}$</td><td>no</td></tr>
</table>

atom table	e_1rr	$e_1r^{\smile}r^{\smile}$	e_2rr	$e_2r^{\smile}r^{\smile}$	rrr	$rrr^{\smile}$	RA
$\begin{array}{c c c c c} \#58 & e_1 & e_2 & r & r^{\smile} \\ \hline e_1 & e_1 & 0 & r & r^{\smile} \\ e_2 & 0 & e_2 & r & r^{\smile} \\ r & r & r & r^{\smile} & 1' \\ r^{\smile} & r^{\smile} & r^{\smile} & 1' & r \end{array}$	e_1rr	$e_1r^{\smile}r^{\smile}$	e_2rr	$e_2r^{\smile}r^{\smile}$	$\dots$	$rrr^{\smile}$	no
$\begin{array}{c c c c c} \#59 & e_1 & e_2 & r & r^{\smile} \\ \hline e_1 & e_1 & 0 & r & r^{\smile} \\ e_2 & 0 & e_2 & 0 & 0 \\ r & r & 0 & r+r^{\smile} & -e_2 \\ r^{\smile} & r^{\smile} & 0 & -e_2 & r+r^{\smile} \end{array}$	e_1rr	$e_1r^{\smile}r^{\smile}$	$\dots$	$\dots$	rrr	$rrr^{\smile}$	yes
$\begin{array}{c c c c c} \#60 & e_1 & e_2 & r & r^{\smile} \\ \hline e_1 & e_1 & 0 & 0 & r^{\smile} \\ e_2 & 0 & e_2 & r & 0 \\ r & r & 0 & r+r^{\smile} & -e_1 \\ r^{\smile} & 0 & r^{\smile} & -e_2 & r+r^{\smile} \end{array}$	$\dots$	$e_1r^{\smile}r^{\smile}$	e_2rr	$\dots$	rrr	$rrr^{\smile}$	no
$\begin{array}{c c c c c} \#61 & e_1 & e_2 & r & r^{\smile} \\ \hline e_1 & e_1 & 0 & r & r^{\smile} \\ e_2 & 0 & e_2 & r & 0 \\ r & r & 0 & r+r^{\smile} & 1 \\ r^{\smile} & r^{\smile} & r^{\smile} & -e_2 & r+r^{\smile} \end{array}$	e_1rr	$e_1r^{\smile}r^{\smile}$	e_2rr	$\dots$	rrr	$rrr^{\smile}$	no
$\begin{array}{c c c c c} \#62 & e_1 & e_2 & r & r^{\smile} \\ \hline e_1 & e_1 & 0 & r & r^{\smile} \\ e_2 & 0 & e_2 & r & r^{\smile} \\ r & r & r & r+r^{\smile} & 1 \\ r^{\smile} & r^{\smile} & r^{\smile} & 1 & r+r^{\smile} \end{array}$	e_1rr	$e_1r^{\smile}r^{\smile}$	e_2rr	$e_2r^{\smile}r^{\smile}$	rrr	$rrr^{\smile}$	no

B.3 Atoms: three fragment identity

The identity is $e_1 + e_2 + e_3$.

atom table	e_1aa	e_2aa	e_3aa	aaa	RA
$\begin{array}{c c c c c} \#63 & e_1 & e_2 & e_3 & a \\ \hline e_1 & e_1 & 0 & 0 & a \\ e_2 & 0 & e_2 & 0 & 0 \\ e_3 & 0 & 0 & e_3 & 0 \\ a & a & 0 & 0 & e_1 \end{array}$	e_1aa	$\dots$	$\dots$	$\dots$	yes

<table>
<tr><th>atom table</th><th>e_1aa</th><th>e_2aa</th><th>e_3aa</th><th>aaa</th><th>RA</th></tr>
<tr><td><table>
<tr><th>#64</th><th>e_1</th><th>e_2</th><th>e_3</th><th>a</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>0</td><td>a</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>a</td></tr>
<tr><td>e_3</td><td>0</td><td>0</td><td>e_3</td><td>0</td></tr>
<tr><td>a</td><td>a</td><td>a</td><td>0</td><td>$e_1 + e_2$</td></tr>
</table></td><td>e_1aa</td><td>e_2aa</td><td>...</td><td>...</td><td>no</td></tr>
<tr><td><table>
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<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>a</td></tr>
<tr><td>e_3</td><td>0</td><td>0</td><td>e_3</td><td>a</td></tr>
<tr><td>a</td><td>a</td><td>a</td><td>a</td><td>$1'$</td></tr>
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<tr><td><table>
<tr><th>#66</th><th>e_1</th><th>e_2</th><th>e_3</th><th>a</th></tr>
<tr><td>e_1</td><td>e_1</td><td>0</td><td>0</td><td>a</td></tr>
<tr><td>e_2</td><td>0</td><td>e_2</td><td>0</td><td>0</td></tr>
<tr><td>e_3</td><td>0</td><td>0</td><td>e_3</td><td>0</td></tr>
<tr><td>a</td><td>a</td><td>0</td><td>0</td><td>$e_1 + a$</td></tr>
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</table>

B.4 Atoms: four fragment identity

The identity is $e_1 + e_2 + e_3 + e_4$. Algebra #69 is the only such nonassociative algebra with four atoms. It is also a relation algebra.

#69	e_1	e_2	e_3	e_4
e_1	e_1	0	0	0
e_2	0	e_2	0	0
e_3	0	0	e_3	0
e_4	0	0	0	e_4

B.5 Atoms: atomic identity and three symmetric

The identity is atomic. A cycle involving identity is consistent if and only if it is $1'1'1'$, $1'aa$, $1'bb$ or $1'cc$.

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atom table	*aaa*	*bbb*	*ccc*	*abb*	*baa*	*acc*	*caa*	*bcc*	*cbb*	*abc*	RA
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$\begin{array}{c\|cccc} \#102 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1' & b+c & b+c \\ b & b & b+c & 1'+a & a \\ c & c & b+c & a & 1'+a \end{array}$	…	…	…	*abb*	…	*acc*	…	…	…	*abc*	no
$\begin{array}{c\|cccc} \#103 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & b+c & b+c \\ b & b & b+c & 1'+a & a \\ c & c & b+c & a & 1'+a \end{array}$	*aaa*	…	…	*abb*	…	*acc*	…	…	…	*abc*	yes 28_{65}
$\begin{array}{c\|cccc} \#104 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+b & 0' & b+c \\ b & b & 0' & 1'+a & a \\ c & c & b+c & a & 1'+a \end{array}$	…	…	…	*abb*	*baa*	*acc*	…	…	…	*abc*	no
$\begin{array}{c\|cccc} \#105 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -c & 0' & b+c \\ b & b & 0' & 1'+a & a \\ c & c & b+c & a & 1'+a \end{array}$	*aaa*	…	…	*abb*	*baa*	*acc*	…	…	…	*abc*	no
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$\begin{array}{c\|cccc} \#107 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & 0' \\ b & b & 0' & 1'+a & a \\ c & c & 0' & a & 1'+a \end{array}$	*aaa*	…	…	*abb*	*baa*	*acc*	*caa*	…	…	*abc*	yes 32_{65}
$\begin{array}{c\|cccc} \#108 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & 0 & 0 \\ b & b & 0 & 1'+b & 0 \\ c & c & 0 & 0 & 1' \end{array}$	*aaa*	*bbb*	…	…	…	…	…	…	…	…	no

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<tr><th>atom table</th><th>aaa</th><th>bbb</th><th>ccc</th><th>abb</th><th>baa</th><th>acc</th><th>caa</th><th>bcc</th><th>cbb</th><th>abc</th><th>RA</th></tr>
<tr><td><table><tr><th>#109</th><th>$1'$</th><th>a</th><th>b</th><th>c</th></tr><tr><td>$1'$</td><td>$1'$</td><td>a</td><td>b</td><td>c</td></tr><tr><td>a</td><td>a</td><td>$-c$</td><td>a</td><td>0</td></tr><tr><td>b</td><td>b</td><td>a</td><td>$1'+b$</td><td>0</td></tr><tr><td>c</td><td>c</td><td>0</td><td>0</td><td>$1'$</td></tr></table></td><td>aaa</td><td>bbb</td><td>…</td><td>…</td><td>baa</td><td>…</td><td>…</td><td>…</td><td>…</td><td>…</td><td>no</td></tr>
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<tr><td><table><tr><th>#112</th><th>$1'$</th><th>a</th><th>b</th><th>c</th></tr><tr><td>$1'$</td><td>$1'$</td><td>a</td><td>b</td><td>c</td></tr><tr><td>a</td><td>a</td><td>$-a$</td><td>a</td><td>a</td></tr><tr><td>b</td><td>b</td><td>a</td><td>$1'+b$</td><td>0</td></tr><tr><td>c</td><td>c</td><td>a</td><td>0</td><td>$1'$</td></tr></table></td><td>…</td><td>bbb</td><td>…</td><td>…</td><td>baa</td><td>…</td><td>caa</td><td>…</td><td>…</td><td>…</td><td>no</td></tr>
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atom table	*aaa*	*bbb*	*ccc*	*abb*	*baa*	*acc*	*caa*	*bcc*	*cbb*	*abc*	RA
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$\begin{array}{ccccc} \#126 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+c & b+c & a+b \\ b & b & b+c & -c & a \\ c & c & a+b & a & 1' \end{array}$	…	*bbb*	…	*abb*	…	…	*caa*	…	…	*abc*	no
$\begin{array}{ccccc} \#127 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & b+c & a+b \\ b & b & b+c & -c & a \\ c & c & a+b & a & 1' \end{array}$	*aaa*	*bbb*	…	*abb*	…	…	*caa*	…	…	*abc*	no
$\begin{array}{ccccc} \#128 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & 0' & a+b \\ b & b & 0' & -c & a \\ c & c & a+b & a & 1' \end{array}$	…	*bbb*	…	*abb*	*baa*	…	*caa*	…	…	*abc*	no
$\begin{array}{ccccc} \#129 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1 & 0' & a+b \\ b & b & 0' & -c & a \\ c & c & a+b & a & 1' \end{array}$	*aaa*	*bbb*	…	*abb*	*baa*	…	*caa*	…	…	*abc*	no
$\begin{array}{ccccc} \#130 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & 0 & c \\ b & b & 0 & 1'+b & 0 \\ c & c & c & 0 & 1'+a \end{array}$	*aaa*	*bbb*	…	…	…	*acc*	…	…	…	…	no
$\begin{array}{ccccc} \#131 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+b & a & c \\ b & b & a & 1'+b & 0 \\ c & c & c & 0 & 1'+a \end{array}$	…	*bbb*	…	…	*baa*	*acc*	…	…	…	…	no
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atom table	aaa	bbb	ccc	abb	baa	acc	caa	bcc	cbb	abc	RA
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$\begin{array}{ccccc} \#134 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -b & 0 & a+c \\ b & b & 0 & 1'+b & 0 \\ c & c & a+c & 0 & 1'+a \end{array}$	aaa	bbb	$\dots$	$\dots$	$\dots$	acc	caa	$\dots$	$\dots$	$\dots$	no
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$\begin{array}{ccccc} \#236 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & -a & a+b & a+c \\ b & b & a+b & -b & b+c \\ c & c & a+c & b+c & -c \end{array}$	…	…	…	*abb*	*baa*	*acc*	*caa*	*bcc*	*cbb*	…	yes 21_{65}

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$\begin{array}{c\|cccc} \#246 & 1' & a & b & c \\ \hline 1' & 1' & a & b & c \\ a & a & 1'+a & 0 & 0 \\ b & b & 0 & 1'+b & 0 \\ c & c & 0 & 0 & 1'+c \end{array}$	*aaa*	*bbb*	*ccc*	…	…	…	…	…	…	…	no
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B.6 Atoms: atomic identity, one symmetric and one nonsymmetric

The identity is atomic. A cycle involving identity is consistent if and only if it is $1'1'1'$, $1'aa$, $r1'r$ or $1'rr$.

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$\begin{array}{c\|cccc} \#349 & 1' & a & r & r^{\smile} \\ \hline 1' & 1' & a & r & r^{\smile} \\ a & a & 1 & 0' & 0' \\ r & r & 0' & a+r^{\smile} & 1'+a \\ r^{\smile} & r^{\smile} & 0' & 1'+a & a+r \end{array}$	aaa	…	$rrr^{\smile}$	arr	rar	raa	rra	yes 35_{37}
$\begin{array}{c\|cccc} \#350 & 1' & a & r & r^{\smile} \\ \hline 1' & 1' & a & r & r^{\smile} \\ a & a & 1' & 0 & 0 \\ r & r & 0 & r+r^{\smile} & -a \\ r^{\smile} & r^{\smile} & 0 & -a & r+r^{\smile} \end{array}$	…	rrr	$rrr^{\smile}$	…	…	…	…	no
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</table></td><td>…</td><td>rrr</td><td>$rrr^{\smile}$</td><td>arr</td><td>rar</td><td>…</td><td>rra</td><td>no</td></tr>
</table>

<table>
<tr><th>atom table</th><th>aaa</th><th>rrr</th><th>$rrr^{\smile}$</th><th>arr</th><th>rar</th><th>raa</th><th>rra</th><th>RA</th></tr>
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<tr><td>a</td><td>a</td><td>$1' + a$</td><td>$r + r^{\smile}$</td><td>$r + r^{\smile}$</td></tr>
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<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$r + r^{\smile}$</td><td>1</td><td>$0'$</td></tr>
</table>
</td><td>aaa</td><td>rrr</td><td>$rrr^{\smile}$</td><td>arr</td><td>rar</td><td>$\ldots$</td><td>rra</td><td>yes
22_{37}</td></tr>
<tr><td>
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<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$0'$</td><td>1</td><td>$0'$</td></tr>
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36_{37}</td></tr>
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<tr><td>r</td><td>r</td><td>$0'$</td><td>$0'$</td><td>1</td></tr>
<tr><td>$r^{\smile}$</td><td>$r^{\smile}$</td><td>$0'$</td><td>1</td><td>$0'$</td></tr>
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37_{37}</td></tr>
</table>

Appendix C

Code

C.1 Sage code for atomic algebra class

This code contains the AtomicAlgebra class used for testing algebras defined on atoms for isomorphisms and axiom satisfaction. We view the composition operation as a matrix with rows and columns labelled by atoms.

```
# Intended for use with AlgebraGenerator.sage
# This class is used to represent and examine algebras on atom tables.
# It is intended to be used for nonassociative algebras, but this is not
    assumed.
class AtomicAlgebra:

    # A human-readable description of each relation algebra axiom.
    AXIOMS = {
"R01": "+-commutativity: x + y = y + x",
"R02": "+-associativity: x + (y + z) = (x + y) + z",
"R03": "Huntington's axiom: -(-x + -y) + -(-x + y) = x",
"R04": ";-associativity: x;(y;z) = (x;y);z",
"R05": ";-distributivity: (x + y);z = x;z + y;z",
"R06": "identity law: x;1' = x",
"R07": "converse-involution: con(con(x)) = x",
"R08": "converse-distributivity: con(x + y) = con(x) + con(y)",
"R09": "converse-involutive distributivity: con(x;y) = con(y);con(x)",
"R10": "Tarski/De Morgan axiom: con(x); -(x;y) + -y = y", "WA": "((id . x)
    . top) . top = (id . x) . (top . top)", "SA": "(x . top) . top = x .
    (top . top)",
"WA" : "((id . x) . top) . top = (id . x) . (top . top)",
"SA" : "(x . top) . top = x . (top . top)"
```

```
}

    # Given an atom table as a string, convert it to a matrix (list of
        lists).
    @classmethod
    def stringToAtomicTable(cls, matrixString):
        M1 = matrixString.strip()[1:-1]
        M2 = M1.strip()[1:-1]
        M3 = [line.split(',') for line in M2.split('],[')]
        M4 = [[Set(entry.split("+"))-Set(['0']) for entry in line] for line
            in M3]
        return M4

    # Give a human readable report on a list of failed axioms, eg. ["R01",
        "R02", "R07"].
    @classmethod
    def reportFailedAxioms(cls, failedAxioms):
        for axiom in failedAxioms:
            print("Fails axiom " + axiom + ": " + cls.AXIOMS[axiom] + ".")

    # Given a map between atoms as a dictionary, returns a map that works on
        unions of atoms.
    @classmethod
    def atomFunction(cls, atomMap, element):
        if type(element) == str:
            return atomMap[element]
        else:
            return Set([cls.atomFunction(atomMap, x) for x in element])

    # Check if a map between atom structures preserves composition.
    # This is required for the function to be an isomorphism.
    @classmethod
    def preservesComposition(cls, algebra1, algebra2, atomMap):
        preservesComposition = True
        for x, y in itertools.product(algebra1.atoms, repeat = 2):
            if cls.atomFunction(atomMap, algebra1.compose(x, y)) !=
                algebra2.compose(cls.atomFunction(atomMap, x),
                cls.atomFunction(atomMap, y)):
                preservesComposition = False
                break
        return preservesComposition
```

```
# Checks if a given algebra is isomorphic to self.
# If creating4AtomAlgebras, we’re assuming that our algebras are coming
    from the gen4Atoms function.
# If so, we can assume some additional structure about the converses.
# This isn’t necessary, but it does speed up the isomorphism checking.
# Can also return a list of isomorphisms, but this isn’t recommended.
def isIsomorphic(self, algebra2, creating4AtomAlgebras = False,
    returnIsomorphisms = False):
    # First we check that the algebras are the same size.
    if self.nAtoms != algebra2.nAtoms:
        return False
    # Next we check that the converse pairs match in number and
        structure.
    converses1 = self.conversePairs
    nonSelfConversePairs1 = len(converses1)
    selfConverses1 = [x[0] for x in converses1 if x[0] == x[1]]
    nonSelfConversePairs1 = [x for x in converses1 if x[0] != x[1]]
    converses2 = algebra2.conversePairs
    nonSelfConversePairs2 = len(converses2)
    selfConverses2 = [x[0] for x in converses2 if x[0] == x[1]]
    nonSelfConversePairs2 = [x for x in converses2 if x[0] != x[1]]
    if len(selfConverses1) != len(selfConverses2):
        return False
    # Enumerate all possible functions respecting converse
    # First we check if we are creating4AtomAlgebras, so we might make
        additional assumptions.
    # Note the small number of possible converse structures.
    if creating4AtomAlgebras and self.identity == Set([’a’]) and
        algebra2.identity == Set([’a’]):
        if len(selfConverses1) == 4:
            possibleIsomorphisms = [
                {’a’: ’a’, ’b’: ’c’, ’c’: ’b’, ’d’: ’d’},
                {’a’: ’a’, ’b’: ’b’, ’c’: ’d’, ’d’: ’c’},
                {’a’: ’a’, ’b’: ’d’, ’c’: ’c’, ’d’: ’b’},
                {’a’: ’a’, ’b’: ’c’, ’c’: ’d’, ’d’: ’b’},
                {’a’: ’a’, ’b’: ’d’, ’c’: ’b’, ’d’: ’c’}
                ]
        elif len(selfConverses1) == 2:
            possibleIsomorphisms = [
                {’a’: ’a’, ’b’: ’b’, ’c’: ’c’, ’d’: ’d’},
                {’a’: ’a’, ’b’: ’b’, ’c’: ’d’, ’d’: ’c’},
                ]
```

```
            else:
                raise ValueError("Unexpected converse structure. Assumes
                    either all atoms are symmetric, or only converse pair is
                    ('c','d').")
        # If we are not creating4AtomAlgebras, then we must check for
            isomorphisms by brute force.
        else:
            # First enumerate all possible ways to map symmetric atoms from
                the first algebra
            # to self converse atoms from the second algebra.
            possibleSelfConverseMaps = []
            for perm in itertools.permutations(selfConverses2):
                possibleSelfConverseMaps.append(zip(selfConverses1, perm))
            # Now enumerate all possible ways to map converse pairs from the
                first algebra
            # to converse pairs from the second algebra.
            possibleConversePairMaps = []
            for perm1 in list(itertools.product(*[[x,x[::-1]] for x in
                nonSelfConversePairs2])):
                for perm2 in itertools.permutations(perm1):
                    map = []
                    pairing = zip(nonSelfConversePairs1, perm2)
                    for pair in pairing:
                        map.append((pair[0][0], pair[1][0]))
                        map.append((pair[0][1], pair[1][1]))
                    possibleConversePairMaps.append(map)
            # Now combine them to generate all maps respecting the converse
                structure.
            possibleIsomorphisms = []
            for selfConverseMap, conversePairMap in
                itertools.product(possibleSelfConverseMaps,
                possibleConversePairMaps):
                possibleIsomorphisms.append(selfConverseMap + conversePairMap)
            possibleIsomorphisms = [dict(x) for x in possibleIsomorphisms]
        # Assume that the algebras are not isomorphic.
        areIsomorphic = False
        isomorphisms = []
        # Go through all the maps that preserve converse structure to test
            if they also preserve composition.
        # If so, they are isomorphic.
        # If we want to enumerate all isomorphisms, check all maps.
            Otherwise, break if an isomorphism is found.
```

```
        for possibleIsomorphism in possibleIsomorphisms:
            if self.preservesComposition(self, algebra2, possibleIsomorphism):
                areIsomorphic = True
                if not returnIsomorphisms:
                    break
                else:
                    isomorphisms.append(possibleIsomorphism)
        if areIsomorphic and returnIsomorphisms:
            return areIsomorphic, isomorphisms
        else:
            return areIsomorphic

    # Create an algebra from a table of atoms, which gives compositions, and
        a converse structure.
    # An atom table is a list of lists, with each entry a Set (as distinct
        from set) of atoms.
    # The set of atoms is interpreted as a union. Atoms are 'a', 'b', 'c',
        etc.
    # The converse pair is a list of 2-tuples of atoms.
    # If 'a' is converse to 'b', write as ('a','b').
    # If 'a' is symmetric, write as ('a', 'a').
    # Can also give converses as a dictionary.
    # Algebra may not necessarily meet all the axioms.
    def __init__(self, atomTable, conversePairs = None):
        if type(atomTable) == str:
            atomTable = self.stringToAtomTable(atomTable)
        # If no converses given assume all atoms are symmetric.
        if conversePairs == None:
            self.conversePairs = [(x,x) for x in self.atoms]
        # Can also give converses as a dictionary.
        if type(conversePairs) == dict:
            self.conversePairs = []
            for pair in conversePairs.items():
                if pair not in self.conversePairs and pair[::-1] not in
                    self.conversePairs:
                    self.conversePairs.append(pair)
        else:
            self.conversePairs = conversePairs
        # Set up the basic properties of the algebra.
        self.atomTable = atomTable
        self.nAtoms = len(atomTable[0])
        self.atoms = [Set([chr(i + 97)]) for i in range(self.nAtoms)]
```

```
        self._nonIdentityAtoms = None
        self.top = Set([chr(i+97) for i in range(self.nAtoms)])
        self.zero = Set()
        self.elements = [Combinations(list(self.top),n).list() for n in
            range(self.nAtoms+1)]
        self.elements = list(itertools.chain.from_iterable(self.elements))
        self.elements = [Set(element) for element in self.elements]
        self.nElements = 2**self.nAtoms
        self.nNonZeroElements = self.nElements - 1
        # We may want to call on a converse from a dictionary.
        # So here we construct a dictionary of converses from the converse
            pairs.
        self.atomConverses = dict()
        for atom in self.atoms:
            for conversePair in self.conversePairs:
                if atom[0] in conversePair:
                    self.atomConverses[atom[0]] =
                        conversePair[~(conversePair.index(atom[0]))]
                    break
        self._identity = None
        self._semigroup = None
        # properties
        self._isNA = None
        self._satisfiesWAaxiom = None
        self._isWA = None
        self._satisfiesSAaxiom = None
        self._isSA = None
        self._satisfiesAssociativity = None
        self._isRA = None
        self._consistentAtomTriples = None
        self._consistentMirrorFreeAtomTriples = None
        self._isRepresentable = None
        self._representation = None

    # Turns a single atom ’a’ into a Set([’a’]).
    def makeSet(self, x):
        if type(x) == str:
            x = Set([x])
        if type(x) != type(Set()):
            raise TypeError(’An element of the algebra needs to be either a
                set of atoms or a string representing a single atom.’)
        return x
```

```
# Define composition of atoms or sets of atoms using the atom table.
# We allow for inputs of single atoms, but every element is properly
# viewed as a set of atoms.
def compose(self, x, y):
    x = self.makeSet(x)
    y = self.makeSet(y)
    # Composition with the 0 element
    if x == Set() or y == Set():
        output = Set()
    else:
        output = Set()
        for i, j in itertools.product(x, y):
            rowPos = ord(i) - 97
            colPos = ord(j) - 97
            try:
                output = output.union(self.atomTable[rowPos][colPos])
            except IndexError:
                "Out of bounds: composition "+ str(x) + "*" + str(y) + "
                    contains a non-atomic element."
    return output

# Define intersection as set intersection.
def intersection(self, x, y):
    x = self.makeSet(x)
    y = self.makeSet(y)
    return x.intersection(y)

# Define union as set union.
def union(self, x, y):
    x = self.makeSet(x)
    y = self.makeSet(y)
    return x.union(y)

# Define converse using the converse dictionary we made earlier.
def converse(self, x):
    x = self.makeSet(x)
    return Set([self.atomConverses[atom] for atom in x])

# Define complement as set complement relative to the top elemenet (set
    of all atoms).
def complement(self, x):
```

```
        x = self.makeSet(x)
        return self.top.difference(x)

    # Return the identity of an algebra if it exists, otherwise returns None
    # If the identity element is not already recorded, will run through all
        elements and check for identity property.
    @property
    def identity(self):
        if self._identity == None:
            for candidateId in self.elements:
                isId = True
                for atom in self.atoms:
                    if self.compose(candidateId, atom) != atom or
                        self.compose(atom, candidateId) != atom:
                        isId = False
                        break
                if isId:
                    self._identity = candidateId
                    break
        return self._identity

    # All non-identity atoms.
    @property
    # Return a list of atoms which are not the identity atom.
    def nonIdentityAtoms(self):
        if self._nonIdentityAtoms == None:
            if self.identity == None:
                return self.atoms
            else:
                self._nonIdentityAtoms = [x for x in self.atoms if x !=
                    self.identity]
        return self._nonIdentityAtoms

    # Determines if the algebra generated by the atom table is a
        nonassociative algebra.
    # Due to the construction, not all axioms need to be checked.
    # Can control the amount of reporting done on failed axioms, if any.
    def isNA(self, whatFails = False, report = False):
        if report:
            whatFails = True
        if self._isNA == None or whatFails == True:
            self._isNA = True
```

```
failedAxioms = []
# Axiom R01: +-commutativity: x + y = y + x
# Axiom R02: +-associativity: x + (y + z) = (x + y) + z
# Axiom R03: Huntington's axiom: -(-x + -y) + -(-x + y) = x
for x,y in itertools.product(self.atoms, repeat = 2):
    firstTerm = self.complement(self.union(self.complement(x),
        self.complement(y)))
    secondTerm = self.complement(self.union(self.complement(x),
        y))
    if self.union(firstTerm, secondTerm) != x:
        failedAxioms.append("R03")
        break
# Axiom R05: ;-distributivity: (x + y);z = x;z + y;z
#for x,y,z in itertools.product(self.atoms, repeat = 3):
#    if self.compose(self.union(x,y), z) !=
    self.union(self.compose(x, z), self.compose(y, z)):
#        failedAxioms.append("R05")
#        break
# Axiom R06: identity law: x;1' = x
if self.identity == None:
    failedAxioms.append("R06")
# Axiom R07: converse-involution: con(con(x)) = x
#    - should not be needed if converse pairs are correctly
    defined.
for x in self.atoms:
    if self.converse(self.converse(x)) != x:
        failedAxioms.append("R07")
        break
# Axiom R08: converse-distributivity: con(x + y) = con(x) + con(y)
for x,y in itertools.product(self.atoms, repeat = 2):
    if self.converse(self.union(x,y)) !=
        self.union(self.converse(x), self.converse(y)):
        failedAxioms.append("R09")
        break
# Axiom R09: converse-involutive distributivity: con(x;y) =
    con(y);con(x)
for x,y in itertools.product(self.atoms, repeat = 2):
    if self.converse(self.compose(x,y)) !=
        self.compose(self.converse(y), self.converse(x)):
        failedAxioms.append("R09")
        break
# Axiom R10: Tarski/De Morgan axiom: con(x); -(x;y) + -y = y
```

```
        for x,y in itertools.product(self.atoms, repeat = 2):
            if self.union(self.compose(self.converse(x),
                self.complement(self.compose(x,y))), self.complement(y))
                != self.complement(y):
                failedAxioms.append("R10")
                break
        if len(failedAxioms) > 0:
            self._isNA = False
    if report:
        self.reportFailedAxioms(failedAxioms)
        return self._isNA
    elif whatFails and not report:
        return (self._isNA, failedAxioms)
    else:
        return self._isNA

# Determines if the algebra generated by the atom table satisfies the
    weakly associative axiom.
# Axiom WA: ((id . x) . top) . top = (id . x) . (top . top)
@property
def satisfiesWAaxiom(self):
    if self._satisfiesWAaxiom == None:
        if self.identity == None:
            self._satisfiesWAaxiom = False
        else:
            self._satisfiesWAaxiom = True
            for x in self.atoms:
                LHS = self.compose(self.compose(
                        self.intersection(self.identity, x), self.top),
                            self.top)
                RHS = self.compose(self.compose(
                        self.intersection(self.identity, x), self.top),
                            self.compose(self.top, self.top))
                if LHS != RHS:
                    self._satisfiesWAaxiom = False
                    break
    return self._satisfiesWAaxiom

# Determines if the algebra generated by the atom table is a weakly
    associative algebra.
# The algebra must be an nonassociative algebra and satisfy the weakly
    associative axiom.
```

```
def isWA(self, whatFails = False, report = False):
    if report:
        whatFails = True
    if whatFails == True:
        self._isWA = True
        failedAxioms = []
        failedAxioms.extend(self.isNA(True,False)[1])
        if self.satisfiesWAaxiom == False:
            failedAxioms.append("WA")
        if len(failedAxioms) > 0:
            self._isWA = False
    elif self._isWA == None:
        self._isWA = (self.isNA() and self.satisfiesWAaxiom)
    if report:
        self.reportFailedAxioms(failedAxioms)
        return self._isWA
    elif whatFails and not report:
        return (self._isWA, failedAxioms)
    else:
        return self._isWA

# Determines if the algebra generated by the atom table satisfies the
    semiassociative axiom.
# Axiom SA: (x . top) . top = x . (top . top)"
@property
def satisfiesSAaxiom(self):
    if self._satisfiesSAaxiom == None:
        self._satisfiesSAaxiom = True
        for x in self.atoms:
            if self.compose(self.compose(x, self.top), self.top) !=
                self.compose(self.compose(x, self.top),
                self.compose(self.top, self.top)):
                self._satisfiesSAaxiom = False
                break
    return self._satisfiesSAaxiom

# Determines if the algebra generated by the atom table is a
    semiassociative algebra.
# The algebra must be an nonassociative algebra and satisfy the
    semiassociative axiom.
def isSA(self, whatFails = False, report = False):
    if report:
```

```
            whatFails = True
        if whatFails == True:
            self._isSA = True
            failedAxioms = []
            failedAxioms.extend(self.isWA(True,False)[1])
            if self.satisfiesSAaxiom == False:
                failedAxioms.append("SA")
            if len(failedAxioms) > 0:
                self._isSA = False
        elif self._isSA == None:
            self._isSA = (self.isNA() and self.satisfiesSAaxiom)
        if report:
            self.reportFailedAxioms(failedAxioms)
            return self._isSA
        elif whatFails and not report:
            return (self._isSA, failedAxioms)
        else:
            return self._isSA

    # Determines if the algebra generated by the atom table has an
        associative composition operation.
    # Axiom R04: ;-associativity: x;(y;z) = (x;y);z."
    @property
    def satisfiesAssociativity(self):
        if self._satisfiesAssociativity == None:
            self._satisfiesAssociativity = True
            for i, j, k in itertools.product(self.elements, repeat = 3):
                if self.compose(self.compose(i,j), k) != self.compose(i,
                    self.compose(j,k)):
                    self._satisfiesAssociativity = False
                    break
        return self._satisfiesAssociativity

    # Determines if the algebra generated by the atom table is a relation
        algebra.
    # Must be an associative nonassociative algebra.
    def isRA(self, whatFails = False, report = False):
        if report:
            whatFails = True
        if whatFails == True:
            self._isRA = True
            failedAxioms = []
```

```
            failedAxioms.extend(self.isSA(True,False)[1])
            if self.satisfiesAssociativity == False:
                failedAxioms.append("R04")
            if len(failedAxioms) > 0:
                self._isRA = False
        elif self._isRA == None:
            self._isRA = (self.isNA() and self.satisfiesAssociativity)
        if report:
            self.reportFailedAxioms(failedAxioms)
            return self._isRA
        elif whatFails and not report:
            return (self._isRA, failedAxioms)
        else:
            return self._isRA
```

C.2 Sage code for brute-force generation of nonassociative algebras

This code contains the methods for generating all nonassociative algebras on three atoms by brute force. We generate every possible composition table and pair it with every possible converse structure before checking the results against the nonassociative algebra axioms.

The `generateIntegralAtomTables` method generates every possible composition table of a given size n. The `hasID` option forces the first atom to act as an identity atom, and so only the bottom $(n-1)\times(n-1)$ square of the composition table needs to be added. To make the generation faster, we separate out the case in which the identity is not integral.

```
# Should be loaded with AtomicAlgebra.sage and AlgebraGenerator.sage
import itertools

# Generate all possible atom tables given number of atoms.
def generatePossibleAtomTables(size, hasID = False):
    atoms = [Set([chr(i + 97)]) for i in range(size)]
    n = size - int(hasID)
    # By interior, we mean the part of the composition table/matrix to the
    # bottom-right of a fixed identity atom, or the whole table/matrix if no
    # identity atom is fixed.
```

```
    possibleInteriors = []
    possibleCells = [Set(cell) for cell in list(powerset([atom[0] for atom
        in atoms]))]
    possibleInteriorLists = [list(x) for x in
        list(itertools.product(possibleCells, repeat = n**2))]
    for interiorList in possibleInteriorLists:
        interior = [interiorList[i:i+n] for i in range(0, len(interiorList),
            n)]
        if hasID:
            interior = [[atoms[i+1]] + interior[i] for i in range(n)]
            interior = [atoms] + interior
        possibleInteriors.append(interior)
    return possibleInteriors

# Generate all atomic nonassociative algebras with 3 atoms.
# Because this is easier than the 4 atom algebras, we do not construct the
# atom tables with cycles.
# Instead, we generate all potential atom tables and test them against the
    axioms.
def gen3Atoms():
    return genNonIdentity3AtomicAlgebras() + genIdentity3AtomicAlgebras()

# Generate all atomic nonassociative algebras with 3 atoms, in which the
# identity is the ’a’ atom.
def genIdentity3AtomicAlgebras():
    print("Generating 3 atom algebras with atomic identity.")
    possibleInteriors = generatePossibleAtomTables(3, True)
    possibleConverseStructures = [[(’a’,’a’), (’b’,’c’)], [(’a’,’a’),
        (’b’,’b’), (’c’,’c’)]]
    algebras = []
    nPossibleInteriors = len(possibleInteriors)
    # The counter is used to track progress and display a progress bar.
    counter = 1
    for interior in possibleInteriors:
        progress = float(counter) / nPossibleInteriors
        for converseStructure in possibleConverseStructures:
            validAlgebra = True
            newAlgebra = AtomicAlgebra(interior, converseStructure)
            if not newAlgebra.isNA():
                validAlgebra = False
            else:
                for algebra in algebras:
```

```
                    if newAlgebra.isIsomorphic(algebra):
                        validAlgebra = False
            if validAlgebra: algebras.append(newAlgebra)
        progressPercent = str(round(progress*100, 2))
      text = "\rProgress: " + progressPercent + "%"
      sys.stdout.write(text)
      sys.stdout.flush()
        counter += 1
    return algebras

# Generate all atomic nonassociative algebras with 3 atoms, in which the
# identity is NOT an atom.
def genNonIdentity3AtomicAlgebras():
    print("Generating 3 atom algebras without atomic identity.")
    twoFragmentsTable = [[Set([’a’]), Set()], [Set(), Set([’b’])]]
    atoms = [Set([chr(i + 97)]) for i in range(3)]
    possibleInteriors = []
    possibleCells = [Set(cell) for cell in list(powerset([atom[0] for atom
        in atoms]))]
    for x1, x2, x3, x4, x5 in itertools.product(possibleCells, repeat=5):
        possibleInterior = [twoFragmentsTable[0] + [x1],
            twoFragmentsTable[1] + [x2], [x3, x4, x5]]
        possibleInteriors.append(possibleInterior)
    converseStructure = [(’a’,’a’), (’b’,’b’), (’c’,’c’)]
    algebras = []
    nPossibleInteriors = len(possibleInteriors)
    # The counter is used to track progress and display a progress bar.
    counter = 1
    for interior in possibleInteriors:
        progress = float(counter) / nPossibleInteriors
        validAlgebra = True
        newAlgebra = AtomicAlgebra(interior, converseStructure)
        if not newAlgebra.isNA():
            validAlgebra = False
        else:
            for algebra in algebras:
                if newAlgebra.isIsomorphic(algebra):
                    validAlgebra = False
        if validAlgebra: algebras.append(newAlgebra)
        progressPercent = str(round(progress*100, 2))
      text = "\rProgress: " + progressPercent + "%"
      sys.stdout.write(text)
```

```
    sys.stdout.flush()
      counter += 1
  return algebras
```

C.3 Sage code for generating nonassociative algebras on four atoms

This code contains the methods for generating all four atom nonassociative algebras. A brute force method would be computationally infeasible for more than three atoms, so a method employing cycles as in [58] is used instead.

The greatest computational difficulty is in checking for isomorphisms. In the AtomicAlgebra class, we encoded a restricted set of converse pairs to be used when generating algebras on four atoms. This is because we are generating the algebras in specific ways. For example, the atom is always either an identity or subidentity atom, and so is always symmetric. This speeds up the generation.

The algebras on four atoms are generate in six parts:

1. algebras with a two-fragment identity and all atoms symmetric,
2. algebras with a two-fragment identity and one converse pair,
3. algebras with a three-fragment identity and all atoms symmetric,
4. algebras with a four-fragment identity (there is only one up to isomorphism),
5. algebras with an atomic identity and all atoms symmetric, and
6. algebras with an atomic identity and one converse pair.

To ensure that these don't overlap, the `generateAlgebrasFromFixedEntries` method rejects an algebra if the identiy element grows, that is, if the input is for algebras with a two-fragment identity but an outputted algebra has a three-fragment identity.

```
import itertools

# Four atoms.
atoms = ['a','b','c','d']
# If the identity is atomic it will be represented by 'a'.
# This fixes the left-most column and uppermost row of the composition
    table of atoms.
```

```
# A fixed entry is represented by a consistent triple, eg. ('a','a','a') or
    ('a','b','b').
atomicIdentity = [('a', atom, atom) for atom in atoms] + [(atom, 'a', atom)
    for atom in atoms[1:]]
# The identity can also be a union of atoms. Here we consider two atoms,
    assumed 'a' and 'b', unioning to identity.
twoFragmentIdentity = [(atom, atom, atom) for atom in atoms[:2]] + [(atom1,
    atom2, 0) for atom1, atom2 in itertools.product(atoms[:2], repeat=2) if
    atom1 != atom2]
# Here we consider three atoms, assumed 'a', 'b' and 'c', unioning to
    identity.
threeFragmentIdentity = [(atom, atom, atom) for atom in atoms[:3]] +
    [(atom1, atom2, 0) for atom1, atom2 in itertools.product(atoms[:3],
    repeat=2) if atom1 != atom2]
# Here we consider four atoms unioning to identity.
fourFragmentIdentity = [(atom, atom, atom) for atom in atoms] + [(atom1,
    atom2, 0) for atom1, atom2 in itertools.product(atoms, repeat=2) if
    atom1 != atom2]
# All 10 possible converse structures.
# We don't need all 10. We only need to consider 3 cases:
    # All self-converse.
    # Just 2 self-converse.
    # No self-converse.
# All others will be isomorphic or irrelevant, eg. no self-converse atoms
    => no identity element.
converses = [
        {'a': 'a', 'b': 'b', 'c': 'c', 'd': 'd'}, # all self-converse
        {'a': 'b', 'b': 'a', 'c': 'c', 'd': 'd'}, # c,d self-converse, (a,b)
        {'a': 'a', 'b': 'c', 'c': 'b', 'd': 'd'}, # a,d self-converse, (b,c)
        {'a': 'a', 'b': 'b', 'c': 'd', 'd': 'c'}, # a,b self-converse, (c,d)
        {'a': 'c', 'b': 'b', 'c': 'a', 'd': 'd'}, # b,d self-converse, (a,c)
        {'a': 'a', 'b': 'd', 'c': 'c', 'd': 'b'}, # a,c self-converse, (b,d)
        {'a': 'd', 'b': 'b', 'c': 'c', 'd': 'a'}, # b,c self-converse, (a,d)
        {'a': 'b', 'b': 'a', 'c': 'd', 'd': 'c'}, # (a,b), (c,d)
        {'a': 'd', 'b': 'c', 'c': 'b', 'd': 'a'}, # (a,d), (b,c)
        {'a': 'c', 'b': 'd', 'c': 'a', 'd': 'b'} # (a,c), (b,d)
        ]

# Given a triple and a converse structure, generate the cycle including
    that triple.
# This is an implementation of the relation algebra concept of a Peircean
    transform.
```

```
# Cycle generated by (x,y,z) is:
    # [ (x,y,z), (con(x),z,y),
        (y,con(z),con(x)),(con(y),con(x),con(z)),(con(z),x,con(y)),
        (z,con(y),x) ]
# A triple in a cycle is consistent if and only if all triples in the cycle
    are consistent.
def genCycle(triple, converse):
    x, y, z = triple
    cycle = []
    cycle.append(triple)
    cycle.append((converse[x], z, y))
    cycle.append((y, converse[z], converse[x]))
    cycle.append((converse[y], converse[x], converse[z]))
    cycle.append((converse[z], x, converse[y]))
    cycle.append((z, converse[y], x))
    cycle.sort()
    # Remove duplicates.
    return list(set(cycle))

# Given a converse structure, partition the triples of elements into cycles.
def genCyclePartition(converse):
    parts = []
    for triple in itertools.product(atoms, repeat = 3):
        cycle = genCycle(triple, converse)
        if cycle not in parts: parts.append(cycle)
    return parts

# Fix an entry of an atom table to either a single atom or 0.
# Input is a list of tuples of fixed entries, eg. [('a','a','b')] will fix
    the upper-left entry to 'b' and only 'b'.
# Output is a list of necessary cycles, a list of forbidden cycles, and a
    list of optional (remaining) cycles.
# Any algebra with consistent cycles at least those that are necessary, and
    none of those that are forbidden,
# will have the desired fixed entry.
def fixEntries(entriesToFix, cycles):
    cyclesToSort = copy(cycles)
    necessaryCycles = []
    forbiddenCycles = []
    # a, b, c in entriesToFix means that a;b = c is the composition we want
        to fix.
    for a, b, c in entriesToFix:
```

```
        # If c == 0, then a;b composes to nothing.
        if c == 0:
            for cycle in cyclesToSort:
                if (a,b) in [triple[:2] for triple in cycle]:
                    if cycle not in forbiddenCycles:
                        forbiddenCycles.append(cycle)
        else:
            for cycle in cyclesToSort:
                # If the cycle contains a;b composing to c, it is necessary.
                if (a,b,c) in cycle:
                    if cycle not in necessaryCycles:
                        necessaryCycles.append(cycle)
                # If the cycle contains a;b composing to something other than
                    c, it is forbidden.
                elif (a,b) in [triple[:2] for triple in cycle]:
                    if cycle not in forbiddenCycles:
                        forbiddenCycles.append(cycle)
    # A cycle that is neither necessary or forbidden is optional.
    optionalCycles = [cycle for cycle in cycles if cycle not in
        necessaryCycles and cycle not in forbiddenCycles]
    return necessaryCycles, forbiddenCycles, optionalCycles

# Create an algebra from a list of cycles to be included, and a converse
    structure.
# First creates an atom table from the cycles, then creates an instance of
    the AtomicAlgebra class
def create4AtomAlgebraFromCycles(cycles, converse):
    # Create an empty atomTable
    atomTable = [[Set() for i in range(4)] for j in range(4)]
    # Fill the atom table.
    for cycle in cycles:
        for triple in cycle:
            x, y, z = triple
            rowPos = ord(x) - 97
            colPos = ord(y) - 97
            # For every triple in every cycle, set the relevant entry of the
                atomTable to correspond to the cycle.
            atomTable[rowPos][colPos] =
                atomTable[rowPos][colPos].union(Set([z]))
    newAlgebra = AtomicAlgebra(atomTable, converse)
    return newAlgebra
```

```
# Generate a list of nonassociative algebras with an atomTable with desired
    fixed entries and converse structure.
# Returned list contains no two isomorphic algebras.
def generateAlgebrasFromFixedEntries(entriesToFix, converse):
    # First generate all of the cycles from the desired converse structure.
    cycles = genCyclePartition(converse)
    # Then generate the necessary, forbidden and optional cycles fixing the
        desired entries.
    necessaryCycles, forbiddenCycles, optionalCycles =
        fixEntries(entriesToFix, cycles)
    # Take the powerset of optional cycles. This is the set of choices of
        optional cycles to include.
    pset = powerset(optionalCycles)
    # Add the necessary cycles to every choice of optional cycles. Each
        possible cycle set is an algebra.
    possibleCycleSets = [necessaryCycles + choice for choice in pset]
    algebras = []
    # The counter is used to track progress and display a progress bar.
    counter = 1
    nCycleSets = len(possibleCycleSets)
    for cycleSet in possibleCycleSets:
        progress = float(counter) / nCycleSets
        # Create the algebra
        newAlgebra = create4AtomAlgebraFromCycles(cycleSet, converse)
        validAlgebra = True
        # The algebra is not valid if it is not a nonassociative algebra.
        if not newAlgebra.isNA():
            validAlgebra = False
        # Strictly speaking, we don't need to separate the cases in which
            the identity is two, three, or four fragments.
        # We do this so that the algebras are ordered by this property.
        elif entriesToFix == twoFragmentIdentity and
            len(newAlgebra.identity) > 2:
            validAlgebra = False
        elif entriesToFix == threeFragmentIdentity and
            len(newAlgebra.identity) > 3:
            validAlgebra = False
        else:
            # Check to see if the new algebra is isomorphic to an algebra we
                have already constructed.
            for algebra in algebras:
                if newAlgebra.isIsomorphic(algebra, True):
```

```
                    validAlgebra = False
                    # If an isomorphic algebra is found, we don’t need to
                        check the rest.
                    break
        # If the algebra is valid, add it to the list of algebras.
        if validAlgebra: algebras.append(newAlgebra)
        # Update the progress bar
        progressPercent = str(round(progress*100, 2))
      text = "\rProgress: " + progressPercent + "%"
      sys.stdout.write(text)
      sys.stdout.flush()
        counter += 1
    return algebras

def gen4Atoms():
    # Generate all algebras and report on progress along the way.
    # We generate the algebras in 6 cases according to identity and converse
        structure.
    # This reduces the number of isomorphism checks needed.
    algebras = []
    print("Generating 4 atom algebras with two-fragment identity, all atoms
        self-converse.")
    algebras1 = generateAlgebrasFromFixedEntries(twoFragmentIdentity,
        converses[0])
    print("Found " + str(len(algebras1)) + " non-isomorphic algebras.")
    print("Generating 4 atom algebras with two-fragment identity, only 2
        atoms self-converse.")
    algebras2 = generateAlgebrasFromFixedEntries(twoFragmentIdentity,
        converses[3])
    print("Found " + str(len(algebras2)) + " non-isomorphic algebras.")
    print("Generating 4 atom algebras with three-fragment identity, all
        atoms self-converse.")
    algebras3 = generateAlgebrasFromFixedEntries(threeFragmentIdentity,
        converses[0])
    print("Found " + str(len(algebras3)) + " non-isomorphic algebras.")
    print("Generating 4 atom the algebra with four-fragment identity.")
    algebras4 = generateAlgebrasFromFixedEntries(fourFragmentIdentity,
        converses[0])
    print("Found " + str(len(algebras4)) + " non-isomorphic algebras.")
    print("Generating 4 atom algebras with atomic identity, all atoms
        self-converse.")
    algebras5 = generateAlgebrasFromFixedEntries(atomicIdentity,
```

```
        converses[0])
    print("Found " + str(len(algebras5)) + " non-isomorphic algebras.")
    print("Generating 4 atom algebras with atomic identity, only 2 atoms
        self-converse.")
    algebras6 = generateAlgebrasFromFixedEntries(atomicIdentity,
        converses[3])
    print("Found " + str(len(algebras6)) + " non-isomorphic algebras.")
    return algebras1 + algebras2 + algebras3 + algebras4 + algebras5 +
        algebras6
```

C.4 Example code for nonrepresentability of non-associative algebras

This is the output from Prover9 [60] used to confirm the nonrepresentability of algebra #146 on four atoms: $1', a, b, c$.

```
==================== Prover9 ====================

Prover9 (64) version 2009-11A, November 2009.

==================== end of head ====================

==================== INPUT ====================

formulas(assumptions).
x = y | A(x,y) | B(x,y) | C(x,y).
A(x,y) -> x != y & -B(x,y) & -C(x,y).
B(x,y) -> x != y & -A(x,y) & -C(x,y).
C(x,y) -> x != y & -B(x,y) & -A(x,y).
(exists x exists y A(x,y)).
(exists x exists y B(x,y)).
(exists x exists y C(x,y)).
A(x,y) -> A(y,x).
B(x,y) -> B(y,x).
C(x,y) -> C(y,x).
A(x,z) & A(z,y) -> -A(x,y).
(exists x exists y exists z (B(x,z) & A(z,y) & A(x,y))).
C(x,z) & A(z,y) -> -A(x,y).
A(x,z) & B(z,y) -> -B(x,y).
(exists x exists y exists z (A(x,z) & B(z,y) & C(x,y))).
(exists x exists y exists z (A(x,z) & C(z,y) & C(x,y))).
(exists x exists y exists z (B(x,z) & B(z,y) & B(x,y))).
C(x,z) & B(z,y) -> -B(x,y).
B(x,z) & C(z,y) -> -C(x,y).
C(x,z) & C(z,y) -> -C(x,y).
end_of_list.

==================== end of input ====================

==================== PROOF ====================
```

```
% Proof 1 at 759.29 (+ 11.30) seconds.
% Length of proof is 308.
% Level of proof is 75.
% Maximum clause weight is 24.000.
% Given clauses 38528.

1 A(x,y) −> x != y & −B(x,y) & −C(x,y) # label(non_clause). [assumption].
2 B(x,y) −> x != y & −A(x,y) & −C(x,y) # label(non_clause). [assumption].
3 C(x,y) −> x != y & −B(x,y) & −A(x,y) # label(non_clause). [assumption].
7 A(x,y) −> A(y,x) # label(non_clause). [assumption].
8 B(x,y) −> B(y,x) # label(non_clause). [assumption].
9 C(x,y) −> C(y,x) # label(non_clause). [assumption].
10 A(x,z) & A(z,y) −> −A(x,y) # label(non_clause). [assumption].
11 (exists x exists y exists z (B(x,z) & A(z,y) & A(x,y))) # label(non_clause). [assumption].
12 C(x,z) & A(z,y) −> −A(x,y) # label(non_clause). [assumption].
13 A(x,z) & B(z,y) −> −B(x,y) # label(non_clause). [assumption].
15 (exists x exists y exists z (A(x,z) & C(z,y) & C(x,y))) # label(non_clause). [assumption].
17 C(x,z) & B(z,y) −> −B(x,y) # label(non_clause). [assumption].
18 B(x,z) & C(z,y) −> −C(x,y) # label(non_clause). [assumption].
19 C(x,z) & C(z,y) −> −C(x,y) # label(non_clause). [assumption].
20 x = y | A(x,y) | B(x,y) | C(x,y). [assumption].
21 −A(x,y) | y != x. [ clausify (1) ].
22 −A(x,y) | −B(x,y). [ clausify (1) ].
23 −A(x,y) | −C(x,y). [ clausify (1) ].
24 −B(x,y) | y != x. [ clausify (2) ].
25 −B(x,y) | −C(x,y). [ clausify (2) ].
26 −C(x,y) | y != x. [ clausify (3) ].
30 −A(x,y) | A(y,x). [ clausify (7) ].
31 −B(x,y) | B(y,x). [ clausify (8) ].
32 −C(x,y) | C(y,x). [ clausify (9) ].
33 −A(x,y) | −A(y,z) | −A(x,z). [ clausify (10) ].
34 B(c7,c9). [ clausify (11) ].
35 A(c9,c8). [ clausify (11) ].
36 A(c7,c8). [ clausify (11) ].
37 −C(x,y) | −A(y,z) | −A(x,z). [ clausify (12) ].
38 −A(x,y) | −B(y,z) | −B(x,z). [ clausify (13) ].
42 A(c13,c15). [ clausify (15) ].
43 C(c15,c14). [ clausify (15) ].
44 C(c13,c14). [ clausify (15) ].
48 −C(x,y) | −B(y,z) | −B(x,z). [ clausify (17) ].
49 −B(x,y) | −C(y,z) | −C(x,z). [ clausify (18) ].
50 −C(x,y) | −C(y,z) | −C(x,z). [ clausify (19) ].
70 C(x,y) | y = x | A(y,x) | B(y,x). [ resolve (32,a,20,d) ].
74 B(c9,c7). [ resolve (34,a,31,a) ].
75 c9 != c7. [ resolve (34,a,24,a) ].
76 −C(c7,c9). [ur(25,a,34,a) ].
77 −A(c7,c9). [ur(22,b,34,a) ].
78 −A(c9,x) | −A(x,c8). [ resolve (35,a,33,c) ].
81 A(c8,c9). [ resolve (35,a,30,a) ].
82 −B(c9,c8). [ resolve (35,a,22,a) ].
83 c9 != c8. [ resolve (35,a,21,a), flip (a) ].
84 −C(c9,c8). [ur(23,a,35,a) ].
85 −A(c7,x) | −A(x,c8). [ resolve (36,a,33,c) ].
86 −A(x,c7) | −A(x,c8). [ resolve (36,a,33,b) ].
88 A(c8,c7). [ resolve (36,a,30,a) ].
90 c8 != c7. [ resolve (36,a,21,a) ].
92 −C(c7,c8). [ur(23,a,36,a) ].
94 −A(x,y) | −A(z,y) | z = x | A(z,x) | B(z,x). [ resolve (37,a,20,d) ].
95 −C(c9,c7). [ur(37,b,36,a,c,35,a) ].
```

```
96 −B(c8,x) | −B(c7,x).  [resolve(38,a,36,a)].
97 −B(c8,x) | −B(c9,x).  [resolve(38,a,35,a)].
122 A(c15,c13).  [resolve(42,a,30,a)].
123 −B(c13,c15).  [resolve(42,a,22,a)].
124 c15 != c13.  [resolve(42,a,21,a)].
126 −A(c14,x) | −A(c15,x).  [resolve(43,a,37,a)].
127 C(c14,c15).  [resolve(43,a,32,a)].
128 c15 != c14.  [resolve(43,a,26,a), flip(a)].
129 −B(c15,c14).  [resolve(43,a,25,b)].
131 −A(c14,x) | −A(c13,x).  [resolve(44,a,37,a)].
132 C(c14,c13).  [resolve(44,a,32,a)].
133 c14 != c13.  [resolve(44,a,26,a)].
134 −B(c13,c14).  [resolve(44,a,25,b)].
135 −A(c13,c14).  [resolve(44,a,23,b)].
150 −B(c14,x) | −B(c13,x).  [resolve(48,a,44,a)].
151 −B(c14,x) | −B(c15,x).  [resolve(48,a,43,a)].
154 −B(x,y) | −B(z,y) | z = x | A(z,x) | B(z,x).  [resolve(48,a,20,d)].
156 −B(x,c13) | −C(x,c14).  [resolve(49,b,44,a)].
157 −B(x,c15) | −C(x,c14).  [resolve(49,b,43,a)].
160 −B(x,y) | −C(x,z) | y = z | A(y,z) | B(y,z).  [resolve(49,b,20,d)].
162 −B(c15,x) | −C(x,c14).  [resolve(49,c,43,a)].
165 −B(x,y) | −C(y,z) | x = z | A(x,z) | B(x,z).  [resolve(49,c,20,d)].
166 −B(c15,c13).  [ur(49,b,44,a,c,43,a)].
171 −C(x,y) | −C(z,y) | z = x | A(z,x) | B(z,x).  [resolve(50,a,20,d)].
194 −A(c8,x) | −A(x,c9).  [resolve(81,a,33,c)].
195 −C(c8,c9).  [ur(23,a,81,a)].
196 −A(c8,x) | −A(x,c7).  [resolve(88,a,33,c)].
197 −C(c8,c7).  [ur(23,a,88,a)].
208 −A(c14,c13).  [ur(37,a,43,a,c,122,a)].
210 −B(c14,x) | −C(x,c15).  [resolve(127,a,49,c)].
211 −B(x,c14) | −C(x,c15).  [resolve(127,a,49,b)].
214 −B(c14,x) | −C(x,c13).  [resolve(132,a,49,c)].
215 −B(x,c14) | −C(x,c13).  [resolve(132,a,49,b)].
216 −B(c14,c13).  [resolve(132,a,25,b)].
226 −A(x,c7) | c8 = x | A(x,c8) | B(x,c8).  [resolve(94,a,88,a), flip(b)].
231 −A(x,c8) | c7 = x | A(x,c7) | B(x,c7).  [resolve(94,a,36,a), flip(b)].
236 −A(x,c7) | c8 = x | A(c8,x) | B(c8,x).  [resolve(94,b,88,a)].
237 −A(x,c9) | c8 = x | A(c8,x) | B(c8,x).  [resolve(94,b,81,a)].
241 −A(x,c8) | c7 = x | A(c7,x) | B(c7,x).  [resolve(94,b,36,a)].
242 −A(x,c8) | c9 = x | A(c9,x) | B(c9,x).  [resolve(94,b,35,a)].
248 −B(x,c7) | c9 = x | A(x,c9) | B(x,c9).  [resolve(154,a,74,a), flip(b)].
254 −B(x,c9) | c7 = x | A(x,c7) | B(x,c7).  [resolve(154,a,34,a), flip(b)].
266 −B(x,c9) | c7 = x | A(c7,x) | B(c7,x).  [resolve(154,b,34,a)].
268 −B(c14,x) | c13 = x | A(x,c13) | B(x,c13).  [resolve(160,b,132,a), flip(b)].
269 −B(c14,x) | c15 = x | A(x,c15) | B(x,c15).  [resolve(160,b,127,a), flip(b)].
271 −B(x,y) | y = z | A(y,z) | B(y,z) | z = x | A(z,x) | B(z,x).  [resolve(160,b,70,a)].
274 −B(c15,x) | c14 = x | A(x,c14) | B(x,c14).  [resolve(160,b,43,a), flip(b)].
279 −B(x,c14) | c15 = x | A(x,c15) | B(x,c15).  [resolve(165,b,127,a), flip(b)].
288 −C(x,c15) | c14 = x | A(x,c14) | B(x,c14).  [resolve(171,a,127,a), flip(b)].
292 −C(x,c14) | c13 = x | A(x,c13) | B(x,c13).  [resolve(171,a,44,a), flip(b)].
298 −C(x,c15) | c14 = x | A(c14,x) | B(c14,x).  [resolve(171,b,127,a)].
310 c14 = x | A(x,c14) | B(x,c14) | c15 = x | A(x,c15) | B(x,c15).  [resolve(288,a,20,d), flip(d)].
315 −B(x,c13) | c14 = x | A(c14,x) | B(c14,x).  [resolve(156,b,70,a)].
316 −B(x,c15) | c14 = x | A(c14,x) | B(c14,x).  [resolve(157,b,70,a)].
319 c13 = x | A(x,c13) | B(x,c13) | c14 = x | A(c14,x) | B(c14,x).  [resolve(292,a,70,a)].
324 −B(c15,x) | c14 = x | A(c14,x) | B(c14,x).  [resolve(162,b,70,a)].
330 c14 = x | A(c14,x) | B(c14,x) | c15 = x | A(c15,x) | B(c15,x).  [resolve(298,a,70,a)].
351 −B(c14,x) | c15 = x | A(c15,x) | B(c15,x).  [resolve(210,b,70,a)].
352 −B(x,c14) | c15 = x | A(c15,x) | B(c15,x).  [resolve(211,b,70,a)].
353 −B(c14,x) | c13 = x | A(c13,x) | B(c13,x).  [resolve(214,b,70,a)].
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354 −B(x,c14) | c13 = x | A(c13,x) | B(c13,x). [resolve(215,b,70,a)].
360 c7 = x | A(c7,x) | B(c7,x) | c9 = x | A(x,c9) | B(x,c9). [resolve(271,a,74,a), flip(d)].
526 c14 = x | A(x,c14) | B(x,c14) | c15 = x | B(x,c15) | A(c15,x). [resolve(310,e,30,a)].
737 c13 = c8 | A(c8,c13) | B(c8,c13) | c14 = c8 | B(c14,c8) | c14 = c7 | A(c14,c7) | B(c14,c7).
    [resolve(319,e,231,a), flip(f)].
751 c13 = c8 | A(c8,c13) | B(c8,c13) | c14 = c8 | B(c14,c8) | −A(c14,c7). [resolve(319,e,86,b)].
1018 c14 = x | A(c14,x) | B(c14,x) | c15 = x | B(c15,x) | A(x,c15). [resolve(330,e,30,a)].
1388 c7 = x | A(c7,x) | B(c7,x) | c9 = x | B(x,c9) | c8 = x | A(c8,x) | B(c8,x). [resolve(360,e,237,a)].
1392 c7 = x | A(c7,x) | B(c7,x) | c9 = x | B(x,c9) | −A(c8,x). [resolve(360,e,194,b)].
2710 c14 = x | B(x,c14) | c15 = x | B(x,c15) | A(c15,x) | A(c14,x). [resolve(526,b,30,a)].
6907 c14 = x | B(c14,x) | c15 = x | B(c15,x) | A(x,c15) | A(x,c14). [resolve(1018,b,30,a)].
12555 c13 = c8 | A(c8,c13) | B(c8,c13) | c14 = c8 | B(c14,c8) | c14 = c7 | B(c14,c7).
    [resolve(737,g,751,f),merge(h),merge(i),merge(j),merge(k),merge(l)].
15752 c14 = x | B(x,c14) | c15 = x | B(x,c15) | A(c14,x) | A(x,c15). [resolve(2710,e,30,a)].
19024 c14 = x | B(c14,x) | c15 = x | B(c15,x) | A(x,c14) | −B(c15,y) | −B(x,y). [resolve(6907,e,38,a)].
19027 c14 = x | B(c14,x) | c15 = x | B(c15,x) | A(x,c14) | −A(c15,y) | −A(x,y). [resolve(6907,e,33,a)].
21660 c14 = x | B(x,c14) | c15 = x | B(x,c15) | A(c14,x) | −A(x,y) | −A(y,c15). [resolve(15752,f,33,c)].
21809 c7 = x | A(c7,x) | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(c8,x).
    [resolve(1388,g,1392,f),merge(h),merge(i),merge(j),merge(k),merge(l)].
25450 c13 = c8 | B(c8,c13) | c14 = c8 | B(c14,c8) | c14 = c7 | B(c14,c7) | A(c13,c8). [resolve(12555,b,30,a)].
28494 c7 = x | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(c8,x) | A(x,c7). [resolve(21809,b,30,a)].
32464 c7 = x | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(c8,x) | A(c8,x).
    [resolve(28494,g,236,a),merge(g),merge(i)].
32466 c7 = x | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(c8,x) | −A(c8,x). [resolve(28494,g,196,b)].
34678 c7 = x | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(c8,x).
    [resolve(32466,g,32464,g),merge(g),merge(h),merge(i),merge(j),merge(k),merge(l)].
34766 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x) | A(c7,x). [resolve(34678,d,266,a),merge(f),merge(h)].
34812 c7 = x | B(c7,x) | c9 = x | B(x,c9) | c8 = x | B(x,c8). [resolve(34678,f,31,a)].
34911 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x) | A(x,c7). [resolve(34766,f,30,a)].
34968 c7 = x | c9 = x | B(x,c9) | c8 = x | B(x,c8) | B(x,c7). [resolve(34812,b,31,a)].
35062 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x) | A(c8,x). [resolve(34911,f,236,a),merge(f),merge(h)].
35064 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x) | −A(c8,x). [resolve(34911,f,196,b)].
35115 c7 = x | c9 = x | c8 = x | B(x,c8) | B(x,c7) | A(x,c7). [resolve(34968,c,254,a),merge(f),merge(h)].
35290 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x) | A(x,c8). [resolve(35062,f,30,a)].
35291 c7 = x | B(c7,x) | c9 = x | c8 = x | B(c8,x).
    [resolve(35064,f,35062,f),merge(f),merge(g),merge(h),merge(i),merge(j)].
35292 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | A(c13,c8) | B(c13,c8).
    [resolve(35291,e,354,a), flip(a), flip(c), flip(d)].
35293 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | A(c15,c8) | B(c15,c8).
    [resolve(35291,e,352,a), flip(a), flip(c), flip(d)].
35306 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | A(c8,c15) | B(c8,c15).
    [resolve(35291,e,279,a), flip(a), flip(c), flip(d)].
35330 c7 = x | B(c7,x) | c9 = x | c8 = x | B(x,c8). [resolve(35291,e,31,a)].
35332 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | A(c15,c7) | B(c15,c7).
    [resolve(35330,b,352,a), flip(a), flip(b), flip(c)].
35338 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | A(c14,c7) | B(c14,c7).
    [resolve(35330,b,315,a), flip(a), flip(b), flip(c)].
35369 c7 = x | c9 = x | c8 = x | B(x,c8) | B(x,c7). [resolve(35330,b,31,a)].
35375 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | −B(c13,c8). [resolve(35330,e,150,a), flip(a), flip(c), flip(d)].
35404 c7 = x | c9 = x | c8 = x | B(x,c7) | B(c8,x). [resolve(35369,d,31,a)].
35573 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | A(c13,c7) | B(c13,c7).
    [resolve(35404,d,353,a), flip(a), flip(b), flip(c)].
35574 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c7 | A(c15,c7) | B(c15,c7).
    [resolve(35404,d,351,a), flip(a), flip(b), flip(c)].
35590 c7 = x | c9 = x | c8 = x | B(c8,x) | A(x,c9) | B(x,c9). [resolve(35404,d,248,a),merge(e)].
35596 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | −B(c13,c7). [resolve(35404,d,150,a), flip(a), flip(b), flip(c)].
35856 c7 = x | c9 = x | c8 = x | B(c8,x) | B(x,c9) | A(c8,x). [resolve(35590,e,237,a),merge(f),merge(h)].
35860 c7 = x | c9 = x | c8 = x | B(c8,x) | B(x,c9) | −A(c8,x). [resolve(35590,e,194,b)].
35943 c7 = x | c9 = x | c8 = x | B(c8,x) | B(x,c9) | −A(x,y) | −A(c8,y). [resolve(35856,f,33,a)].
35945 c7 = x | c9 = x | c8 = x | B(c8,x) | B(x,c9).
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    [resolve(35860,f,35856,f),merge(f),merge(g),merge(h),merge(i),merge(j)].
35984 c7 = x | c9 = x | c8 = x | B(x,c9) | −B(c7,x).  [resolve(35945,d,96,a)].
36008 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | A(c13,c9) | B(c13,c9).
    [resolve(35945,e,353,a), flip(a), flip(b), flip(c)].
36009 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | A(c15,c9) | B(c15,c9).
    [resolve(35945,e,351,a), flip(a), flip(b), flip(c)].
36022 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | A(c9,c15) | B(c9,c15).
    [resolve(35945,e,269,a), flip(a), flip(b), flip(c)].
36029 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | −B(c13,c9).  [resolve(35945,e,150,a), flip(a), flip(b), flip(c)].
36631 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | B(c13,c8) | −A(c8,c15).
    [resolve(35292,f,21660,f),unit_del(g,133),unit_del(h,134),unit_del(i,124),unit_del(j,123),unit_del(k,208)].
36643 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | B(c15,c8) | −A(c14,c8).  [resolve(35293,f,126,b)].
36955 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c15,c7) | −A(c14,c7).  [resolve(35332,f,126,b)].
36998 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c14,c7) | −A(c13,c7).  [resolve(35338,f,131,a)].
37331 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | B(c13,c7) | −A(c15,c7).
    [resolve(35573,f,19027,g),unit_del(g,133),unit_del(h,216),unit_del(i,124),unit_del(j,166),unit_del(k,135)].
37962 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | B(c13,c9) | −A(c9,c15).
    [resolve(36008,f,21660,f),unit_del(g,133),unit_del(h,134),unit_del(i,124),unit_del(j,123),unit_del(k,208)].
37974 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | B(c15,c9) | −A(c14,c9).  [resolve(36009,f,126,b)].
38710 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | B(c13,c8) | c15 = c8 | B(c8,c15).
    [resolve(36631,g,35306,f),merge(g),merge(h),merge(i),merge(j)].
38716 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | B(c15,c8) | B(c8,c14).
    [resolve(36643,g,35290,f), flip(g), flip(i), flip(j),merge(g),merge(h),merge(i),merge(j)].
38819 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c15,c7) | B(c14,c7).
    [resolve(36955,g,35115,f), flip(g), flip(h), flip(i),merge(g),merge(h),merge(i),merge(j)].
38837 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c14,c7) | B(c13,c7).
    [resolve(36998,g,35115,f), flip(g), flip(h), flip(i),merge(g),merge(h),merge(i),merge(j)].
38961 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | B(c13,c7) | c15 = c7 | B(c15,c7).
    [resolve(37331,g,35574,f),merge(g),merge(h),merge(i),merge(j)].
39267 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | B(c13,c9) | c15 = c9 | B(c9,c15).
    [resolve(37962,g,36022,f),merge(g),merge(h),merge(i),merge(j)].
39275 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | B(c15,c9) | B(c14,c9).
    [resolve(37974,g,35590,e), flip(g), flip(h), flip(i),merge(g),merge(h),merge(i),merge(j)].
39709 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | B(c8,c14) | A(c8,c14).
    [resolve(38716,f,274,a),merge(g),merge(i)].
39724 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c14,c7) | A(c14,c7).
    [resolve(38819,f,324,a),merge(g),merge(i)].
39732 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c13,c7) | A(c13,c7).
    [resolve(38837,f,353,a),merge(g),merge(i)].
39874 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | B(c14,c9) | A(c14,c9).
    [resolve(39275,f,324,a),merge(g),merge(i)].
39936 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | B(c8,c14).
    [resolve(39709,g,35064,f), flip(g), flip(i), flip(j),merge(g),merge(h),merge(i),merge(j),merge(k)].
39937 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c15 = c8 | B(c14,c8).  [resolve(39936,f,31,a)].
39960 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c14,c7) | A(c14,c8).
    [resolve(39724,g,226,a), flip(g),merge(g),merge(i)].
39998 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c13,c7) | A(c13,c8).
    [resolve(39732,g,226,a), flip(g),merge(g),merge(i)].
40161 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c15 = c9 | B(c14,c9).
    [resolve(39874,g,35943,f), flip(g), flip(h), flip(i),merge(g),merge(h),
                                        merge(i),merge(j),merge(k),unit_del(g,81)].
40291 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c14,c7) | −A(c14,c7).  [resolve(39960,g,86,b)].
40327 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c13,c7) | −A(c13,c7).  [resolve(39998,g,86,b)].
40590 c14 = c7 | c14 = c9 | c14 = c8 | B(c14,c8) | c15 = c7 | B(c14,c7).
    [resolve(40291,g,39724,g),merge(g),merge(h),merge(i),merge(j),merge(k),merge(l)].
40591 c14 = c7 | c14 = c9 | c14 = c8 | c15 = c7 | B(c14,c7) | B(c8,c14).  [resolve(40590,d,31,a)].
40609 c13 = c7 | c13 = c9 | c13 = c8 | B(c13,c8) | c14 = c7 | B(c13,c7).
    [resolve(40327,g,39732,g),merge(g),merge(h),merge(i),merge(j),merge(k),merge(l)].
51904 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | B(c8,c15).
    [resolve(38710,f,35375,e),merge(h),merge(i),merge(j),merge(k)].
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51906 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | B(c15,c8). [resolve(51904,g,31,a)].
51908 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | −B(c14,c8). [resolve(51906,g,151,b)].
51912 c14 = c7 | B(c7,c14) | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8.
    [resolve(51908,g,39937,f),merge(g),merge(h),merge(i),merge(j),merge(k)].
51913 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | B(c14,c9).
    [resolve(51912,b,35984,e), flip(f), flip(g), flip(h),merge(f),merge(g),merge(h)].
51926 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | B(c9,c14). [resolve(51913,f,31,a)].
51930 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c8 | c15 = c8 | −B(c8,c14). [resolve(51926,f,97,b)].
51944 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | B(c13,c7) | c15 = c7 | −B(c14,c7).
    [resolve(38961,h,151,b)].
52022 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | c15 = c9 | B(c9,c15).
    [resolve(39267,f,36029,e),merge(h),merge(i),merge(j),merge(k)].
52024 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | c15 = c9 | B(c15,c9). [resolve(52022,g,31,a)].
52028 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | c15 = c9 | −B(c14,c9). [resolve(52024,g,151,b)].
52031 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c9 | c15 = c9.
    [resolve(52028,g,40161,f),merge(g),merge(h),merge(i),merge(j),merge(k)].
52032 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c15 = c9 | c13 = c8 | c15 = c8.
    [resolve(52031,d,51930,f),merge(f),merge(g),merge(h)].
52051 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c13 = c8 | c15 = c8 | −B(c9,c14). [para(52032(e,1),129(a,1))].
52082 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c13 = c8 | c15 = c8.
    [resolve(52051,g,51926,f),merge(g),merge(h),merge(i),merge(j),merge(k)].
52089 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c13 = c8 | −B(c8,c14). [para(52082(f,1),129(a,1))].
52147 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c13 = c8 | c15 = c9.
    [resolve(52089,f,52031,d),merge(f),merge(g),merge(h),merge(i)].
52186 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c9 | c13 = c8.
    [para(52147(f,1),52082(f,1)),merge(f),merge(g),merge(h),merge(i),merge(j),unit_del(f,83)].
52204 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | −A(c9,x) | −A(c13,x). [para(52186(b,1),131(a,1))].
52296 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c8,c13) | B(c14,c8) | B(c14,c7).
    [resolve(52204,f,25450,g),merge(f),merge(h),merge(j),unit_del(e,35)].
52312 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c8,c13) | B(c14,c7).
    [para(52186(b,1),52296(f,1)),merge(e),merge(f),merge(g),merge(h),unit_del(f,82)].
52316 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c8,c13) | B(c7,c14). [resolve(52312,f,31,a)].
52317 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c7,c14) | A(c14,c8) | B(c14,c8).
    [resolve(52316,e,315,a),merge(f)].
52347 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c7,c14) | B(c14,c8) | A(c7,c14).
    [resolve(52317,f,241,a), flip(g),merge(g),merge(i)].
52381 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c7,c14) | B(c14,c8).
    [para(52186(b,1),52347(g,2)),merge(e),merge(f),merge(g),merge(h),unit_del(g,77)].
52386 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c7,c14).
    [para(52186(b,1),52381(f,1)),merge(e),merge(f),merge(g),merge(h),unit_del(f,82)].
52393 c14 = c7 | c14 = c8 | c13 = c9 | c13 = c8 | B(c14,c7). [resolve(52386,e,31,a)].
60173 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | B(c13,c7) | c15 = c7.
    [resolve(51944,h,40591,e),merge(h),merge(i),merge(j),merge(k),merge(l)].
60179 c14 = c7 | c14 = c9 | c14 = c8 | B(c8,c14) | c13 = c7 | c15 = c7.
    [resolve(60173,f,35596,e),merge(g),merge(h),merge(i),merge(j)].
60180 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c15 = c7 | c13 = c8 | c15 = c8.
    [resolve(60179,d,51930,f),merge(f),merge(g),merge(h)].
60198 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c15 = c7 | c13 = c8 | −B(c8,c14). [para(60180(g,1),129(a,1))].
60239 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c15 = c7 | c13 = c8.
    [resolve(60198,g,60179,d),merge(g),merge(h),merge(i),merge(j),merge(k)].
60248 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c13 = c8 | −B(c7,c14). [para(60239(e,1),129(a,1))].
60338 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c13 = c8 | c15 = c8.
    [resolve(60248,f,51912,b),merge(f),merge(g),merge(h),merge(i)].
60381 c14 = c7 | c14 = c9 | c14 = c8 | c13 = c7 | c13 = c8.
    [para(60338(f,1),60239(e,1)),merge(f),merge(g),merge(h),merge(i),merge(k),unit_del(f,90)].
60382 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | C(c15,c9). [para(60381(b,1),43(a,2))].
60393 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | −B(c9,x) | c13 = x | A(x,c13) | B(x,c13).
    [para(60381(b,1),268(a,1))].
60400 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | −A(c9,x) | −A(c13,x). [para(60381(b,1),131(a,1))].
60422 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | −B(c9,x) | c13 = x | A(c13,x) | B(c13,x).
```

```
    [para(60381(b,1),353(a,1))].
60480 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | A(c7,c13) | B(c7,c13).  [factor(60393,c,f),unit_del(e,74)].
60481 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | A(c13,c7) | B(c13,c7).  [factor(60422,c,f),unit_del(e,74)].
60494 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | −B(c9,x) | −B(c15,x).  [resolve(60382,e,48,a)].
60525 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c8,c13) | B(c14,c8) | B(c14,c7).
    [resolve(60400,f,25450,g),merge(f),merge(h),merge(j),unit_del(e,35)].
60549 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c7,c13) | A(c13,c7).  [resolve(60480,e,30,a)].
60552 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c13,c7) | A(c8,c13) | B(c8,c13).
    [resolve(60481,e,236,a), flip(f),merge(f)].
60554 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c13,c7) | −A(c8,c13).  [resolve(60481,e,196,b)].
60576 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c7,c13) | −A(c13,x) | −A(x,c7).  [resolve(60549,f,33,c)].
60604 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c8,c13) | B(c14,c7).
    [para(60381(b,1),60525(f,1)),merge(e),merge(f),merge(g),merge(h),unit_del(f,82)].
60609 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c8,c13) | −B(c13,c7).  [resolve(60604,f,150,a)].
60622 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c13,c7) | B(c8,c13).
    [resolve(60552,f,60554,f),merge(g),merge(h),merge(i),merge(j),merge(k)].
60624 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c8,c13).
    [resolve(60622,e,60609,f),merge(f),merge(g),merge(h),merge(i),merge(j)].
60625 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | A(c14,c8) | B(c14,c8).  [resolve(60624,e,315,a),merge(e)].
60629 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | −B(c7,c13).  [resolve(60624,e,96,a)].
60643 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c14,c8) | A(c7,c14) | B(c7,c14).
    [resolve(60625,e,241,a), flip(f),merge(f)].
60649 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c14,c8) | −A(c7,c14).  [resolve(60625,e,85,b)].
60678 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c14,c8) | B(c7,c14).
    [resolve(60643,f,60649,f),merge(g),merge(h),merge(i),merge(j),merge(k)].
60685 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | B(c7,c14).
    [para(60381(b,1),60678(e,1)),merge(e),merge(f),merge(g),merge(h),unit_del(e,82)].
60686 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | c15 = c7 | A(c15,c7) | B(c15,c7).  [resolve(60685,e,352,a)].
60694 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | c15 = c7 | B(c15,c7) | B(c7,c13).
    [resolve(60686,f,60576,g),merge(g),merge(h),merge(i),merge(j),unit_del(h,42)].
60717 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | c15 = c7 | B(c7,c13).
    [resolve(60694,f,60494,f),merge(g),merge(h),merge(i),merge(j),unit_del(g,74)].
60718 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8 | c15 = c7.
    [resolve(60717,f,60629,e),merge(f),merge(g),merge(h),merge(i)].
60751 c14 = c7 | c14 = c8 | c13 = c7 | c13 = c8.
    [para(60718(e,1),60382(e,1)),merge(e),merge(f),merge(g),merge(h),unit_del(e,76)].
60752 c14 = c7 | c13 = c7 | c13 = c8 | C(c15,c8).  [para(60751(b,1),43(a,2))].
60753 c14 = c7 | c13 = c7 | c13 = c8 | C(c13,c8).  [para(60751(b,1),44(a,2))].
60756 c14 = c7 | c13 = c7 | c13 = c8 | c15 != c8.  [para(60751(b,1),128(a,2))].
60757 c14 = c7 | c13 = c7 | c13 = c8 | −B(c15,c8).  [para(60751(b,1),129(a,2))].
60759 c14 = c7 | c13 = c7 | c13 = c8 | −B(c13,c8).  [para(60751(b,1),134(a,2))].
61279 c14 = c7 | c13 = c7 | c13 = c8 | c15 = c7 | c15 = c9 | c15 = c8 | B(c15,c7).
    [resolve(60757,d,35369,d), flip(d), flip(e), flip(f)].
61283 c14 = c7 | c13 = c7 | c13 = c8 | c13 = c9 | B(c13,c7).
    [resolve(60759,d,40609,d),merge(d),merge(f),merge(g)].
61290 c14 = c7 | c13 = c7 | c13 = c8 | c13 = c9 | −B(c15,c7).
    [resolve(61283,e,19024,g),unit_del(e,133),unit_del(f,216),unit_del(g,124),unit_del(h,166),unit_del(i,135)].
61325 c14 = c7 | c13 = c7 | c13 = c8 | c15 = c7 | c15 = c9 | c15 = c8 | c13 = c9.
    [resolve(61279,g,61290,e),merge(g),merge(h),merge(i)].
61372 c14 = c7 | c13 = c7 | c13 = c8 | c15 = c7 | c15 = c8 | c13 = c9.
    [para(61325(e,1),60752(d,1)),merge(g),merge(h),merge(i),unit_del(g,84)].
61373 c14 = c7 | c13 = c7 | c13 = c8 | c15 = c7 | c13 = c9.
    [resolve(61372,e,60756,d),merge(f),merge(g),merge(h)].
61411 c14 = c7 | c13 = c7 | c13 = c8 | c13 = c9.
    [para(61373(d,1),60752(d,1)),merge(e),merge(f),merge(g),unit_del(e,92)].
61412 c13 = c7 | c13 = c8 | c13 = c9 | C(c15,c7).  [para(61411(a,1),43(a,2))].
61423 c13 = c7 | c13 = c8 | c13 = c9 | −B(c7,x) | c15 = x | A(x,c15) | B(x,c15).  [para(61411(a,1),269(a,1))].
61430 c13 = c7 | c13 = c8 | c13 = c9 | −B(c7,x) | −B(c13,x).  [para(61411(a,1),150(a,1))].
61451 c13 = c7 | c13 = c8 | c13 = c9 | −B(c7,x) | c13 = x | A(c13,x) | B(c13,x).  [para(61411(a,1),353(a,1))].
61489 c13 = c7 | c13 = c8 | c13 = c9 | A(c13,c9) | B(c13,c9).  [factor(61451,c,e),unit_del(d,34)].
```

```
61502 c13 = c7 | c13 = c8 | c13 = c9 | -B(c7,x) | -B(c15,x). [resolve(61412,d,48,a)].
61598 c13 = c7 | c13 = c8 | c13 = c9 | B(c13,c9) | A(c9,c13). [resolve(61489,d,30,a)].
61614 c13 = c7 | c13 = c8 | c13 = c9 | B(c13,c9) | -A(c9,x) | -A(x,c13). [resolve(61598,e,33,c)].
61641 c13 = c7 | c13 = c8 | c13 = c9 | c15 = c9 | A(c9,c15) | B(c9,c15). [resolve(61423,d,34,a)].
61642 c13 = c7 | c13 = c8 | c13 = c9 | c15 = c9 | B(c9,c15) | B(c13,c9).
    [resolve(61641,e,61614,e),merge(f),merge(g),merge(h),unit_del(g,122)].
61653 c13 = c7 | c13 = c8 | c13 = c9 | c15 = c9 | B(c9,c15).
    [resolve(61642,f,61430,e),merge(f),merge(g),merge(h),unit_del(f,34)].
61659 c13 = c7 | c13 = c8 | c13 = c9 | c15 = c9 | B(c15,c9). [resolve(61653,e,31,a)].
61660 c13 = c7 | c13 = c8 | c13 = c9 | c15 = c9.
    [resolve(61659,e,61502,e),merge(e),merge(f),merge(g),unit_del(e,34)].
61697 c13 = c7 | c13 = c8 | c13 = c9. [para(61660(d,1),61412(d,1)),merge(d),merge(e),merge(f),unit_del(d,95)].
61699 c13 = c7 | c13 = c8 | C(c9,c14). [para(61697(c,1),44(a,1))].
62257 c13 = c7 | c13 = c8 | c14 = c7. [para(61697(c,1),60753(d,1)),merge(d),merge(e),unit_del(d,84)].
63013 c13 = c7 | c13 = c8. [para(62257(c,1),61699(c,2)),merge(c),merge(d),unit_del(c,95)].
63015 c13 = c7 | C(c8,c14). [para(63013(b,1),44(a,1))].
63016 c13 = c7 | A(c15,c8). [para(63013(b,1),122(a,2))].
63023 c13 = c7 | c14 != c8. [para(63013(b,1),133(a,2))].
63024 c13 = c7 | -B(c8,c14). [para(63013(b,1),134(a,1))].
64794 c13 = c7 | -A(c14,x) | -A(c8,x). [resolve(63015,b,37,a)].
64955 c13 = c7 | c15 = c9 | A(c9,c15) | B(c9,c15). [resolve(63016,b,242,a),flip(b)].
64959 c13 = c7 | -A(c9,c15). [resolve(63016,b,78,b)].
64968 c13 = c7 | c14 = c7 | c14 = c9 | c14 = c8 | c15 = c7. [resolve(63024,b,60179,d),merge(e)].
65086 c13 = c7 | c15 = c9 | B(c9,c15). [resolve(64955,c,64959,b),merge(d)].
65208 c13 = c7 | c15 = c9 | c14 = c9 | A(c14,c9) | B(c14,c9). [resolve(65086,c,316,a)].
65215 c13 = c7 | c15 = c9 | B(c15,c9). [resolve(65086,c,31,a)].
65339 c13 = c7 | c15 = c9 | -B(c14,c9). [resolve(65215,c,151,b)].
65685 c13 = c7 | c15 = c9 | c14 = c9 | B(c14,c9). [resolve(65208,d,64794,b),merge(e),unit_del(e,81)].
65686 c13 = c7 | c15 = c9 | c14 = c9. [resolve(65685,d,65339,c),merge(d),merge(e)].
66139 c13 = c7 | c14 = c9 | c14 = c7 | c14 = c8. [para(65686(b,1),64968(e,1)),merge(c),merge(e),unit_del(e,75)].
66545 c13 = c7 | c14 = c7 | c14 = c8. [para(66139(b,1),63015(b,2)),merge(d),unit_del(d,195)].
66546 c13 = c7 | c14 = c7. [resolve(66545,c,63023,b),merge(c)].
67556 c13 = c7. [para(66546(b,1),63015(b,2)),merge(b),unit_del(b,197)].
69312 c14 = c7 | c14 = c8 | B(c14,c7).
    [back_rewrite(52393),rewrite([67556(7),67556(10)]),flip(c),flip(d),unit_del(c,75),unit_del(d,90)].
74868 -B(c14,c7). [back_rewrite(216),rewrite([67556(2)])].
74884 c14 != c7. [back_rewrite(133),rewrite([67556(2)])].
74895 C(c7,c14). [back_rewrite(44),rewrite([67556(1)])].
74902 c14 = c8. [back_unit_del(69312),unit_del(a,74884),unit_del(c,74868)].
74915 $F. [back_rewrite(74895),rewrite([74902(2)]),unit_del(a,92)].

==================== end of proof ====================
```

C.5 Python code for generating reduct signatures

Generates the signatures that are reducts of the signature of Tarski's relation algebras along with domain and range operators. 487 unique signatures are generated, 239 of which include composition. Without domain and range operators there are 200 unique signatures, 100 of which include composition.

The full signature used here is the Tarski signature (see Definition 1.1.3) along with the partial order $\leqslant$ and domain and range operators D and R. Thus, the full signature considered is

$$(;, +, \cdot, \leqslant, \breve{}, -, 1', 0, 1, \mathrm{D}, \mathrm{R})$$

In particular, the order in which these operations are displayed in each signature is maintained through the `sigSort` method.

A signature may be capable of expressing operations not explicitly listed. For example, any signature containing a lattice operation $\vee$ or $\wedge$ is equipped with a partial order inherited from that operation. In order to avoid redundancy in the full list, each signature is "completed" according to the rules expressed in Table 1.14. This task is accomplished by the `complete` method.

```
import itertools

# The full signature under consideration. Includes composition, lattice
# operations, a partial order, negation, converse, constants, and domain
# and range operators.
fullsig = [ "comp", "join", "meet", "le", "con", "-", "id", "0", "top",
    "dom", "ran" ]

# Defines a canonical order on the operations of the signature for aesthetic
# reasons.
# (cdot, lor, land, le, -, con, id, 0, top, dom, ran)
sigKey = {"comp" : 0, "join" : 1, "meet" : 2, "le": 3, "con" : 4, "-" : 5,
    "id" : 6, "0" : 7, "top" : 8, "dom" : 9, "ran" : 10 }

# Return all nonempty subsets of a set S.
def findSubLists(S):
    allSubLists = []
    for i in range(0,len(S)+1):
        subList = list(itertools.combinations(S, i))
        subList = [list(j) for j in subList]
        allSubLists += subList
    return allSubLists

# Returns True if every element of the list A is in the list B.
def isSublist(A, B):
    isContained = True
    for a in A:
        if a not in B:
            isContained = False
            break
    return isContained
```

```
# A sorting key used to order the operations in a signature according to the
# canonical order defined above.
def sigSort(x,y):
    if sigKey[x] > sigKey[y]:
        return 1
    elif sigKey[x] == sigKey[y]:
        return 0
    else:
        return -1

# Create a copy of a signature, given as a list, and define new operations
    in
# the signature that can be derived from others. For example, a signature
# capable of expressing join and - is also capable of expressing meet.
def complete(S):
    T  = list(S)
    # Standard operations
    if ("join" in S) or ("meet" in S):
        T.append("le")
    if ("-" in S) and ("join" in S):
        T.append("meet")
    if ("-" in S) and ("meet" in S):
        T.append("join")
    if (("0" in S) and ("-" in S)) or (("join" in S) and ("-" in S)):
        T.append("top")
    if (("top" in S) and ("-" in S)) or (("join" in S) and ("-" in S)):
        T.append("0")
    # Domain and range
    if ("con" in S) and ("ran" in S):
        T.append("dom")
    if ("con" in S) and ("dom" in S):
        T.append("ran")
    if (("top" in S) and ("dom" in S)) or (("top" in S) and ("ran" in S)):
        T.append("id")
    if ("meet" in S) and ("id" in S) and ("con" in S) and ("comp" in S):
        T.append("dom")
        T.append("ran")
    # Remove duplicate operations in the signature, and sort according to the
    # canonical order.
    T = list(set(T))
    T.sort(sigSort)
```

```
    # If no operations have been added, then the signature is already
        complete,
    # so return it. If not, try to complete again.
    if T == S:
        return T
    else:
        return complete(T)

# Generate a list of reducts of the full signature that contain at least the
# minimumSignature. For example, one might be interested in all
# reducts equipped with composition ("comp").
def generateSignatures(minimumSignature = []):
    signatures = []
    # Consider the operations in the full signature that are not in the
        minimum
    # signature. That is, consider the optional operations.
    optionalOperations = [operation for operation in fullsig if operation
        not in minimumSignature]
    # Consider the union of the minimum required signature with every
        possible
    # reduct of the signature of optional operations, and complete as above.
    # Sort by the canonical order, and then add to the output if not already
    # included.
    for partialSignature in findSubLists(optionalOperations):
        S = minimumSignature + partialSignature
        S = complete(S)
        S.sort(sigSort)
        if S not in signatures:
            signatures.append(S)
    signatures.sort(key = len)
    return signatures
```

Bibliography

[1] J. F. Allen. Maintaining knowledge about temporal intervals. *Commun. ACM*, 26(11):832–843, Nov 1983. 11, 62, 70

[2] H. Andréka. On the representation problem of distributive semilattice-ordered semigroups. Technical report, Mathematical Institute of the Hungarian Academy of Sciences, Mar 1988. 21

[3] H. Andréka. Representations of distributive lattice-ordered semigroups with binary relations. *Algebra Universalis*, 28(1):12–25, 1991. 29

[4] H. Andréka and R. D. Maddux. Representations for small relation algebras. *Notre Dame J. Formal Logic*, 35(4):550–562, 1994. 74, 76, 84

[5] H. Andréka and S. Mikulás. Axiomatizability of positive algebras of binary relations. *Algebra Universalis*, 66(1-2):7–34, Aug 2011. 21, 29, 39

[6] F. Backer. Representable relation algebras. Report for a seminar on relation algebras conducted by A. Tarski, mimeographed, University of California, Berkeley, Spring, 1970. 74

[7] M. Bodirsky. Complexity Classification in Infinite-Domain Constraint Satisfaction. *ArXiv e-prints*, Jan 2012, 1201.0856. 70

[8] G. Boole. *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities.* Walton and Maberly, London, 1854. 15

[9] G. Boolos, J. Burgess, and R. Jeffrey. *Computability and Logic.* Cambridge University Press, fourth edition, 2002. 19

[10] D. A. Bredikhin. An abstract characterization of some classes of algebras of binary relations. In *Algebra and number theory, No. 2 (Russian),* 3–19. Kabardino-Balkarsk. Gos. Univ., Nalchik, 1977. 21

[11] D. A. Bredikhin. On reducts of Tarski relation algebras. *Algebra i Logika*, 37(1):3–16,122, 1998. 21, 39

[12] D. A. Bredikhin and B. M. Schein. Representations of ordered semigroups and lattices by binary relations. *Colloq. Math.*, 39(1):1–12, 1978. 39

[13] S. Burris and H. Sankappanavar. A course in universal algebra. https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html. Online version of [14]. 9, 44

[14] S. Burris and H. Sankappanavar. *A Course in Universal Algebra*, volume 78 of *Graduate texts in mathematics*. Springer-Verlag, New York, 1981. 252

[15] P. J. Cameron. The random graph. In *The mathematics of Paul Erdős, II,* 333–351, volume 14 of *Algorithms Combin.* Springer, Berlin, 1997. 99

[16] A. Cayley. On the theory of groups as depending on the symbolic equation $\theta^n = 1$. *Philosophical Magazine*, 7(42):40–47, 1854. 3

[17] A. G. Cohn and S. M. Hazarika. Qualitative spatial representation and reasoning: An overview. *Fundam. Inf.*, 46(1-2):1–29, Jan 2001. 63

[18] J. Corcoran. Aristotle's Prior Analytics and Boole's Laws of Thought. *Hist. Philos. Logic*, 24(4):261–288, 2003. A Festschrift in honour of Professor Ivor Grattan-Guinness. 15

[19] M. Cristani and R. Hirsch. The complexity of constraint satisfaction problems for small relation algebras. *Artificial Intelligence*, 156(2):177–196, 2004. 69, 70, 71, 75, 76, 80, 81, 89, 90, 95, 97, 98, 99

[20] A. De Morgan. On the syllogism, no. iv. and on the logic of relations. *Transactions of the Cambridge Philosophical Society.*, 10(2):331–358, 1864. 15

[21] A. De Morgan. *"On the syllogism" and other logical writings.* Edited, with an introduction, by Peter Heath. Yale University Press, New Haven, Conn., 1966. 6

[22] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer, Heidelberg, fourth edition, 2010. 84

[23] W. Dyck. Gruppentheoretische studien. *Mathematische Annalen*, 20(1):1–44, 1882. 2

[24] T. Evans. Embeddability and the word problem. *J. Lond. Math. Soc.*, 28:76–80, 1953. 25

[25] R. Fraïssé. Sur l'extension aux relations de quelques propriétés des ordres. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:363–388, 1954. 85

[26] J. A. Gallian. *Contemporary Abstract Algebra.* Cengage Learning, seventh edition, 2009. 3

[27] V. S. Garvac′kiĭ. ∩-semigroups of transformations. In *Theory of semigroups and its applications, No. 2 (Russian),* 3–13. Izdat. Saratov. Univ., Saratov, 1971. 58

[28] S. Givant. The calculus of relations as a foundation for mathematics. *Journal of Automated Reasoning*, 37(4):277–322, 2006. 17

[29] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951. 47

[30] M. Hanheide, A. Peters, and N. Bellotto. Analysis of human-robot spatial behaviour applying a qualitative trajectory calculus. In *The 21st IEEE International Symposium on Robot and Human Interactive Communication, IEEE RO-MAN 2012, Paris, France, September 9-13, 2012,* 689–694. IEEE, 2012. 62

[31] J. V. Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, 1967. 255

[32] R. Hirsch. A finite relation algebra with undecidable network satisfaction problem. *Logic Journal of IGPL*, 7(4):547–554, 1999, http://jigpal.oxfordjournals.org/content/7/4/547.full.pdf+html. 72

[33] R. Hirsch. The class of representable ordered monoids has a recursively enumerable, universal axiomatisation but it is not finitely axiomatisable. *Logic Journal of IGPL*, 13(2):159–171, Mar 2005. 8, 21, 39, 40, 41

[34] R. Hirsch. The finite representation property for reducts of relation algebra, 2011. Manuscript. http://www.cs.ucl.ac.uk/fileadmin/UCL-CS/staff/Robin_Hirsch/Papers/domain_fin.pdf. 18, 21

[35] R. Hirsch and I. Hodkinson. Representability is not decidable for finite relation algebras. *Trans. Amer. Math. Soc.*, 353(4):1403–1425, 2001. 14, 21, 39

[36] R. Hirsch and I. Hodkinson. *Relation algebras by games*, volume 147 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 2002. 4, 7, 13, 39

[37] R. Hirsch and M. Jackson. Undecidability of representability as binary relations. *J. Symbolic Logic*, 77(4):1211–1244, Dec 2012. 21, 23, 24, 25, 26, 27

[38] R. Hirsch, M. Jackson, and T. Kowalski. Algebraic foundations for qualitative calculi and networks. *ArXiv e-prints*, Jun 2016, 1606.09140. 61, 71, 72

[39] R. Hirsch and S. Mikulás. Representable semilattice-ordered monoids. *Algebra Universalis*, 57(3):333–370, Nov 2007. 21

[40] W. Hodges. *A Shorter Model Theory.* Cambridge University Press, 1997. 85

[41] J. M. Howie. *Fundamentals of Semigroup Theory*, volume 12 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications. 47

[42] E. V. Huntington. Boolean algebra. A correction to: "New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's *Principia Mathematica*" [43]. *Trans. Amer. Math. Soc.*, 35(2):557–558, 1933. 15

[43] E. V. Huntington. New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's *Principia Mathematica. Trans. Amer. Math. Soc.*, 35(1):274–304, 1933. 15, 254

[44] E. V. Huntington. A second correction to: "New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's *Principia Mathematica*" [43]. *Trans. Amer. Math. Soc.*, 35(4):971, 1933. 15

[45] B. Jónsson and A. Tarski. Representation problems for relation algebras. *Bull. Amer. Math. Soc.*, 54(1):80, 1192, 1948. 12, 17

[46] B. Jónsson and A. Tarski. Boolean algebras with operators II. *American Journal of Mathematics*, 74(1):pp. 127–162, 1952. 61

[47] C. Jordan. *Traité des substitutions et des équations algébriques.* Gauthier-Villars, 1870. 3

[48] J. H. Lee, J. Renz, and D. Wolter. StarVars - Effective reasoning about relative directions. In F. Rossi, editor, *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013.* IJCAI/AAAI, 2013. 67

[49] G. Ligozat. Weak representations of interval algebras. In H. E. Shrobe, T. G. Dietterich, and W. R. Swartout, editors, *Proceedings of the 8th National Conference on Artificial Intelligence. Boston, Massachusetts, July 29 - August 3, 1990, 2 Volumes,* 715–720. AAAI Press / The MIT Press, 1990. 65

[50] G. Ligozat. *Qualitative Spatial and Temporal Reasoning*. ISTE. Wiley, 2013. 65, 66

[51] G. Ligozat and J. Renz. What is a qualitative calculus? A general framework. In C. Zhang, H. W. Guesgen, and W.-K. Yeap, editors, *PRICAI 2004: Trends in Artificial Intelligence,* 53–64, volume 3157 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2004. 61, 66

[52] L. Löwenheim. Über Möglichkeiten im Relativkalkül. *Math. Ann.*, 76(4):447–470, 1915. English translation, "On possibilities in the calculus of relatives" in [31, pp. 228–251]. 16

[53] L. Löwenheim. Einkleidung der Mathematik in Schröderschen Relativkalkul. *J. Symbolic Logic*, 5:1–15, 1940. 16

[54] R. C. Lyndon. The representation of relational algebras. *Ann. of Math. (2)*, 51:707–729, 1950. 12

[55] R. C. Lyndon. The representation of relation algebras. II. *Ann. of Math. (2)*, 63:294–307, 1956. 74

[56] R. D. Maddux. Some varieties containing relation algebras. *Trans. Amer. Math. Soc.*, 272(2):501–526, 1982. 66

[57] R. D. Maddux. The origin of relation algebras in the development and axiomatization of the calculus of relations. *Studia Logica*, 50(3-4):421–455, 1991. 16

[58] R. D. Maddux. *Relation Algebras*, volume 150 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2006. 5, 6, 7, 63, 74, 76, 102, 109, 234

[59] N. Mavridis, N. Bellotto, K. Iliopoulos, and N. Van de Weghe. QTC3D: extending the qualitative trajectory calculus to three dimensions. *CoRR*, abs/1402.3779, 2014. 62

[60] W. McCune. Prover9 and Mace4. http://www.cs.unm.edu/~mccune/prover9/, 2005–2010. 108, 240

[61] R. McKenzie. *The representation of relation algebras*. PhD thesis, University of Colorado, Boulder, 1966. 74

[62] R. McKenzie. Representations of integral relation algebras. *Michigan Math. J.*, 17:279–287, 1970. 12

[63] R. McKenzie and B. M. Schein. Every semigroup is isomorphic to a transitive

semigroup of binary relations. *Trans. Amer. Math. Soc.*, 349(1):271–285, 1997. 51

[64] S. Mikulás. Axiomatizability of algebras of binary relations. In *Classical and new paradigms of computation and their complexity hierarchies,* 187–205, volume 23 of *Trends Log. Stud. Log. Libr.* Kluwer Acad. Publ., Dordrecht, 2004. 27

[65] D. Monk. On representable relation algebras. *Michigan Math. J.*, 11(3):207–210, Sep 1964. 14, 21

[66] M. Neuzerling. Undecidability of representability for lattice-ordered semigroups and ordered complemented semigroups. *Algebra Universalis*, 76(4):431–443, 2016. 23

[67] E. C. Nummela. Cayley's theorem for topological groups. *Amer. Math. Monthly*, 87(3):202–203, 1980. 3

[68] C. S. Peirce. Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic. *Memoirs of the American Academy of Arts and Sciences*, 9(2):317–378, 1873. 16

[69] E. L. Post. *The Two-Valued Iterative Systems of Mathematical Logic.* Annals of Mathematics Studies, no. 5. Princeton University Press, Princeton, N. J., 1941. 70

[70] V. Pratt. Origins of the calculus of binary relations. In *Proceedings of MFCS'93,* 142–155. Springer-Verlag, 1992. 16, 17

[71] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In B. Nebel, C. Rich, and W. R. Swartout, editors, *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92). Cambridge, MA, October 25-29, 1992,* 165–176. Morgan Kaufmann, 1992. 63

[72] D. Rees. On semi-groups. *Proc. Cambridge Philos. Soc.*, 36:387–400, 1940. 51

[73] O. Reingold. Undirected ST-connectivity in log-space. In *Proceedings of the Thirty-seventh Annual ACM Symposium on Theory of Computing,* 376–385, STOC '05, New York, NY, USA, 2005. ACM. 71

[74] J. Renz. A canonical model of the region connection calculus. *Journal of Applied Non-Classical Logics*, 12(3-4):469–494, 2002. 64

[75] J. Renz. *Qualitative Spatial Reasoning with Topological Information.* Lecture Notes in Artificial Intelligence. Springer, 2002. 63

[76] B. Russell. *The Principles of Mathematics.* Cambridge University Press, 1903. 16

[77] T. J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing,* 216–226, STOC '78, New York, NY, USA, 1978. ACM. 70

[78] B. M. Schein. Relation algebras and function semigroups. *Semigroup Forum*, 1(1):1–62, 1970. 20

[79] B. M. Schein. A new representation theorem for completely [0−] simple semigroups. *Semigroup Forum*, 4:312–320, 1972. 51

[80] B. M. Schein. Lectures on semigroups of transformations. In A. F. Lavrik, editor, *Twelve Papers in Logic and Algebra,* 123–181, American Mathematical Society Translations–Series 2. American Mathematical Society, Boston, 1979. 19

[81] B. M. Schein. Representation of subreducts of Tarski relation algebras. In H. Andréka, J. D. Monk, and I. Németi, editors, *Algebraic logic (Budapest, 1988),* 621–635, volume 54 of *Colloq. Math. Soc. János Bolyai.* North-Holland, Amsterdam, 1991. 21, 23, 27, 35

[82] B. M. Schein. Difference semigroups. *Comm. Algebra*, 20(8):2153–2169, 1992. 58

[83] E. Schröder. *Vorlesungen über die Algebra der Logik (exakte Logik)*, volume III. B. G. Teubner, Leipzig, 1895. Reprinted in 1966 by Chelsea Pub. Co. 16

[84] W. A. Stein et al. *Sage Mathematics Software (Version 6.3).* The Sage Development Team, 2014. http://www.sagemath.org.

[85] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936. 15

[86] A. Tarski. On the calculus of relations. *J. Symbolic Logic*, 6:73–89, 1941. xvii, 5, 12, 17

[87] A. Tarski. Contributions to the theory of models. III. *Nederl. Akad. Wetensch. Proc. Ser. A.*, 58:56–64 = Indag. Math. 17, 56–64, 1955. 5

[88] A. Tarski. Equationally complete rings and relation algebras. *Nederl. Akad. Wetensch. Proc. Ser. A.* **59** = *Indag. Math.*, 18:39–46, 1956. 5

[89] A. Tarski and S. Givant. *A formalization of set theory without variables,*

volume 41 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. 17

[90] B. L. van der Waerden. *A history of algebra*. Springer-Verlag, Berlin, 1985. From al-Khwārizmī to Emmy Noether. 2

[91] K. A. Zaretskiĭ. The representation of ordered semigroups by binary relations. *Izv. Vysš. Učebn. Zaved. Matematika*, (6 (13)):48–50, 1959. 18, 21

[92] P. Zhang and J. Renz. Qualitative spatial representation and reasoning in angry birds: The extended rectangle algebra. In C. Baral, G. De Giacomo, and T. Eiter, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the Fourteenth International Conference, KR 2014, Vienna, Austria, July 20-24, 2014*. AAAI Press, 2014. 62

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