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Stability of homogeneous geodesics in low-dimensional Lie groups

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Abstract

In Riemannian geometry, a geodesic is the shortest curve between any two close points. Geodesics are analogues of straight lines in Euclidean geometry, and as such, are probably the most important curves on a Riemannian manifold. In a metric Lie group, geodesics can be described by the famous Euler-Arnold equation, which in practice is a finite-dimensional system of nonlinear ordinary differential equations on the Lie algebra of the Lie group. In this project, we study the Lyapunov stability of certain distinguished geodesics called homogeneous geodesics, which correspond to the stationary solutions of the Euler-Arnold equation. We give a complete classification in case of three-dimensional Lie groups and a partial classification in dimension four.

Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

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Introduction

In this introduction, we will informally explain the scope of this thesis, leaving the precise statements and definitions to later chapters, and then formulate our main results.

The research undertaken in the thesis lies in the overlap of three areas — Lie theory, Riemannian geometry and Ordinary differential equations (Dynamical systems). We study the dynamical behaviour of *geodesics* on Lie groups of low dimensions (of dimension 3 and 4).

Informally, one can think of a Lie group of dimension $n \geq 1$ as a topological group which is at the same time an *n-dimensional manifold*; the latter means that locally a neighbourhood of each point is homeomorphic to an open ball in $\mathbb{R}^n$ (the actual definition requires certain differentiability — see Definition 1.2 in Chapter 1). Naturally, we require group operations to be differentiable. Popular examples of Lie groups include $\mathbb{R}^n$ under addition and matrix groups — e.g., the groups of nonsingular, orthogonal, or unitary matrices under matrix multiplication.

Given a Lie group G, we add to two existing structures, algebraic and topological, the third one, geometric — the metric. We require that the topology defined by that metric coincides with the topology on the Lie group. But to get the most natural class of metrics, we also want geometry to interplay with algebra. To do that, we require that the left multiplication by any $g \in G$ is an *isometry* of the metric. Metrics with this property are called *left-invariant* (we consider not all possible left-invariant metrics, but only those which are *Riemannian* — see Section 1.5 for precise definition). The property of a metric to be left-invariant means that it is sufficient to define it on an arbitrarily small neighbourhood of the identity $e \in G$, or in fact, in the Riemannian case, on the tangent space $\mathfrak{g}$ to G at the identity. It turns out that left-invariant Riemannian metrics on a Lie group are in a one-to-one correspondence with inner products on $\mathfrak{g}$ — given any such inner product we can carry it over the whole group G by its left action on itself (see Section 1.5). The tangent space $\mathfrak{g}$ to the group G at the identity is an n-dimensional space, which carries an additional structure — it is a *Lie algebra*. This means that $\mathfrak{g}$ is equipped with a natural bilinear, skew-symmetric binary operation satisfying the Jacobi identity called the *Lie bracket* (see Section 1.2); informally, this bracket on $\mathfrak{g}$ is obtained by "differentiating" the group structure on G. Popular examples of Lie algebras include the *abelian* Lie algebra with zero bracket, the space $\mathbb{R}^3$ with the bracket defined by the cross product, the space of square matrices with the bracket defined by the commutator of two matrices, etc. Furthermore, it turns out that not only a Lie group G defines its Lie algebra $\mathfrak{g}$ (as the tangent space at the identity with the corresponding Lie bracket), but also, vice versa, any abstractly defined Lie algebra $\mathfrak{g}$ (a space $\mathbb{R}^n$ with the Lie bracket) determines a Lie group to which $\mathfrak{g}$ is the Lie algebra, and moreover, such a Lie group is essentially unique (see Section 1.3).

The above construction effectively reduces the study of geometry of Lie groups with left-invariant metrics to the study of *metric Lie algebras*. The latter is a linear space $\mathbb{R}^n$ with two *independent* structures — the Lie bracket and the inner product.

On a Lie group with a left-invariant Riemannian metric, we have two distinguished classes of curves: the one-dimensional subgroups coming from the algebra of G and the *geodesics* coming from geometry on G. Recall that a geodesic on an arbitrary Riemannian space plays the same role as a straight line in $\mathbb{R}^n$ — it is locally the shortest curve between any two close points. It is known that any geodesic passing through the identity $e \in G$ is uniquely determined by its tangent vector at the identity; it is also known that any one-dimensional subgroup of G is uniquely determined by its tangent vector at the identity. But in general, these curves are different. However, by a result of **[17]**, any left-invariant metric on a Lie group has at least one *homogeneous geodesic* — a geodesic which is also a subgroup.

In this setup, we can now describe the question which we address in this thesis. Although not every geodesic is homogeneous, we ask whether the property of being a homogeneous geodesic is *stable* — is it so that if a tangent vector X at the identity defines a homogeneous geodesic, then any vector close to it will define a geodesic which "converges" to that homogeneous geodesic? This question was first studied in the pioneering paper of Arnold [**1**]. We give a complete answer in the first two nontrivial cases, for 3-dimensional and partly for 4-dimensional metric Lie groups.

Our methodology will be to first translate our question to the language of metric Lie algebras. As we explained above, every Lie group G with a left-invariant metric is locally uniquely determined by its metric Lie algebra $\mathfrak{g}$. Consider a (naturally parameterised) geodesic $\gamma = \gamma(t)$ in G starting at the identity (so that $\gamma(0) = e$), take the unit tangent vector $\dot{\gamma}(t)$ at a point $\gamma(t)$ and translate that vector to $\mathfrak{g}$ by the differential of the left action of the element $\gamma^{-1}(t) \in G$ on G; then we will get a unit vector $X(t) \in \mathfrak{g}$ (see Section 1.6 for details). As a result, a geodesic of G defines a curve on the unit sphere of $\mathfrak{g}$. Such a curve (sometimes called the *hodograph* of γ) satisfies a nonlinear ordinary differential equation known as the *Euler-Arnold equation*:

$$\dot{X} = \mathrm{ad}_X^* X \tag{0.0.1}$$

(see Section 1.6 for details and unexplained notation). If the geodesic γ is homogeneous (i.e., locally is a subgroup), then the corresponding curve $X(t)$ is just a single point — the *stationary solution* of (0.0.1). We say that a homogeneous geodesic (or equivalently, the corresponding stationary point X_0) is *stable*, if that stationary point is a *Lyapunov stable solution* of equation (0.0.1) — that is, any solution $X(t)$ of (0.0.1) which starts close to X_0 remains close to X_0 for all times $t > 0$. Otherwise, the homogeneous geodesic (and the corresponding stationary point) is called *unstable* (see Section 1.7).

In this thesis, we classify all the stationary points and determine which of them are stable and which are unstable for all 3-dimensional metric Lie algebras, and partly for 4-dimensional metric Lie algebras. It is important to emphasise that equation (0.0.1) "lives" on a sphere (the function $\|X\|^2$ is a first integral), so in the 3-dimensional case, we have an autonomous system of ODEs on the two-dimensional unit sphere S^2 in the 3-dimensional space $\mathfrak{g}$, and similarly, on the three-dimensional sphere S^3 in the 4-dimensional case.

The thesis outline is as follows. The first chapter is expository, it provides background results from Lie theory, Riemannian geometry and Stability theory as a foundation for the work later on. The main findings of the thesis are presented in the next three chapters. In Chapter 2, we first give a classification of all metric Lie algebras of dimension 3. There are two large classes of such algebras — *unimodular* and *non-unimodular* (see Section 2.1). In the unimodular case, the classification is given in the cornerstone paper of Milnor **[21]**; the classification in the non-unimodular case is also well known in the literature (see **[20]**; for completeness, we give a full proof in Section 2.1). Next we determine, for each case, the stationary points of the Euler-Arnold equation (0.0.1): we find all unit vectors $X \in \mathfrak{g}$ for which the right-hand side $\mathrm{ad}_X^* X$ is zero. In some cases, the set of stationary points will be infinite (e.g. the equator of S^2 or even the whole sphere S^2). We further treat each case separately to determine the stability of the stationary points. The unimodular case is substantially easier (see Section 2.3); using Milnor's classification, we find that in each case, the Euler-Arnold equation (0.0.1) either can be solved explicitly or admits another first integral (except for $\|X\|^2$) which is a quadratic form in X. Then the trajectories of (0.0.1) lie on the curves in the intersection of the unit sphere S^2 with the certain quadratic surfaces (hyperbolic or elliptic cylinders). This allows us to easily determine which stationary points are stable and which are unstable [1].

The non-unimodular case of 3-dimensional metric Lie algebras is much harder. There are no easy-to-find first integrals. We use several well-known techniques

[1]A substantial part of the work on this case was done during the candidate's AMSI VRS.

from the stability theory of the ODEs. One of them is linearisation: the stability of a stationary point is determined by the eigenvalues of the Jacobian matrix of the right-hand side of (0.0.1) at that point. In several cases, this technique is inconclusive; then we use Lyapunov and Chetaev stability criteria. In one case we need something more — we use a theorem from the original Arnold's paper **[1]** (see Theorem 1.46).

Chapters 3 and 4 follow the same methodology. In Chapter 3, we first introduce the definition of *nilpotent* Lie algebras before providing a classification of nilpotent metric Lie algebras of dimension 4, up to orthogonal isomorphism. We then apply the Euler-Arnold equation (0.0.1) to find homogeneous geodesics, and classify their corresponding stability status. In Chapter 4, we discuss the stability analysis of homogeneous geodesics in 4-dimensional *unimodular* metric Lie algebras with a nontrivial centre. Even though we normally proceed in a case-by-case basis, we notice a very interesting pattern that comes up in both of these algebras. We find that the stability status of homogeneous geodesics determined by the Jacobian condition (see Theorem 1.39) and Arnold's condition (see Theorem 1.46) both depend on a similar function and complement each other. The condition where these two theorems are indecisive is a lot harder and more unconventional, we have to go back to the definition of stability and apply different direct approaches (topological methods, algebraic manipulations, etc.) to test for stability.

The results of the thesis can be summarised in the following theorems. To the best of our knowledge, these results did not appear in the literature before.

0.1. THEOREM. *Let $\mathfrak{g}$ be an unimodular metric Lie algebra of dimension 3. By* **[21]**, *there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ relative to which the Lie bracket is given by*

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2,$$

where we can assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

The following table gives the classification of stable and unstable stationary points of the Euler-Arnold equation (0.0.1) *on the unit sphere $S^2 = S^2(1) \subset \mathfrak{g}$ (the coordinates are given relative to the basis $\{e_1, e_2, e_3\}$).*

Case	*Stationary points*	*Stability*
$\lambda_1 = \lambda_2 = \lambda_3$	*Every point of* S^2	*Stable*
$\lambda_1 > \lambda_2 = \lambda_3$	$(\pm 1, 0, 0)$	*Stable*
	All points on the circle $X_1 = 0$	*Unstable*
$\lambda_1 = \lambda_2 > \lambda_3$	$(0, 0, \pm 1)$	*Stable*
	All points on the circle $X_3 = 0$	*Unstable*
$\lambda_1 > \lambda_2 > \lambda_3$	$(\pm 1, 0, 0)$	*Stable*
	$(0, \pm 1, 0)$	*Unstable*
	$(0, 0, \pm 1)$	*Stable*

TABLE 0.1. Stability of homogeneous geodesics in 3-dimensional unimodular metric Lie algebras.

0.2. THEOREM. *Let $\mathfrak{g}$ be a non-unimodular metric Lie algebra of dimension 3. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ relative to which the Lie bracket is given by*

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

where we can assume $\alpha + \delta > 0$ and $\alpha > 0$.

Denote $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. If $\det(M) = 0$, we can specify the basis $\{e_1, e_2, e_3\}$ in such a way that $\gamma = \delta = 0$. The following table gives the classification of stable and unstable stationary points of the Euler-Arnold equation (0.0.1) *on the unit sphere $S^2 = S^2(1) \subset \mathfrak{g}$ if M is singular.*

β	*Stationary points*	*Stability*
	$(0, 0, \pm 1)$	*Stable*
$\beta = 0$	$\left(-\sqrt{1-c^2}, 0, c\right)$, $\lvert c\rvert < 1$	*Stable*
	$\left(\sqrt{1-c^2}, 0, c\right)$, $\lvert c\rvert < 1$	*Unstable*
$\beta \neq 0$	$\left(-\frac{\sqrt{\alpha^2 - c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}, c\right)$, $\lvert c\rvert < \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$	*Stable*
	$\left(\frac{\sqrt{\alpha^2 - c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}, c\right)$, $\lvert c\rvert \leq \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$	*Unstable*

TABLE 0.2. Stability of homogeneous geodesics in 3-dimensional non-unimodular metric Lie algebras if M is singular.

If $\det(M) \neq 0$*, we can specify the basis* $\{e_1, e_2, e_3\}$ *in such a way that* $\gamma = -\beta$*. The following table gives the classification of stable and unstable stationary points of the Euler-Arnold equation* (0.0.1) *on the unit sphere* $S^2 = S^2(1) \subset \mathfrak{g}$ *if* M *is non-singular (where in the case* $\delta < 0$*, we define* $\rho = \sqrt{\alpha}$*,* $\sigma = \sqrt{-\delta}$*).*

δ		*Stationary points*	*Stability*
$\delta > 0$		$(1, 0, 0)$	*Unstable*
		$(-1, 0, 0)$	*Stable*
$\delta = 0$		$(1, 0, 0)$	*Unstable*
		$(-1, 0, 0)$	*Stable*
		$(0, 0, \pm 1)$	*Unstable*
$\delta < 0$	$\det M < 0$	$(1, 0, 0)$	*Unstable*
		$(-1, 0, 0)$	*Unstable*
		$(\rho^2 + \sigma^2)^{-1/2}\,(0, \pm\sigma, \pm\rho)$	*Stable*
	$\det M > 0$	$(-1, 0, 0)$	*Stable*
		$(\rho^2 + \sigma^2)^{-1/2}\,(0, s_1\sigma, s_2\rho)$ $s_1, s_2 \in \{-1, 1\},\ s_1 s_2 \beta > 0.$	*Stable*
		$(\rho^2 + \sigma^2)^{-1/2}\,(0, s_1\sigma, s_2\rho)$ $s_1, s_2 \in \{-1, 1\},\ s_1 s_2 \beta < 0.$	*Unstable*

TABLE 0.3. Stability of homogeneous geodesics in 3-dimensional non-unimodular metric Lie algebras if M is nonsingular.

0.3. THEOREM. *Let $\mathfrak{g}$ be a nilpotent metric Lie algebra of dimension 4. Then either (1) $\mathfrak{g}$ is abelian, or (2) there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ relative to which the only nonzero Lie brackets (up to skew-symmetry) are given by one of the following:*

$$
\begin{aligned}
&[e_1, e_2] = ce_4, && && c \neq 0,\\
\textit{or}\quad &[e_1, e_2] = ae_3 + be_4, && [e_1, e_3] = ce_4, && a, c \neq 0.
\end{aligned}
$$

The following table gives the classification of stable and unstable stationary points of the Euler-Arnold equation (0.0.1) *on the unit sphere $S^3 = S^3(1) \subset \mathfrak{g}$ (where in the third case, we define $P = aX_2X_3 + bX_2X_4 + cX_3X_4$ and $Q = (aX_3 + bX_4)^2 + c^2X_4^2 + acX_2X_4$).*

<table>
<tr><th>Case</th><th colspan="2">Stationary Point</th><th>Stability</th></tr>
<tr><td>Abelian</td><td colspan="2">Every point of S^3</td><td>Stable</td></tr>
<tr><td rowspan="2">$[e_1, e_2] = ce_4, c \neq 0$</td><td colspan="2">The circle $X_3^2 + X_4^2 = 1$</td><td>Stable</td></tr>
<tr><td colspan="2">The sphere $X_3 = 0$
minus the points $(0, 0, 0, \pm 1)$</td><td>Unstable</td></tr>
<tr><td rowspan="3">$[e_1, e_2] = ae_3 + be_4, [e_1, e_3] = ce_4$,
$a, c \neq 0$</td><td colspan="2">The circle $X_1^2 + X_2^2 = 1$</td><td>Unstable</td></tr>
<tr><td rowspan="2">Intersection of sphere $X_1 = 0$
and $P = 0$</td><td>$Q > 0$ or
$Q = 0$ and $b = 0$</td><td>Stable</td></tr>
<tr><td>$Q < 0$ or
$Q = 0$ and $b \neq 0$</td><td>Unstable</td></tr>
</table>

TABLE 0.4. Stability of homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras.

0.4. THEOREM. *Let $\mathfrak{g}$ be an unimodular metric Lie algebra of dimension 4 with a nontrivial centre. Then either (1) $\mathfrak{g}$ is nilpotent, or (2) there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ relative to which the only nonzero Lie brackets (up to skew-symmetry) are given by*

$$[e_1, e_2] = \lambda_3 e_3 + v_3 e_4, \quad [e_2, e_3] = \lambda_1 e_1 + v_1 e_4, \quad [e_3, e_1] = \lambda_2 e_2 + v_2 e_4,$$

where $\lambda_i, v_i \in \mathbb{R}$. If at least two of λ_i are zeros, then $\mathfrak{g}$ is nilpotent. If at least two of λ_i are equal, say $\lambda_1 = \lambda_2$, then $\mathfrak{g}$ is not necessarily nilpotent, but the Euler-Arnold equation is the same as for the nilpotent metric algebra given by the above brackets, with $\lambda_1 = \lambda_2 = 0$ and λ_3 replaced by $\lambda_3 - \lambda_1$. In both cases, the stability analysis is given in Theorem 0.3.

Assume that λ_i are pairwise non-equal (and hence no more than one λ_i is zero). The following table gives the classification of stable and unstable stationary points of the Euler-Arnold equation (0.0.1) *on the sphere $S^3 \subset \mathfrak{g}$ when $\mathfrak{g}$ is not nilpotent (where we define $\sigma(x) = (s - \lambda_3)(s - \lambda_2)x_1^2 + (s - \lambda_3)(s - \lambda_1)x_2^2 + (s - \lambda_2)(s - \lambda_1)x_3^2$).*

<table>
<tr><th>Case</th><th colspan="2">Stationary Point</th><th>Stability</th></tr>
<tr><td>$x = 0$</td><td colspan="2">$(0, 0, 0, x_4), x_4 \in \mathbb{R}$</td><td>Stable</td></tr>
<tr><td rowspan="5">$x \neq 0$</td><td rowspan="5">$X_0 = x + x_4 e_4$ satisfying $Lx + x_4 v = sx$ where $L = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, λ_i are pairwise nonequal and $x_4, s \in \mathbb{R}$</td><td>$\sigma(x) > 0$</td><td>Stable</td></tr>
<tr><td>$\sigma(x) < 0$</td><td>Unstable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$</td><td>Unstable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s = \lambda_1$ and $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have the same sign</td><td>Stable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s = \lambda_1$ and $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have opposite signs</td><td>Unstable</td></tr>
</table>

TABLE 0.5. Stability of homogeneous geodesics in 4-dimensional unimodular metric Lie algebras with a nontrivial centre.

CHAPTER 1

Preliminaries on Lie theory, Riemannian geometry and Stability theory

In this chapter, we introduce notions and definitions which will provide us with the necessary background for the next three chapters. Sections 1.1-1.3 give a general exposition of abstract Lie groups and Lie algebras, and then these theories are illustrated with concrete examples from matrix Lie groups in Section 1.4. Section 1.5 outlines some main results in Riemannian metrics and Levi-Civita connection, and then introduces left-invariant Riemannian metrics on Lie groups. In Section 1.6, we give the proof of the Euler-Arnold equation for geodesics on a Lie group with a left-invariant metric, which plays the central role in this thesis. Finally, the definition of stationary points and stability of stationary points are given in Section 1.7, together with several well-known techniques from the stability theory of the ODEs that are used in the project.

1.1. Lie groups

In this section, we give the definition of a Lie group, and proceed to construct the Lie algebra of the Lie group as its tangent space at the identity. The references are [**12**] and [**25**].

1.1. Definition. Let M be a topological space with topology $\mathcal{O}$. Then M is called an ***n-dimensional topological manifold***, if the following holds:

1. M is ***Hausdorff***, that is, for all $p, q \in M$ with $p \neq q$ there exists open sets $U, V \subset M$ with $p \in U, q \in V$ and $U \cap V = \emptyset$.
2. The topology of M has a ***countable basis***, that is, there exists a countable subset $\mathcal{B} \subset \mathcal{O}$, such that for every $U \in \mathcal{O}$ there are $B_i \in \mathcal{B}, i \in I$, with
$$U = \bigcup_{i \in I} B_i$$
3. M is ***locally homeomorphic*** to $\mathbb{R}^n$, that is, for all $p \in M$ there exist an open subset $U \subset M$ with $p \in U$, an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $f : U \to V$.

Informally, a manifold is a space such that when you zoom in enough, it looks like a flat Euclidean space. A topological manifold is called ***differentiable*** (of class

C^k) if the open subsets U and the homeomorphisms f in Definition 1.1 can be chosen in such a way that if U_1 and U_2 intersect, then the map $f_2 \circ f_1^{-1}$ between the corresponding domains of $\mathbb{R}^n$ is differentiable of class C^k.

1.2. DEFINITION. A ***Lie group*** is a differentiable manifold which is also a group such that the multiplication map

$$m : G \times G \to G, \quad (x, y) \mapsto xy$$

and the inversion map

$$\iota : G \to G, \quad x \mapsto x^{-1}$$

are differentiable.

1.3. EXAMPLE. The simplest example is the commutative Lie group $\mathbb{R}^n$ under vector addition.

Many important Lie groups belong to families of "classical Lie groups".

1.4. EXAMPLE. The *general linear group* $\mathrm{GL}(n)$ is the group of $n \times n$ invertible real matrices under matrix multiplication. It is an open subset of the space of all $n \times n$ matrices. Multiplication is differentiable because the matrix entries of a product matrix AB are polynomials in the entries of matrices A and B. Inversion is differentiable by Cramer's formula for the inverse. Hence, $\mathrm{GL}(n)$ is a Lie group.

1.5. EXAMPLE. The *special linear group*

$$\mathrm{SL}(n) = \{A \in \mathrm{GL}(n) : \det(A) = 1\},$$

is the group of all volume-preserving linear transformations of $\mathbb{R}^n$. The *orthogonal group*

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n) : AA^T = I\},$$

where A^T is the transpose matrix of A and I is the identity matrix, is the group of norm-preserving linear transformations of $\mathbb{R}^n$. The *special orthogonal group*

$$\mathrm{SO}(n) = \{A \in \mathrm{GL}(n) : AA^T = I \quad \text{and} \quad \det(A) = 1\},$$

is the group of all rotations of $\mathbb{R}^n$; it is a normal subgroup of $\mathrm{O}(n)$ of index 2.

As a Lie group is a differentiable manifold, one can introduce, at each point of it, the tangent space to the Lie group at that point. The tangent space at the identity of the group plays a distinguished role: on top of a usual structure of a linear space it carries the structure of a *Lie algebra* defined from the Lie bracket which is inherited (and canonically constructed) from the multiplication in the Lie group. This correspondence, which we will define rigorously in the next section, assigns to a complicated object (a differentiable manifold with a group structure) a much easier to understand object — a linear space with an additional binary operation. Amazingly, all the information about the Lie group is encoded in the so-constructed Lie algebra (strictly speaking, assuming that the group is simply-connected as a topological manifold).

1.2. Lie algebras and left-invariant vector fields

In this section, we give the definition of a Lie algebra and of the Lie algebra associated with a Lie group. Our exposition is based on [**5**] and [**18**].

Any Lie group G acts on itself by the *left multiplication*. If $g \in G$ is fixed, this action is defined by

$$L_g(h) = gh, \quad \text{for all } h \in G.$$

The map $L_g : G \to G$ is a diffeomorphism, as for each $g \in G$, we can define a differentiable inverse of L_g by

$$L_g^{-1}(h) = g^{-1}h = L_{g^{-1}}(h), \quad \text{for all } h \in G.$$

Suppose that $F : M \to N$ is a differentiable map between two differentiable manifolds M and N. Then for each $m \in M$, we can define a map

$$dF_m : T_mM \to T_{F(m)}N,$$

called the *differential* (the tangent map) of F at m. This is a linear map that pushes forward the tangent space T_mM at a point $m \in M$ to the tangent space $T_{F(m)}N$ at the point $F(m) \in N$. In particular, for any $g \in G$, the map $L_g : G \to G$ induces the map $dL_g : T_hG \to T_{gh}G$. We say that a vector field on the Lie group G is *left-invariant*, if it is invariant relative to the action of L_g. More precisely, we have the following definition.

1.6. DEFINITION. A vector field X on a Lie group G is said to be ***left-invariant*** if for all $g, h \in G$, we have

$$d(L_g)_h(X(h)) = X(L_g(h)) = X(gh).$$

Equivalently, a vector field X is left-invariant iff the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "X"] & G \arrow[d, "X"] \\ TG \arrow[r, "d(L_g)"] & TG \end{tikzcd},$$

for all $g \in G$, where TG is the tangent bundle of G.

Choosing an arbitrary vector $X(e) \in T_eG$, we can set $h = e$ in Definition 1.6 to define a vector field X on G by

$$X(g) = d(L_g)_e(X(e)).$$

This means that there is a unique left-invariant vector field X on G with the prescribed tangent vector at the identity. Conversely, any left-invariant vector field must have $d(L_g)_h(X(h)) = X(gh)$, so for all g, we have $X(g) = d(L_g)_e(X(e))$. Hence, we come to the following conclusion.

1.7. THEOREM. *Let G be a Lie group. Then the vector space of all left-invariant vector fields on G is isomorphic to the tangent space of G at the identity.*

Given a vector field X on a differentiable manifold M we can define, for any differentiable function $f : M \to \mathbb{R}$, the function $X(f)$, the derivative of f in the direction of X.

1.8. DEFINITION. Let M be a differentiable manifold, and let X and Y be differentiable vector fields on M. Then the operator $[X, Y]$, called the ***Lie bracket*** of X and Y, is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{1.2.1}$$

for an arbitrary C^∞-function $f : M \to \mathbb{R}$.

The key fact is that although the right-hand side seemingly involves the second derivatives of f, it in fact only depends on the first derivatives, and so this operator actually gives rise to a new vector field (see **[18]**). Moreover, the so-defined Lie bracket possesses some very nice properties.

1.1. PROPOSITION. *Let X, Y, Z be differentiable vector fields. Then the Lie bracket satisfies the following properties:*

1. *Anti-symmetry:*
$$[X, Y] = -[Y, X].$$
2. *Bilinearity: For $a, b \in \mathbb{R}$,*
$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$
3. *The Jacobi identity:*
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

PROOF. Anti-symmetry is obvious from (1.2.1). For bilinearity, let $f : M \to \mathbb{R}$ be an arbitrary C^∞-function, and $a, b \in \mathbb{R}$. We have

$$\begin{aligned}[aX + bY, Z](f) &= (aX + bY)(Z(f)) - Z((aX + bY)(f)) \\ &= aX(Z(f)) + bY(Z(f)) - aZ(X(f)) - bZ(Y(f)) \\ &= a[X, Z](f) + b[Y, Z](f),\end{aligned}$$

and the second relation follows from anti-symmetry.

The Jacobi identity can be verified directly. We have

$$
\begin{aligned}
&[X,[Y,Z]](f)+[Y,[Z,X]](f)+[Z,[X,Y]](f)\\
&=X[Y,Z](f)-[Y,Z]X(f)+Y[Z,X](f)-[Z,X]Y(f)+Z[X,Y](f)-[X,Y]Z(f)\\
&=XYZ(f)-XZY(f)-YZX(f)+ZYX(f)+YZX(f)-YXZ(f)\\
&-ZXY(f)+XZY(f)+ZXY(f)-ZYX(f)-XYZ(f)+YXZ(f)\\
&=0.
\end{aligned}
$$

□

Taking the properties in Proposition 1.1 as axioms, we obtain the following definition.

1.9. DEFINITION. A ***Lie algebra*** over $\mathbb{R}$ is a real vector space $\mathfrak{g}$, together with a map called the ***Lie bracket***

$$\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},\quad (x,y)\mapsto[x,y]$$

satisfying the following properties:

1. Anti-symmetry: for all $X,Y\in\mathfrak{g}$,
$$[X,Y]=-[Y,X].$$
2. Bilinearity: for any $a,b\in\mathbb{R}$ and for all $X,Y,Z\in\mathfrak{g}$
$$
\begin{aligned}
[aX+bY,Z]&=a[X,Z]+b[Y,Z],\\
[Z,aX+bY]&=a[Z,X]+b[Z,Y].
\end{aligned}
$$
3. The Jacobi identity: for all $X,Y,Z\in\mathfrak{g}$,
$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

The first example of a Lie algebra comes from the above construction: the linear space of all C^1-vector fields on a differentiable manifold, with the Lie bracket defined by (1.2.1) is a Lie algebra, by Proposition 1.1. Note that this Lie algebra is "huge", of infinite dimension. However, if our manifold is a Lie group, we have the following remarkable fact: *the Lie bracket of left-invariant vector fields is again left-invariant* (see **[18]**). This fact justifies the following definition.

1.10. DEFINITION. ***The Lie algebra of a Lie group*** G is the Lie algebra of left-invariant vector fields on G equipped with the bracket defined by (1.2.1).

Combining this with Theorem 1.7, we see that the Lie algebra of a Lie group can be identified with the tangent space to the Lie group at the identity, with the Lie bracket of two vectors defined as the value at the identity of the Lie bracket of the corresponding left-invariant vector fields.

We finish this section with several examples of finite-dimensional Lie algebras. The simplest example is an *abelian* Lie algebra — the Lie bracket of any two elements is zero.

1.11. EXAMPLE. A well-known example of a Lie algebra is $\mathbb{R}^3$ with the Lie bracket defined by the cross product $[x, y] = x \times y$.

1.12. EXAMPLE. Matrix Lie algebras. On the space of $n \times n$ real matrices we can define the Lie bracket as the matrix commutator. Then this space and any of its subspaces closed under the commutator are examples of a Lie algebra. Examples of such subspaces include the subspace of all matrices with zero trace, the subspace of all skew-symmetric matrices, the subspace of upper-triangular matrices, etc. In Section 1.4, we will see that these examples of Lie algebras correspond to the examples of Lie groups which we gave at the end of Section 1.1.

1.3. The exponential map and the adjoint representation

In the previous section we demonstrated a canonical construction which associates to a given Lie group its Lie algebra (viewed as the tangent space to the Lie group at the identity). In this section, we first show how to go in the opposite direction: starting with a Lie algebra we will construct the locally unique Lie group to which it is associated. We then provide a different construction of the correspondence between a Lie group and its Lie algebra which will allow us to explicitly obtain the Lie bracket on the Lie algebra by "differentiating" the group law. Our exposition follows **[5]**, **[7]**, **[12]**, **[18]**, and **[25]**.

1.13. DEFINITION. A ***homomorphism*** between two Lie groups G_1 and G_2 is a map $\phi : G_1 \to G_2$ that is a homomorphism of groups and a differentiable map between the manifolds G_1 and G_2. An ***isomorphism*** ϕ of Lie groups is a bijection such that both ϕ and ϕ^{-1} are homomorphisms of Lie groups.

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra viewed as the tangent space at the identity. We have the following theorem (whose proof can be found e.g. in **[7]**) which establishes a one-to-one correspondence between the set of nonzero elements of $\mathfrak{g}$ and the set of one-dimensional subgroups of G. We view $\mathbb{R}$ as the one-dimensional abelian Lie group by addition.

1.14. THEOREM. *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then, for each $X \in \mathfrak{g}$, there exists a unique homomorphism $\gamma_X : \mathbb{R} \to G$ that is differentiable at $t = 0$ and satisfies $\frac{d}{dt}\gamma_X(t)\big|_{t=0} = X$.*

This theorem motivates the definition of the exponential map.

1.15. DEFINITION. The map $\exp : \mathfrak{g} \to G$, called the ***exponential map*** of G, is defined as follows, for any $X \in \mathfrak{g}$:

$$\exp X = \gamma_X(1),$$

where γ is the homomorphism in Theorem 1.14.

The exponential map of G possesses the following important properties.

1.2. PROPOSITION. *For all $t, s \in \mathbb{R}$ and $X \in \mathfrak{g}$,*

1. $\exp(tX) = \gamma_X(t)$.
2. $\exp(s + t)X = \exp sX \exp tX$.
3. $(\exp tX)^{-1} = \exp(-tX)$.

The significance of the exponential map is that it allows us to go from the Lie algebra $\mathfrak{g}$ back to the Lie group G.

We now introduce a different approach to define the Lie bracket on the Lie algebra of a Lie group (as compared to the approach via left-invariant vector fields in Section 1.2) and show their equivalence.

1.16. DEFINITION. Let G be a Lie group. Then G acts on itself by the conjugation map, that is, for all $g \in G$, we can define

$$C_g : G \to G, \quad C_g(h) = ghg^{-1}.$$

It is easy to see that C_g is a Lie group isomorphism: it is a group automorphism, and it is differentiable together with its inverse, because of differentiability of the group operations. Also, as $C_g(e) = geg^{-1} = e$, the conjugation map preserves the identity, so we can look at the differential of C_g at the identity.

1.17. DEFINITION. The ***adjoint representation*** of a Lie group G is a map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, where $\mathrm{GL}(\mathfrak{g})$ is the group of all invertible linear transformations of the vector space $\mathfrak{g}$, defined for all $g \in G$ by

$$\mathrm{Ad}_g : T_e(G) \to T_e(G), \quad \mathrm{Ad}_g = d(C_g)_e.$$

For $g, g' \in G$, we have

$$C_{gg'}(h) = (gg')h(gg')^{-1} = g(g'hg'^{-1})g^{-1} = C_g \circ C_{g'}.$$

So applying the chain rule to both sides at the identity gives

$$\mathrm{Ad}_{gg'} = \mathrm{Ad}_g \circ \mathrm{Ad}_{g'}.$$

Thus, the map $\mathrm{Ad} : g \to \mathrm{Ad}_g$ is a homomorphism from G to $\mathrm{GL}(\mathfrak{g})$. Also, note that Ad is differentiable, so we can take the differential at the identity.

1.18. DEFINITION. The ***adjoint representation*** of the Lie algebra $\mathfrak{g}$ of a Lie group G is a map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g})$ is the space of all linear transformations of the vector space $\mathfrak{g}$, defined for all $X \in \mathfrak{g}$ by

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \mathrm{ad}_X = d(\mathrm{Ad})_e(X).$$

We now show that the adjoint representation defined above agrees with the definition of the Lie bracket of $\mathfrak{g}$.

1.19. THEOREM. *For all $X, Y \in \mathfrak{g}$, we have* $\mathrm{ad}_X Y = [X, Y]$.

PROOF. First, observe that by Theorem 1.14, if X is a left-invariant vector field on G, then we have

$$Xf(g) = \frac{d}{dt} f(g \exp tX)\Big|_{t=0}$$

for all $g \in G$ and $f \in C^\infty(G)$.

Now, let $g \in G$, $X, Y \in \mathfrak{g}$, and $f \in C^\infty(G)$. Then we have

$$\begin{aligned}
(\mathrm{ad}_X Y(f))(g) &= \left(\frac{d}{dt}(\mathrm{Ad}_{\exp(tX)}Y)f\Big|_{t=0}\right)(g) \\
&= \frac{d}{dt}\left((\mathrm{Ad}_{\exp(tX)}Y)f\right)(g)\Big|_{t=0} \\
&= \frac{d}{dt}\frac{d}{du}f(g\exp(tX)\exp(uY)\exp(-tX))\Big|_{t=u=0}.
\end{aligned}$$

Applying the chain rule, the relation becomes

$$\begin{aligned}
(\mathrm{ad}_X Y(f))(g) &= \frac{d}{dt}\frac{d}{du}f(g\exp(tX)\exp(uY))\Big|_{t=u=0} - \frac{d}{dt}\frac{d}{du}f(g\exp(uY)\exp(-tX))\Big|_{t=u=0} \\
&= \frac{d}{dt}Yf(g\exp tX)\Big|_{t=0} - \frac{d}{du}Xf(g\exp uY)\Big|_{u=0} \\
&= XYf(g) - YXf(g) \\
&= [X,Y]f(g).
\end{aligned}$$

Hence, we have $\mathrm{ad}_X Y = [X,Y]$. $\square$

1.4. Example: matrix Lie groups

We illustrate the constructions from the previous section for matrix Lie groups (see e.g., Examples 1.4 and 1.5). In this case, the exponential map can be calculated explicitly.

1.20. DEFINITION. For an $n \times n$ matrix X, the ***exponential*** of X, denoted e^X or $\exp X$, is defined by the power series

$$e^X = \sum_{n=1}^{\infty} \frac{X^n}{n!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots,$$

where I is the $n \times n$ identity matrix.

It can be shown that this series is absolutely converging. It has the following properties.

1.3. PROPOSITION. *Let X and Y be $n \times n$ matrices. We have*

1. $e^0 = I$.
2. *If X and Y commute, then $e^{X+Y} = e^X e^Y$.*
3. $\det(e^X) = e^{\mathrm{tr}(X)}$.
4. *The curve $t \mapsto e^{tX}$ is differentiable, and*
$$\frac{d}{dt}(e^{tX}) = Xe^{tX}.$$
In particular,
$$\frac{d}{dt}(e^{tX})\big|_{t=0} = X.$$
5. *If X is sufficiently close to the identity, one can define the* **logarithm** *of X by*
$$\log(X) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(X-I)^n = (X-I) - \frac{1}{2}(X-I)^2 + \frac{1}{3}(X-I)^3 - \dots$$
Then the maps $\exp$ *and* $\log$ *are inverse, i.e.,* $\log(e^X) = X$.

1.21. EXAMPLE. Let G be the general linear group $\mathrm{GL}(n)$ (the group of invertible $n \times n$ real matrices). Then the tangent space to this group at the identity is the space $\mathfrak{gl}(n)$ of all $n \times n$ real matrices. Indeed, for any $X \in \mathfrak{gl}(n)$, the curve $\gamma(t) = \exp(tX)$ lies in $\mathrm{GL}(n)$ for all $t \in \mathbb{R}$ by Proposition 1.3(3.). We have $\gamma(0) = I$; differentiating at $t = 0$ we find $\dot\gamma(0) = X$ by Proposition 1.3(4.).
Furthermore, from Definition 1.16 we have $C_A(B) = ABA^{-1}$ for $A, B \in \mathrm{GL}(n)$. In particular, for $B = \exp(tY)$, where $Y \in \mathfrak{gl}(n)$ we obtain

$$\mathrm{Ad}_A(B) = \frac{d}{dt}(A\exp(tY)A^{-1})\big|_{t=0} = AYA^{-1},$$

by Definition 1.17. Then from Definition 1.18, taking $A = \exp(tX)$ for $X \in \mathfrak{gl}(n)$ we find

$$\mathrm{ad}_X Y = \frac{d}{dt}\big(\mathrm{Ad}_{\exp(tX)} Y\big)\big|_{t=0} = \frac{d}{dt}\big(\exp(tX) Y \exp(-tX)\big|_{t=0} = XY - YX,$$

by Proposition 1.3(4.) and the product rule.
Hence by Theorem 1.19, the Lie bracket of the Lie algebra $\mathfrak{gl}(n)$ is defined by the matrix commutator:

$$[X, Y] = XY - YX, \qquad \text{for } X, Y \in \mathfrak{gl}(n).$$

1.22. EXAMPLE. Let G be the special linear group $\mathrm{SL}(n)$ (the group of $n \times n$ real matrices with determinant 1). Then the tangent space to this group at the identity is the space $\mathfrak{sl}(n)$ of all $n \times n$ real matrices of trace zero. Indeed, for any $X \in (n)$, the curve $\gamma(t) = \exp(tX)$ lies in $\mathrm{GL}(n)$ for all $t \in \mathbb{R}$ by Proposition 1.3(3.): we have $\det(e^{tX}) = e^{\mathrm{tr}(tX)} = e^0 = 1$. Its derivative at $t = 0$ is the matrix X by Proposition 1.3(4.). By a similar argument in Example 1.21, we find that the Lie bracket on $\mathfrak{sl}(n)$ is the matrix commutator (note that the commutator of two matrices always has zero trace).

1.23. EXAMPLE. Let G be the special orthogonal group $\mathrm{SO}(n)$ (the group of $n \times n$ real orthogonal matrices with determinant 1). Then the tangent space to this group at the identity is the space $\mathfrak{so}(n)$ of all $n \times n$ skew-symmetric real matrices. Indeed, let $X \in \mathfrak{so}(\mathfrak{n})$. Then, $X^T = -X$, so X and X^T commute, and by Proposition 1.3(2.), we have

$$e^X e^{X^T} = e^{X + X^T} = e^0 = I.$$

Furthermore, as $e^{X^T} = (e^X)^T$ (because $(X^T)^m = (X^m)^T$ for all $m \in \mathbb{N}$), we have

$$I = e^X e^{X^T} = e^X (e^X)^T.$$

Therefore, we conclude that if $X \in \mathfrak{so}(\mathfrak{n})$, then $e^X \in \mathrm{O}(n)$ (is an orthogonal matrix). As the trace of any skew-symmetric matrix is zero, from the previous example we find that $\det e^X = 1$, so in fact, $e^X \in \mathrm{SO}(n)$. Repeating the argument of Example 1.21 we obtain that the Lie bracket on $\mathfrak{so}(n)$ is again the matrix commutator (note that the commutator of two skew-symmetric matrices is again skew-symmetric).

Note that we could have started with the "full" orthogonal group $\mathrm{O}(n)$ instead. The above construction would then give us the same Lie algebra $\mathfrak{so}(n)$. This illustrates the fact that two different Lie groups can have the same Lie algebra. However, the difference between such groups is purely topological (so that they coincide in a neighbourhood of the identity); in this particular example, the group $\mathrm{O}(n)$ is disconnected, and $\mathrm{SO}(n)$ is its connected component which contains the identity.

1.5. Preliminaries from Riemannian geometry

We give some main definitions and results from Riemannian geometry that are used in later sections. The references are **[2]**, **[10]**, **[19]**, and **[24]**.

1.24. DEFINITION. Let V be a vector space over $\mathbb{R}$. An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbb{R}$ that has the following properties:

1. Positive definite: for all $v \in V$,

$$\langle v, v\rangle \geq 0; \text{ and } \langle v, v\rangle = 0 \text{ if and only if } v = 0.$$

2. Symmetric: for any $u, v \in V$

$$\langle u, v\rangle = \langle v, u\rangle.$$

3. Linear: for all $u, v, w \in V$ and $a, b \in \mathbb{R}$,

$$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle.$$

1.25. DEFINITION. ***A Riemannian metric*** on a differentiable manifold M is a correspondence that associates to each point p of M an inner product $\langle \cdot, \cdot\rangle_p$ on the tangent space $T_p(M)$, which varies differentiably in the following sense: for every pair of differentiable vector fields X, Y in a neighborhood of p, the map $p \mapsto \langle X_p, Y_p\rangle$ is differentiable. A differential manifold with a Riemannian metric is called a ***Riemannian manifold***.

1.26. DEFINITION. Let M and N be Riemannian manifolds. A diffeomorphism $\phi : M \mapsto N$ is called an ***isometry*** if

$$\langle X, Y\rangle_p = \langle d\phi_p(X), d\phi_p(Y)\rangle_{\phi(p)}, \quad \text{for all } p \in M,\ X, Y \in T_p(M).$$

Riemannian metrics are abundant "in nature". The most trivial example is the Euclidean metric on $\mathbb{R}^n$ whose value at each $x \in \mathbb{R}^n$ is just the usual dot product on $T_x(\mathbb{R}^n)$ (the latter can be identified with $\mathbb{R}^n$). Taking any differentiable surface in $\mathbb{R}^3$, we can turn it into a Riemannian manifold by defining the inner product of two tangent vectors at any point as their dot product in $\mathbb{R}^3$; this example can be easily generalised for arbitrary dimensions.

Another important class of Riemannian metrics is the class of left-invariant metrics on a Lie group. As a Lie group is a differentiable manifold as well as a group, we can choose a Riemannian metric that links its geometric structure with its group properties. More specifically, we choose a Riemannian metric so that the left multiplications $L_g : G \to G$ are isometries for all $g \in G$.

1.27. DEFINITION. A Riemannian metric on a Lie group G is called ***left-invariant*** if

$$\langle X, Y\rangle_h = \langle d(L_g)_h(X), d(L_g)_h(Y)\rangle_{gh}, \quad \text{for all } g, h \in G,\ X, Y \in T_h(G).$$

As a left-invariant metric has such a large isometry group, it is not surprising that it is completely determined by its value at a single point (for which we can choose the identity). We have the following proposition (in which we use the identification between left-invariant vector fields and the tangent space to a Lie group at the identity — see Theorem 1.7).

1.4. PROPOSITION. *There is a bijective correspondence between left-invariant metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.*

PROOF. If the metric on G is left-invariant, then for all $g \in G$ and all $X, Y \in T_g(G)$, due to the left-invariance of the vector fields X, Y and the metric, we have

$$\langle X, Y\rangle_g = \langle d(L_{g^{-1}})_g(X), d(L_{g^{-1}})_g(Y)\rangle_e = \langle X_e, Y_e\rangle = \langle X, Y\rangle_e.$$

Thus, $\langle X, Y\rangle$ defines an inner product on $\mathfrak{g} = T_e(G)$.

Conversely, let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and consider

$$\langle X, Y\rangle_g = \langle d(L_{g^{-1}})_g(X), d(L_{g^{-1}})_g(Y)\rangle_e$$

for all $g \in G$ and all $X, Y \in T_g(G)$. It can be easily checked that this inner product induces a left-invariant metric on G. Hence, the result follows as desired. $\square$

1.28. DEFINITION. Let $\mathfrak{g}$ be a Lie algebra and G be the corresponding Lie group. A ***metric Lie algebra*** $(\mathfrak{g}, \langle .,.\rangle)$ is a Lie algebra $\mathfrak{g}$ together with a Euclidean inner product $\langle .,.\rangle$ on $\mathfrak{g}$. This inner product $\langle .,.\rangle$ on $\mathfrak{g}$ induces a left-invariant metric on the Lie group G by Theorem 1.4.

Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on a differentiable manifold M.

1.29. DEFINITION. ***An (affine) connection*** on a differentiable manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

denoted by $(X, Y) \mapsto \nabla_X Y$ that satisfies the following properties:

1. $\nabla_{f_1 X + f_2 Y} = f_1 \nabla_X(Z) + f_2 \nabla_Y(Z)$,
2. $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$,
3. $\nabla_X(fY) = f\nabla_X(Y) + X(f)Y$,

for all $X, Y, Z \in \mathfrak{X}(M)$ and $f_1, f_2 \in C^1(M)$. $\nabla_X Y$ is called the ***covariant derivative*** of Y in the direction of X.

Informally speaking, the connection is a way to define the directional derivative of a vector field on a manifold. Clearly there are many possible connections on a manifold, however, in the case of a Riemannian manifold, it is advantageous to choose a particular connection ∇ that reflects the properties of its Riemannian metric.

1.30. DEFINITION. Given any metric $\langle \cdot, \cdot \rangle$ on a differentiable manifold M, a connection ∇ on M is called ***compatible*** with the metric, or a ***metric connection*** if it satisfies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

It turns out that a Riemannian manifold can admit different metric connections, so compatibility is still not sufficient to identify a unique connection on such a manifold.

1.31. DEFINITION. A connection ∇ on a differentiable manifold M is called ***symmetric*** if it satisfies

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

Now these two conditions guarantee a unique connection on a manifold.

1.32. THEOREM (Fundamental Theorem of Riemannian geometry). *Let M be any Riemannian manifold. Then there exists a unique connection ∇ on M that is compatible with the metric of M and is symmetric. This connection is called the **Levi-Civita** connection on M.*

PROOF. We derive a formula for ∇. Suppose that ∇ is such a connection, and let $X, Y, Z \in \mathfrak{X}(M)$. Then by compatibility of ∇, we have

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Then, by symmetry of ∇, we have

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle, \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle. \end{aligned}$$

Adding the first two equations and subtracting the third yields

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$$
$$= 2\langle \nabla_X Y, Z\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle.$$

Solving for $\langle \nabla_X Y, Z\rangle$, we have

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$$
$$- \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle.$$

This formula is called ***Koszul's formula***. Thus, we obtain an expression that involves only the metric and the Lie bracket, but no ∇, which proves uniqueness. It is easy to check that ∇ so defined is indeed a metric compatible, symmetric affine connection, hence Koszul's formula also gives existence. □

1.33. DEFINITION. Let M be a differentiable manifold and let $\gamma : I \to M$ be a curve in M. Then a ***vector field along the curve*** γ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.

An obvious example is the tangent vector field $\dot\gamma(t)$, which is a vector along $\gamma(t)$. Now as we have a vector field, we see that it is also possible to find the covariant derivative along the curve.

1.5. PROPOSITION. *Let M be a Riemannian manifold with the Levi-Civita connection ∇, and let $\gamma : I \to M$ be a curve in M. Then there exists a unique operator that associates to a vector field V another vector field $\dot V(t) = \frac{DV}{dt}$ along the curve γ that satisfies the following properties:*

1. *For any function $f \in C^1$ on I,*
$$\frac{D}{dt}(fV)(t) = \dot f(t)V(t) + f(t)\dot V(t),$$
2. *If V is induced by a vector field $Y \in \mathfrak{X}(M)$, that is, $V(t) = Y(\gamma(t))$ for all $t \in I$, then*
$$\frac{DV}{dt} = \nabla_{\dot\gamma(t)} Y.$$

The vector field $\frac{DV}{dt}$ is called the **covariant derivative of V along γ**.

1.34. DEFINITION. Let M be a Riemannian manifold equipped with a connection ∇. A curve $\gamma : I \to M$ is called a ***geodesics*** on M if

$$\frac{D\dot\gamma(t)}{dt} = \nabla_{\dot\gamma(t)}\dot\gamma(t) = 0.$$

1.35. EXAMPLE. This definition suggests that geodesics are curves with zero "acceleration" in some sense. For example, the geodesics of $\mathbb{R}^n$ are the straight lines with constant velocities:

$$\frac{D\dot{\gamma}(t)}{dt} = 0 \quad \text{iff} \quad \ddot{\gamma}(t) = 0.$$

The most important property of geodesics is that it is locally a length-minimising curve between any two points, and conversely, all locally length-minimising curves are geodesics (see [**19**]).

1.6. Euler-Arnold equation for geodesics on a Lie group

In this section, we deduce the Euler-Arnold equation, which plays a central role in our work. Our exposition follows the material in **[1]**, **[9]**, **[10]**, and **[22]**, to which we also refer the readers for further details.

We begin by introducing the adjoint of the operator ad. Let $\mathfrak{g}$ be a metric Lie algebra with the inner product $\langle\cdot,\cdot\rangle$. For $X, Y, Z \in \mathfrak{g}$, define

$$\langle Z, \mathrm{ad}_X Y\rangle = \langle \mathrm{ad}^*_X Z, Y\rangle.$$

The map $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ X \mapsto \mathrm{ad}^*_X$ is called the ***coadjoint representation*** of $\mathfrak{g}$.

1.6. PROPOSITION. ad^* *is bi-linear, that is,* $\mathrm{ad}^*_{\mathrm{X}}\mathrm{Y}$ *is linear in both arguments* $X, Y \in \mathfrak{g}$.

PROOF. Let X, Y_1, Y_2 and $Z \in \mathfrak{g}$ and $a_1, a_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\langle a_1Y_1 + a_2Y_2, [X, Z]\rangle &= a_1\langle Y_1, [X, Z]\rangle + a_2\langle Y_2, [X, Z]\rangle \\
\implies \quad \langle a_1Y_1 + a_2Y_2, \mathrm{ad}_X Z\rangle &= a_1\langle Y_1, \mathrm{ad}_X Z\rangle + a_2\langle Y_2, \mathrm{ad}_X Z\rangle \\
\implies \langle \mathrm{ad}^*_X(a_1Y_1 + a_2Y_2), Z\rangle &= a_1\langle \mathrm{ad}^*_X Y_1, Z\rangle + a_2\langle \mathrm{ad}^*_X Y_2, Z\rangle \\
\implies \langle \mathrm{ad}^*_X(a_1Y_1 + a_2Y_2), Z\rangle &= \langle a_1\mathrm{ad}^*_X Y_1 + a_2\mathrm{ad}^*_X Y_2, Z\rangle.
\end{aligned}$$

As this is true for any Z, we get $\mathrm{ad}^*_X(a_1Y_1 + a_2Y_2) = a_1\mathrm{ad}^*_X Y_1 + a_2\mathrm{ad}^*_X Y_2$, which proves linearity by Y. Similarly, we have $\mathrm{ad}^*_{a_1X_1+a_2X_2}Y = a_1\mathrm{ad}^*_{X_1}Y + a_2\mathrm{ad}^*_{X_2}Y$ for any X_1, X_2 and $Y \in \mathfrak{g}$ and any $a_1, a_2 \in \mathbb{R}$, which proves the linearity by X. $\square$

The following proposition gives the formula for the Levi-Civita connection of a left-invariant metric on a Lie group.

1.7. PROPOSITION. *Let G be a Lie group with a left-invariant metric $\langle\cdot,\cdot\rangle$, and let X, Y, Z be left-invariant vector fields. Then we have*

$$\nabla_X Y = \tfrac{1}{2}\left([X, Y] - \mathrm{ad}^*_X Y - \mathrm{ad}^*_Y X\right).$$

PROOF. Let G be a Lie group with a left-invariant metric $\langle\cdot,\cdot\rangle$, and let X, Y, Z be left-invariant vector fields. Then, by Proposition 1.4, $\langle\cdot,\cdot\rangle_g$ evaluated on left-invariant vector fields does not depend on $g \in G$. Therefore, we have

$$X\langle Y, Z\rangle = 0, \quad Y\langle X, Z\rangle = 0, \quad Z\langle X, Y\rangle = 0.$$

Now, substituting this into the Koszul's formula in Theorem 1.32, we obtain

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle &= -\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \\
&= \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle \\
&= \langle [X, Y], Z\rangle - \langle \mathrm{ad}_Y Z, X\rangle - \langle \mathrm{ad}_X Z, Y\rangle \\
&= \langle [X, Y], Z\rangle - \langle Z, \mathrm{ad}^*_Y X\rangle - \langle Z, \mathrm{ad}^*_X Y\rangle \\
&= \langle [X, Y], Z\rangle - \langle \mathrm{ad}^*_X Y, Z\rangle - \langle \mathrm{ad}^*_Y X, Z\rangle,
\end{aligned}$$

as required. □

We are ready to prove the Euler-Arnold equation. Let G be a Lie group with a left-invariant metric and let $\mathfrak{g}$ be the corresponding metric Lie algebra. For a C^2 curve $\gamma : I \to G$, define the curve $X : I \to \mathfrak{g}$ by $X(t) = dL_{\gamma(t)^{-1}}\dot\gamma(t)$ (so that $X(t)$ is the result of translation, by the differential of the left action, of the vector $\dot\gamma(t)$ from the point $\gamma(t)$ to the identity). The curve $X(t)$ lies in the Lie algebra $\mathfrak{g}$ of G and is sometimes called the *hodograph* of γ. In this notation, we have the following.

1.36. THEOREM. *The curve $\gamma(t)$ is a naturally parameterised geodesic on a Lie group G if and only if*

$$\dot X = \mathrm{ad}^*_X X, \tag{1.6.1}$$

Equation (1.6.1) *is called the* **Euler-Arnold equation**.

PROOF. We prove the "only if" part; the "if" part follows by reversing the argument. Let G be a Lie group with a left-invariant metric $\langle \cdot, \cdot\rangle$ and $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields of G. Let $\gamma = \gamma(t)$ be a geodesic on G, with t a natural parameter, and let $\dot\gamma(t) = \frac{d\gamma}{dt}$ be the unit tangent vector field along γ. Denote E_i an orthonormal basis in $\mathfrak{g}$ relative to the inner product $\langle \cdot, \cdot\rangle$; extend each E_i to a left-invariant vector field on G and let $E_i(t)$ be the value of that vector field at the point $\gamma(t)$. We can decompose $\dot\gamma(t) = \sum_i X_i(t)E_i(t)$, where $X_i(t) : \mathbb{R} \to \mathbb{R}$ are C^1 functions. Note that in this notation, $X(t) = \sum_i X_i(t)E_i$. As γ is a geodesic, by Definition 1.34, we have

$$\begin{aligned}
0 = \nabla_{\dot\gamma(t)}\dot\gamma(t) &= \nabla_{\dot\gamma(t)} \sum_i X_i E_i \\
&= \sum_i \dot X_i E_i + \sum_i X_i \nabla_X E_i \\
&= \sum_i \dot X_i E_i + \sum_i X_i \nabla_{\sum_j X_j E_j} E_i \\
&= \sum_i \dot X_i E_i + \sum_{ij} X_i X_j \nabla_{E_j} E_i.
\end{aligned}$$

Note that E_i are left-invariant vector fields, so by Proposition 1.7, we have

$$\nabla_{E_j} E_i = \frac{1}{2}([E_j, E_i] - \mathrm{ad}^*_{E_j} E_i - \mathrm{ad}^*_{E_i} E_j).$$

Thus, the last equation becomes

$$0 = \sum_i \dot{X}_i E_i + \frac{1}{2} \sum_{ij} X_i X_j ([E_j, E_i] - \mathrm{ad}^*_{E_j} E_i - \mathrm{ad}^*_{E_i} E_j).$$

By Proposition 1.6, we have

$$\begin{aligned}\sum_{i,j} X_i X_j \mathrm{ad}^*_{E_j} E_i &= \sum_i \sum_j X_i X_j \mathrm{ad}^*_{E_j} E_i = \sum_i X_i \mathrm{ad}^*_{\sum X_j E_j} E_i \\ &= \sum_i X_i \mathrm{ad}^*_X E_i = \mathrm{ad}^*_X (\sum_i X_i E_i) = \mathrm{ad}^*_X X.\end{aligned}$$

So we have

$$\begin{aligned}0 &= \dot{X} + \frac{1}{2}[X, X] - \frac{1}{2}\mathrm{ad}^*_X X - \frac{1}{2}\mathrm{ad}^*_X X \\ &= \dot{X} - \mathrm{ad}^*_X X,\end{aligned}$$

which yields the Euler-Arnold equation. □

1.6.1. REMARK. Equation (1.6.1) always has a first integral $\frac{1}{2}\|X\|^2$. This is easy to prove since we have

$$\left(\frac{1}{2}\|X\|^2\right)^{\cdot} = \left(\frac{1}{2}\langle X, X\rangle\right)^{\cdot} = \langle \dot{X}, X\rangle = \langle \mathrm{ad}^*_X X, X\rangle = \langle X, \mathrm{ad}_X X\rangle = 0.$$

In the 3-dimensional case, this gives

$$X_1^2 + X_2^2 + X_3^2 = \text{constant},$$

and in the 4-dimensional case,

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = \text{constant},$$

where $X_i = X_i(t)$ are the components of $X(t)$ relative to an orthonormal basis for $\mathfrak{g}$.

1.7. Stability of stationary points of autonomous systems of ODEs

Continuing from the previous section, we introduce the definition of stationary points from Dynamical systems theory that is equivalent to homogeneous geodesics on Lie groups. We also describe the condition of Lyapunov stability of stationary points. The references are **[4]**, **[11]**, and **[16]**.

Consider an autonomous system of ordinary differential equations: for $X : I \to \Omega \subset \mathbb{R}^n, f : \Omega \to \mathbb{R}^n \in C^1$,

$$\dot{X} = f(X), \tag{1.7.1}$$

whose right-hand side is independent of t. The points X_0 where f vanishes, i.e., where $f(X_0) = 0$ are called ***stationary points*** of the system. Clearly, if X_0 is a stationary point, then $X(t) = X_0$ is a solution of (1.7.1). If we look at the Euler-Arnold equation (1.6.1), then by definition, the stationary points can be found by equating the expression on the right-hand side to zero. Geometrically, the stationary points of the Euler-Arnold equation (1.6.1) correspond to geodesics on the metric Lie group whose tangent vector is left-invariant, that is, to geodesics of the form $\exp(tX)$. Thus such geodesics are 1-dimensional subgroups of G (see Theorem 1.14 and Definition 1.15). They are called ***homogeneous geodesics***.

1.37. DEFINITION. A stationary point X_0 of (1.7.1) is called ***(Lyapunov) stable*** if for any $\epsilon > 0$, there exists a $\epsilon' > 0$ such that for any solution $X(t)$, we have

$$\|X(0) - X_0\| < \epsilon' \Rightarrow \|X(t) - X_0\| < \epsilon, \quad \text{for all } \ t \geq 0.$$

Otherwise the point X_0 is called ***(Lyapunov) unstable***.

Informally speaking, a stationary point of the system is stable if the solutions starting at nearby points remain close to it for all future times. The question of stability is extremely significant in real-world physical applications, where systems are constantly subject to small changes and it is often impossible to measure a perfect initial condition. A stable stationary point allows some small perturbations, which means that it corresponds to a steady state observed in a realistic system. On the other hand, even small errors in the initial state would force the solutions to move away from an unstable stationary point.

We begin the stability analysis by assuming that the system (1.7.1) is linear and homogeneous, that is,

$$\dot{X} = AX, \tag{1.7.2}$$

where A is a constant $n \times n$ matrix, and that the stationary point of interest is the zero solution $X(t) = 0$.

1.38. THEOREM. *Given the system* (1.7.2)*, denote* $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{C}$ *the eigenvalues of the matrix* A *(note that* $m \leq n$*).*

(i) If $\mathrm{Re}(\lambda_i) < 0$ *for all* $i = 1, 2, \cdots, m$*, then the stationary point of* (1.7.2) *is stable.*

(ii) If $\mathrm{Re}(\lambda_i) > 0$ *for at least one* $i = 1, 2, \cdots, m$*, then the stationary point of* (1.7.2) *is unstable.*

The proof can be found in [**11**]. As our system (1.6.1) from the Euler-Arnold equation is nonlinear, we will now describe a technique known as *linearisation* for nonlinear systems. Suppose we have an autonomous system (1.7.1) and X_0 is a stationary point of this system, i.e., $f(X_0) = 0$. Consider a perturbation X near X_0, that is, $X = X_0 + Z$, where Z is small. The equation (1.7.1) becomes

$$\dot{Z} = f(X_0 + Z).$$

Taking the Taylor expansion of the right-hand side at X_0 gives

$$\dot{Z} = f(X_0) + J(X_0)Z + o(Z),$$

where $O(Z)$ denotes higher-order terms in the expansion, and $J(X_0)$ is the ***Jacobian matrix*** of the system evaluated at X_0, defined by

$$J(X_0) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \cdots & \frac{\partial f_2}{\partial X_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1} & \frac{\partial f_n}{\partial X_2} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix}\right|_{X=X_0}.$$

As X_0 is a stationary point, we have $f(X_0) = 0$, and for solutions close to the stationary point, $O(Z)$ is negligible. We obtain the linearised system of (1.7.1) at X_0:

$$\dot{Z} = J(X_0)Z.$$

We can then apply Theorem 1.38 to get the following important result.

1.39. THEOREM. *Let* X_0 *be a stationary point of the system* (1.7.1)*, and let* $J(X_0)$ *be the Jacobian matrix of* f *at* X_0*. Then:*

(i) If all the eigenvalues of $J(X_0)$ *have negative real parts, then the point* X_0 *is stable.*

(ii) If at least one eigenvalue of $J(X_0)$ *has a positive real part, the point* X_0 *is unstable.*

In the 2-dimensional case ($n = 2$), the criteria given in Theorem 1.39 can be expressed in a nicer way. Recall that for a 2×2 matrix $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is given by

$$\det(J - \lambda I) = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \mathrm{tr}(J)\lambda + \det(J),$$

where $\mathrm{tr}(J)$ and $\det(J)$ are the trace and the determinant of the matrix J, respectively. The eigenvalues of J are $\frac{-\mathrm{tr}(J) \pm \sqrt{\mathrm{tr}^2(J) - 4\det(J)}}{2}$, and Theorem 1.39 can be restated as follows.

1.40. THEOREM. *Let $n = 2$. Suppose X_0 is a stationary point of the system* (1.7.1), *and let $J_0 = J(X_0)$ be the 2×2 Jacobian matrix of f at X_0. Then*

(i) If $\det(J_0) < 0$, then the point X_0 is unstable.
(ii) If $\det(J_0) > 0$, then the point X_0 is unstable if $\operatorname{tr}(J_0) > 0$ or stable if $\operatorname{tr}(J_0) < 0$.

We need to emphasise that both Theorem 1.39 and 1.40 are not applicable if the Jacobian matrix has zero eigenvalues, because in that case, stability can depend on the higher order terms in the Taylor expansion. In the case of zero eigenvalues, we discuss another method, known as the *Lyapunov's direct method*. The name is inspired by the fact that we can determine the stability of a stationary point without any knowledge of the solution of the system, but rather via a suitable auxiliary function.

1.41. THEOREM. *Let X_0 be a stationary point of the system* (1.7.1). *Let $V : D \to \mathbb{R}$ be a differentiable function defined on an open set D containing X_0. Suppose further that*

(1) V is positive definite *on D, i.e., $V(X_0) = 0$, and $V(X) > 0$ for all $X \in D \setminus \{X_0\}$;*
(2) The derivative $\dot{V}$ of V is negative definite *on D, i.e., $\dot{V}(X_0) = 0$, and $\dot{V}(X) < 0$ for all $X \in D \setminus \{X_0\}$.*

Then X_0 is stable.

The function V is called a ***Lyapunov function***. The proof of the theorem can be found in **[4]**, **[11]** or **[23]**.

1.42. EXAMPLE. Consider the following dynamical system

$$\begin{cases} \dot{X}_1 = -2X_1X_2^2 - X_1^3 \\ \dot{X}_2 = -X_2 + X_1^2X_2 \end{cases}.$$

The only stationary point for this system is the origin $(X_1, X_2) = (0, 0)$. To investigate its stability, consider the quadratic Lyapunov function

$$V = X_1^2 + X_2^2.$$

It is clear that V is positive definite on the entire space $\mathbb{R}^2$. In addition, the derivative of V given by

$$\dot{V} = -2X_2^2 - 2X_1^2(X_1^2 + X_2^2)$$

is negative definite. By Theorem 1.41, we conclude that $(X_1, X_2) = (0, 0)$ is a stable stationary point. Note that the Jacobian matrix of the system at $(X_1, X_2) = (0, 0)$ is $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, whose determinant is zero. Hence Theorem 1.40 is not applicable.

In some cases, we need a stronger version of this theorem known as the *Chetaev Instability Theorem* [**6**]:

1.43. THEOREM. *Let X_0 be a stationary point of the system* (1.7.1). *Let $V(X)$ be a differentiable function defined on a neighbourhood of X_0 and let Q be a connected component of the set $\{X : V(X) > 0\}$. Suppose further that*

(1) X_0 lies on the boundary of Q;

(2) There exists a neighbourhood D of X_0 such that $\dot{V} > 0$ for all $X \in D \cap Q$.

Then X_0 is unstable.

The function V is called a ***Chetaev function***.

1.44. EXAMPLE. Consider the following dynamical system

$$\begin{cases} \dot{X}_1 = X_1^2 + 2X_2^5 \\ \dot{X}_2 = X_1 X_2^2 \end{cases}.$$

Again, the only stationary point for this system is the origin $(X_1, X_2) = (0, 0)$. To investigate its stability, consider the Chetaev function

$$V = X_1^2 - X_2^4,$$

which is positive inside the region Q bounded by

$$X_1 = X_2^2 \quad \text{and} \quad X_1 = -X_2^2.$$

Clearly the origin lies on the boundary of Q, and if we let D be the right half-plane, then $\dot{V} = 2X_1^3 > 0$ for all $(X_1, X_2) \in D \cap Q$. By Theorem 1.43, we conclude that $(X_1, X_2) = (0, 0)$ is an unstable stationary point.

There is another stability theorem by Arnold [**1**, Théorème 4] that we use frequently in the project. This theorem concerns a special type of stationary points in a Lie group.

1.45. DEFINITION. Suppose $X_0 \in \mathfrak{g}$ is a stationary point and $Y \in \mathfrak{g}$ is an arbitrary point. Denote $B(X, Y) = \mathrm{ad}_Y^* X$. Then X_0 is called *regular* if the dimension of the linear space

$$L_{X_0} = \{Y \in \mathfrak{g} \,:\, B(X_0, Y) = 0\}$$

is locally constant.

1.46. THEOREM. *Suppose X_0 is a regular stationary point. If the quadratic form*

$$\Phi(Y) = \|B(X_0, Y)\|^2 + \langle [Y, X_0], B(X_0, Y) \rangle$$

is positive definite on the orthogonal complement to the space L_{X_0}, then the point X_0 is stable.

CHAPTER 2

Stability of homogeneous geodesics in 3-dimensional metric Lie algebras

In Chapter 1, we have given a survey of related theories necessary for our understanding of stability of homogeneous geodesics on a Lie group. We now begin to discuss the main findings of the project in this chapter. In Section 2.1, we give the classification of 3-dimensional metric Lie algebras. In Section 2.2, we proceed to apply the Euler-Arnold equation (1.6.1) to find all the homogeneous geodesics of the Lie algebra, and classify their stability status respectively for unimodular Lie algebras in Section 2.3 and for non-unimodular Lie algebras in Sections 2.4-2.5.

2.1. Classification of 3-dimensional metric Lie algebras

The classifications which we obtain in this section will be the starting point of the proof of the first two main theorems of this thesis. We note that both these classifications are not new; the classification in the unimodular case was obtained in **[21]** and is also known in the non-unimodular case (see e.g. **[20]**); for completeness we provide the full proof in both cases.

It is important to emphasise that the classification of real 3-dimensional Lie algebras up to isomorphism is a classical result obtained by Bianchi in 1898 (see **[3]** for English translation). We will classify 3-dimensional *metric* Lie algebras up to *isometric isomorphism*.

We start with 3-dimensional unimodular metric Lie algebras. Recall that a metric Lie algebra is a Lie algebra endowed with a positive definite inner product $\langle\cdot,\cdot\rangle$. Given a Lie algebra $\mathfrak{g}$, for any $X \in \mathfrak{g}$, the linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_X Y = [X, Y]$ by Theorem 1.19.

2.1. DEFINITION. A Lie algebra $\mathfrak{g}$ is said to be ***unimodular*** if $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$.

2.2. LEMMA. **[21**, Lemma 4.1**]** *Let $\mathfrak{g}$ be a 3-dimensional metric Lie algebra with the inner product $\langle\cdot,\cdot\rangle$. Then there exists a unique linear map $L : \mathfrak{g} \to \mathfrak{g}$ such that the Lie bracket is given by*

$$[X, Y] = L(X \times Y),$$

where $\times$ is the cross product in the underlying 3-dimensional Euclidean space of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is unimodular if and only if the map L is symmetric relative to $\langle\cdot,\cdot\rangle$.

PROOF. Choose a positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ for $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and define the linear map $L : \mathfrak{g} \to \mathfrak{g}$ by $L(e_1) = [e_2, e_3], L(e_2) = [e_3, e_1], L(e_3) = [e_1, e_2]$. Then $L(e_i \times e_j) = [e_i, e_j]$, hence $L(X \times Y) = [X, Y]$ by bilinearity of both the Lie bracket and the cross product.
Introduce $\alpha_{ij} \in \mathbb{R}$, where $i, j \in \{1, 2, 3\}$, by

$$L(e_i) = \sum\nolimits_{j=1}^{3} \alpha_{ij} e_j.$$

Then we have:

$$\begin{aligned}
&\mathrm{ad}_{e_1}(e_1) = [e_1, e_1] = 0,\\
&\mathrm{ad}_{e_1}(e_2) = [e_1, e_2] = L(e_3) = \alpha_{31}e_1 + \alpha_{32}e_2 + \alpha_{33}e_3,\\
&\mathrm{ad}_{e_1}(e_3) = [e_1, e_3] = -[e_3, e_1] = -L(e_2) = -\alpha_{21}e_1 - \alpha_{22}e_2 - \alpha_{23}e_3.
\end{aligned}$$

We obtain

$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & \alpha_{31} & -\alpha_{21} \\ 0 & \alpha_{32} & -\alpha_{22} \\ 0 & \alpha_{33} & -\alpha_{23} \end{pmatrix},$$

and so $\mathrm{tr}(\mathrm{ad}_{e_1}) = \alpha_{32} - \alpha_{23}$.
Likewise, we get $\mathrm{tr}(\mathrm{ad}_{e_2}) = \alpha_{13} - \alpha_{31}$ and $\mathrm{tr}(\mathrm{ad}_{e_3}) = \alpha_{21} - \alpha_{12}$.
Thus, $\mathfrak{g}$ is unimodular if and only if the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

of L relative to our basis is symmetric: $\alpha_{ij} = \alpha_{ji}$ for every i, j, that is, if L is a symmetric linear map. $\square$

If L is symmetric, then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of eigenvectors of L, so that $Le_i = \lambda_i e_i$ for $i = 1, 2, 3$ where λ_i are the corresponding eigenvalues of L. Then we have

$$\begin{aligned}
[e_1, e_2] &= Le_3 = \lambda_3 e_3,\\
[e_2, e_3] &= Le_1 = \lambda_1 e_1,\\
[e_3, e_1] &= Le_2 = \lambda_2 e_2.
\end{aligned} \tag{2.1.1}$$

These equations give Milnor's classification of 3-dimensional unimodular metric Lie algebras: every such algebra is uniquely determined by a choice of three real numbers $\lambda_1, \lambda_2, \lambda_3$.

2.1.1. REMARK. The three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are defined up to orientation. This means that if we change the orientation of our basis to the opposite, then L changes to $-L$, and so we can switch sign of all three eigenvalues without changing the properties of the Lie algebra. Furthermore, by scaling the vectors of our basis we can make every nonzero λ_i to be equal to its sign (but of course, we lose orthonormality). This gives Milnor's classification of 3-dimensional unimodular Lie algebras up to isomorphism (no metric!): we can choose the sign, $+, -$ or 0 for every element of the triple $(\lambda_1, \lambda_2, \lambda_3)$, up to cyclic permutation and simultaneous change of the sign to the opposite.

In conclusion, we have 6 classes of 3-dimensional unimodular Lie algebras as follows in the table.

Signs of $(\lambda_1, \lambda_2, \lambda_3)$	Possible variations
$(0,0,0)$	
$(+,0,0)$	$(0,+,0),(0,0,+),(-,0,0),(0,-,0),(0,0,-)$
$(+,+,0)$	$(0,+,+),(+,0,+),(-,-,0),(0,-,-),(-,0,-)$
$(+,-,0)$	$(0,+,-),(-,0,+),(-,+,0),(0,-,+),(+,0,-)$
$(+,+,+)$	$(-,-,-)$
$(+,+,-)$	$(-,+,+),(+,-,+),(-,-,+),(+,-,-),(-,+,-)$

TABLE 2.1. Classes of 3-dimensional unimodular Lie algebras.

To obtain the classification for 3-dimensional *non-unimodular* metric Lie algebras, we use the map L constructed in Lemma 2.2 and the Jacobi identity. This classification is also known in the literature (see e.g., **[20]**).

2.3. LEMMA. *Let $\mathfrak{g}$ be a 3-dimensional non-unimodular metric Lie algebra with the inner product $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ such that the Lie brackets in $\mathfrak{g}$ are given by*

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

where $\alpha + \delta > 0$.

PROOF. Choose an arbitrary orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$. By Lemma 2.2, we have $Le_1 = [e_2, e_3]$, $Le_2 = [e_3, e_1]$, and $Le_3 = [e_1, e_2]$. Then the Jacobi identity gives

$$\begin{aligned} & [e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] = 0 \\ \Longrightarrow \quad & L(Le_1 \times e_1) + L(Le_2 \times e_2) + L(Le_3 \times e_3) \quad = 0. \end{aligned}$$

Now denote $\alpha_{ij},\ i, j \in \{1, 2, 3\}$, the entries of the matrix L. Note that $L(e_i) = \sum_{j=1}^{3} \alpha_{ij} e_j$, and so

$$\begin{aligned} Le_1 \times e_1 &= \alpha_{13} e_2 - \alpha_{12} e_3, \\ Le_2 \times e_2 &= -\alpha_{23} e_1 + \alpha_{21} e_3, \\ Le_3 \times e_3 &= \alpha_{32} e_1 - \alpha_{31} e_2. \end{aligned}$$

Thus, from the Jacobi identity we obtain

$$L(\alpha_{13}e_2 - \alpha_{12}e_3) + L(-\alpha_{23}e_1 + \alpha_{21}e_3) + L(\alpha_{32}e_1 - \alpha_{31}e_2) = 0.$$

By linearity of L we obtain

$$\begin{aligned} \alpha_{13}Le_2 - \alpha_{12}Le_3 - \alpha_{23}Le_1 + \alpha_{21}Le_3 + \alpha_{32}Le_1 - \alpha_{31}Le_2 &= 0 \\ \implies \quad (\alpha_{23} - \alpha_{32})Le_1 + (\alpha_{31} - \alpha_{13})Le_2 + (\alpha_{12} - \alpha_{21})Le_3 &= 0 \end{aligned}$$

So if we let

$$v = \begin{pmatrix} \alpha_{23} - \alpha_{32} \\ \alpha_{31} - \alpha_{13} \\ \alpha_{12} - \alpha_{21} \end{pmatrix},$$

the latter equation is equivalent to $Lv = 0$. As $\mathfrak{g}$ is non-unimodular, by Lemma 2.2, $\alpha_{ij} \neq \alpha_{ji}$ for at least one pair (i, j). Thus, $v \neq 0$. So we can specify our basis in such a way that e_1 is a unit vector in the direction of v, so that $v = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}$, where $c \neq 0$.

Then $\alpha_{31} - \alpha_{13} = \alpha_{12} - \alpha_{21} = 0$ and the equation $Lv = 0$ gives

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} = 0 \implies c \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{pmatrix} = 0 \implies \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{pmatrix} = 0.$$

Therefore we obtain

$$\begin{aligned} [e_1, e_2] &= \alpha_{32}e_2 + \alpha_{33}e_3, \\ [e_1, e_3] &= -\alpha_{22}e_2 - \alpha_{23}e_3, \\ [e_2, e_3] &= 0. \end{aligned}$$

By changing e_1 to $-e_1$ if necessary, we can assume that $-c = \alpha_{32} - \alpha_{23} > 0$, and the proof is complete up to changing the notation. $\square$

2.2. Homogeneous geodesics in 3-dimensional metric Lie algebras

In this section, we will write down the Euler-Arnold equation for geodesics in 3-dimensional metric Lie algebras and find the stationary points. This was independently done in **[20]**.

We begin with 3-dimensional unimodular Lie algebras. Consider the six cases in Table 2.1.

1. $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$. This is an abelian Lie algebra, so we have $[X, Y] = 0$ for any X, Y.
   Hence
$$\dot{X}(t) = \mathrm{ad}^*_X X = 0. \tag{2.2.1}$$

2. $(\lambda_1, \lambda_2, \lambda_3) : (+, 0, 0)$. If $X = X(t)$ is the hodograph of a geodesic γ on the Lie group G of the algebra $\mathfrak{g}$ and $X = X_1e_1 + X_2e_2 + X_3e_3$ is its decomposition relative to the basis e_i, we obtain
$$\begin{aligned} X(t) &= X_1e_1 + X_2e_2 + X_3e_3 \\ \implies \mathrm{ad}_{X(t)} &= X_1\mathrm{ad}_{e_1} + X_2\mathrm{ad}_{e_2} + X_3\mathrm{ad}_{e_3} \\ &= \begin{pmatrix} 0 & -\lambda_1X_3 & \lambda_1X_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$
   Equation (1.6.1) gives
$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda_1X_1X_3 \\ \lambda_1X_1X_2 \end{pmatrix} \implies \begin{cases} \dot{X}_1 = 0 \\ \dot{X}_2 = -\lambda_1X_1X_3 \\ \dot{X}_3 = \lambda_1X_1X_2 \end{cases}. \tag{2.2.2}$$

3. $(\lambda_1, \lambda_2, \lambda_3) : (+, +, 0)$. Computing in a similar manner, we obtain
$$\begin{cases} \dot{X}_1 = \lambda_2X_2X_3 \\ \dot{X}_2 = -\lambda_1X_1X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2)X_1X_2 \end{cases}, \text{ for } \lambda_1, \lambda_2 > 0. \tag{2.2.3}$$

4. $(\lambda_1, \lambda_2, \lambda_3) : (+, -, 0)$. We obtain a system similar to system (2.2.3), but with the condition that $\lambda_1 > 0$ and $\lambda_2 < 0$:
$$\begin{cases} \dot{X}_1 = \lambda_2X_2X_3 \\ \dot{X}_2 = -\lambda_1X_1X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2)X_1X_2 \end{cases}, \text{ for } \lambda_1 > 0, \lambda_2 < 0. \tag{2.2.4}$$

5. $(\lambda_1, \lambda_2, \lambda_3) : (+, +, +)$. The system of ODEs is

$$\begin{cases} \dot{X}_1 = (\lambda_2 - \lambda_3) X_2 X_3 \\ \dot{X}_2 = (\lambda_3 - \lambda_1) X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} \quad \text{, for } \lambda_1, \lambda_2, \lambda_3 > 0. \tag{2.2.5}$$

6. $(\lambda_1, \lambda_2, \lambda_3) : (+, +, -)$. We obtain a system similar to system (2.2.5), but with the condition that $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$:

$$\begin{cases} \dot{X}_1 = (\lambda_2 - \lambda_3) X_2 X_3 \\ \dot{X}_2 = (\lambda_3 - \lambda_1) X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} \quad \text{, for } \lambda_1, \lambda_2 > 0, \lambda_3 < 0. \tag{2.2.6}$$

We find the stationary points for each of these six systems by equating all the expressions on the right-hand sides to zero. Recall that by Remark 1.6.1, these systems have a first integral $\frac{1}{2}\|X\|^2$, which means that if the initial condition lies on a sphere, then so does the whole solution curve. Hence, it is sufficient to study its behaviour on the unit sphere $S^2 = S^2(1) \subset \mathbb{R}^3$.

1. For (2.2.1), $\dot{X} = 0$, so all points on the sphere S^2 are stationary points.
2. For system (2.2.2), stationary solutions are given by

$$X_1 = 0 \quad \text{or} \quad X_2 = X_3 = 0,$$

   so on S^2 we have a pair of antipodal points $(\pm 1, 0, 0)$ and the whole circle $X_1 = 0$.
3. For systems (2.2.3) and (2.2.6), in case $\lambda_1 \neq \lambda_2$ we have three pairs of antipodal points $(\pm 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$. If $\lambda_1 = \lambda_2$, we get a pair of antipodal points $(0, 0, \pm 1)$ and the circle $X_3 = 0$.
4. For system (2.2.4), we get three pairs of antipodal points $(\pm 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$.
5. For system (2.2.5), if $\lambda_i,\ i = 1, 2, 3$, are pairwise nonequal, we have three pairs of antipodal points $(\pm 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$. If $\lambda_i = \lambda_j \neq \lambda_k$, where $i, j, k \in \{1, 2, 3\}$, we get a pair of antipodal points $X_i = X_j = 0$ and the circle $X_k = 0$. If $\lambda_1 = \lambda_2 = \lambda_3$, all points on the sphere S^2 are stationary.

This completes the search for homogeneous geodesics in 3-dimensional unimodular metric Lie algebras.

We continue with 3-dimensional non-unimodular metric Lie algebras. Let $\mathfrak{g}$ be a 3-dimensional non-unimodular metric Lie algebra with the inner product $\langle\cdot,\cdot\rangle$. Then by Lemma 2.3, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ such that the Lie brackets in $\mathfrak{g}$ are given by

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

where $\alpha + \delta > 0$. We compute

$$\begin{aligned}
\mathrm{ad}_{e_1}(e_1) &= [e_1, e_1] = 0, \\
\mathrm{ad}_{e_1}(e_2) &= [e_1, e_2] = \alpha e_2 + \beta e_3, \\
\mathrm{ad}_{e_1}(e_3) &= [e_1, e_3] = -\gamma e_2 + \delta e_3.
\end{aligned}$$

Thus, we obtain $\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}$. Similarly, $\mathrm{ad}_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix}$ and $\mathrm{ad}_{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ -\gamma & 0 & 0 \\ -\delta & 0 & 0 \end{pmatrix}$.

Hence, we obtain

$$\begin{aligned}
\mathrm{ad}_X &= X_1 \mathrm{ad}_{e_1} + X_2 \mathrm{ad}_{e_2} + X_3 \mathrm{ad}_{e_3} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ -\alpha X_2 - \gamma X_3 & \alpha X_1 & \gamma X_1 \\ -\beta X_2 - \delta X_3 & \beta X_1 & \delta X_1 \end{pmatrix},
\end{aligned}$$

which gives

$$\begin{aligned}
\mathrm{ad}^*_X &= X_1 \mathrm{ad}^*_{e_1} + X_2 \mathrm{ad}^*_{e_2} + X_3 \mathrm{ad}^*_{e_3} \\
&= \begin{pmatrix} 0 & -\alpha X_2 - \gamma X_3 & -\beta X_2 - \delta X_3 \\ 0 & \alpha X_1 & \beta X_1 \\ 0 & \gamma X_1 & \delta X_1 \end{pmatrix}.
\end{aligned}$$

So, the Euler-Arnold equation (1.6.1) gives

$$\begin{aligned}
\dot{X} &= \mathrm{ad}^*_X X \\
\Longrightarrow \quad \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} &= \begin{pmatrix} 0 & -\alpha X_2 - \gamma X_3 & -\beta X_2 - \delta X_3 \\ 0 & \alpha X_1 & \beta X_1 \\ 0 & \gamma X_1 & \delta X_1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \\
&= \begin{pmatrix} (-\beta - \gamma) X_2 X_3 - \alpha X_2^2 - \delta X_3^3 \\ \alpha X_1 X_2 + \beta X_1 X_3 \\ \gamma X_1 X_2 + \delta X_1 X_3 \end{pmatrix}.
\end{aligned}$$

If we let $x = X_1$, $u = \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$, and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the last equation can be written in a more compact form as the following autonomous system of differential equations:

$$\begin{cases} \dot{x} = -\langle Mu, u \rangle \\ \dot{u} = xMu \end{cases}. \tag{2.2.7}$$

We find the stationary points of system (2.2.7) on the unit sphere S^2 given by

$$x^2 + \|u\|^2 = 1 \Leftrightarrow X_1^2 + X_2^2 + X_3^2 = 1.$$

The right-hand side of (2.2.7) is always zero when $u = 0$. This gives two antipodal stationary points $(\pm 1, 0, 0)$, independent of the matrix M. To find other stationary points, we consider several cases depending on the matrix M.

1. Suppose that the matrix M is singular, i.e., there is a vector $u \neq 0$ such that $Mu = 0$. Note that the kernel of M must be 1-dimensional, as otherwise we would have $M = 0$, which contradicts the fact from Lemma 2.3 that $\mathrm{tr}(M) = \alpha + \delta > 0$. As M is singular, we have $\det(M) = \det(M^T) = 0$, so the transpose matrix M^T of M is also singular. Thus, there exists a unit vector u such that $M^T u = 0$. We can then specify the basis in the (u_1, u_2)-plane relative to which $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we have $M = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$, with $\alpha > 0$. This leads to 2 subcases.

   (i) Suppose that $\beta = 0$, so that $M = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$. The system (2.2.7) becomes

   $$\begin{cases} \dot{X}_1 = -\alpha X_2^2 \\ \dot{X}_2 = \alpha X_2 X_3 \\ \dot{X}_3 = 0 \end{cases}. \tag{2.2.8}$$

   Setting the right-hand side to 0, we find $X_2 = 0$. This gives the circle of stationary points $X_1^2 + X_3^2 = 1$.

   (ii) Suppose that $\beta \neq 0$, so that $M = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$. The system (2.2.7) becomes

   $$\begin{cases} \dot{X}_1 = -(\alpha X_2 + \beta X_3) X_2 \\ \dot{X}_2 = (\alpha X_2 + \beta X_3) X_1 \\ \dot{X}_3 = 0 \end{cases}. \tag{2.2.9}$$

   The right-hand side is 0 when $X_1 = X_2 = 0$ or when $\alpha X_2 + \beta X_3 = 0$. So the set of stationary points is the pair of antipodal points $(0, 0, \pm 1)$ and the circle in the intersection of the plane $\alpha X_2 + \beta X_3 = 0$ and the unit sphere S^2.

Note that the antipodal points $(\pm 1, 0, 0)$ found earlier lie on the circles in both subcases.

2. Suppose that M is nonsingular. Assuming $u \neq 0$, the stationary points of system (2.2.7) satisfy $\langle Mu, u\rangle = 0$ and $x = 0$. Denote M^S the *symmetrisation* of the matrix M, that is, $M^S = \frac{1}{2}(M + M^T)$. We have

$$M^S = \frac{1}{2}\left(\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} + \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}\right) = \begin{pmatrix}\alpha & \frac{\beta+\gamma}{2} \\ \frac{\beta+\gamma}{2} & \delta\end{pmatrix}.$$

Moreover, we have

$$\langle M^S u, u\rangle = (\beta + \gamma)u_1 u_2 + \alpha u_1^2 + \delta u_2^2 = \langle Mu, u\rangle.$$

We can now choose an orthonormal basis in the (u_1, u_2)-plane relative to which the symmetric matrix M^S is diagonal, so that $M^S = \begin{pmatrix}\alpha & 0 \\ 0 & \delta\end{pmatrix}$. Thus the equation $\langle Mu, u\rangle = 0$ is equivalent to $\langle M^S u, u\rangle = 0 \iff \alpha u_1^2 + \delta u_2^2 = 0$. Furthermore, by Lemma 2.3, at least one of α or δ must be positive, so up to relabelling the basis we can assume $\alpha > 0$. Then the stationary points depend on the value of δ. We consider three subcases.

   (i) $\delta > 0$. Then the only solution to $\alpha u_1^2 + \delta u_2^2 = 0$ is $u_1 = u_2 = 0$. Hence the only possible stationary points are the antipodal points $(\pm 1, 0, 0)$.

   (ii) $\delta = 0$. Then $u_1 = 0$, this gives the points $(0, 0, \pm 1)$. So there are two pairs of antipodal stationary points: $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$.

   (iii) $\delta < 0$. Denote $\rho = \sqrt{\alpha}$, $\sigma = \sqrt{-\delta}$ (note that $\rho, \sigma > 0$). Then we get $\rho^2 u_1^2 - \sigma^2 u_2^2 = 0$. Substituting this into $u_1^2 + u_2^2 = 1$ gives $(u_1, u_2) = (\rho^2 + \sigma^2)^{-1/2}(0, \pm\sigma, \pm\rho)$. Hence there are three pairs of antipodal stationary points: $(\pm 1, 0, 0)$ and $(\rho^2 + \sigma^2)^{-1/2}(0, \pm\sigma, \pm\rho)$.

The following table summarises our findings for homogeneous geodesics on the unit sphere in 3-dimensional non-unimodular metric Lie algebras.

Case	Form of M	Stationary Points on S^2
$\det M = 0$	$M = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \alpha > 0$	The circle $X_1^2 + X_3^2 = 1,\ X_2 = 0.$
	$M = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \alpha > 0, \beta \neq 0$	1 pair of antipodal points $(0, 0, \pm 1)$ and the circle in the intersection of the plane $\alpha X_2 + \beta X_3 = 0$ and S^2.
$\det M \neq 0$	$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix}, \alpha, \delta > 0$	1 pair of antipodal points: $(\pm 1, 0, 0)$.
	$M = \begin{pmatrix} \alpha & \beta \\ -\beta & 0 \end{pmatrix}, \alpha > 0$	2 pairs of antipodal points: $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$.
	$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix} = \begin{pmatrix} \rho^2 & \beta \\ -\beta & -\sigma^2 \end{pmatrix}$, where $\alpha > 0 > \delta$	3 pairs of antipodal points: $(\pm 1, 0, 0)$ and $(\rho^2 + \sigma^2)^{-1/2}(0, \pm\sigma, \pm\rho)$.

TABLE 2.2. Homogeneous geodesics on the unit sphere in 3-dimensional non-unimodular metric Lie algebras.

2.3. Stability analysis: unimodular case. Proof of Theorem 0.1

In this section, we give the stability analysis of stationary points in 3-dimensional unimodular metric Lie algebras and prove Theorem 0.1. The proof goes on a case-by-case basis using the classification of stationary points from systems (2.2.1)-(2.2.6) obtained in Section 2.2, page 38.

1. For (2.2.1), all points are stationary points and are stable.
2. System (2.2.2) gives

$$\begin{cases} \dot{X}_1 = 0 \\ \dot{X}_2 = -\lambda_1 X_1 X_3 \\ \dot{X}_3 = \lambda_1 X_1 X_2 \end{cases} \Rightarrow \begin{cases} X_1 = c \\ X_2 = -c_1 \sin(\lambda_1 ct) + c_2 \cos(\lambda_1 ct) \\ X_3 = c_1 \cos(\lambda_1 ct) + c_2 \sin(\lambda_1 ct) \end{cases} .$$

- The stationary points $(\pm 1, 0, 0)$ are the "north pole" and the "south pole" of the unit sphere S^2. Let $X_0 = (1, 0, 0)$ and consider the solution $X(t)$ with $X(0) = (c, c_2, c_1)$. We have

$$\|X(0) - X_0\| = \|X(t) - X_0\| = \sqrt{(1-c)^2 + c_1^2 + c_2^2},$$

  for all $t \in \mathbb{R}$. So if we choose $\epsilon' = \epsilon$, the stability condition of Definition 1.37 will be satisfied. This shows that X_0 (and similarly $-X_0$) is stable. This fact agrees with the observation that every solution to (2.2.2) travels on the "parallels" of the sphere, so the distance to the poles remains constant all the time.
- The stationary point on the "equator" of the sphere given by $X_1 = 0$ is unstable. Indeed, fixing a stationary point X_0 on the equator and choosing a point $X(0)$ close to it and not lying on the equator, we find that the solution $X(t)$ travels on the parallel of the sphere and does not remain in a neighbourhood of X_0, it will eventually be close to the antipodal point $-X_0$.

3. System (2.2.3) gives

$$\begin{cases} \dot{X}_1 = \lambda_2 X_2 X_3 \\ \dot{X}_2 = -\lambda_1 X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} , \text{for } \lambda_1, \lambda_2 > 0.$$

   We have

$$\left(\lambda_1 X_1^2 + \lambda_2 X_2^2\right)^{\cdot} = 2\lambda_1\lambda_2 X_1 X_2 X_3 - 2\lambda_1\lambda_2 X_1 X_2 X_3 = 0.$$

It follows that the function $\lambda_1 X_1^2 + \lambda_2 X_2^2$ is a first integral, and so every solution of this system lies on the intersections between the sphere S^2 and the elliptic cylinder $\lambda_1 X_1^2 + \lambda_2 X_2^2 = c_1$ (see Figure 2.1–the axis X_3 points "up").

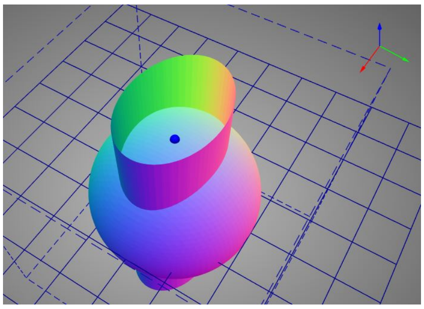


FIGURE 2.1. Intersection of S^2 with the elliptic cylinder.

We see that the stationary point $(0, 0, 1)$ shown in Figure 2.1 (and its antipodal point) is stable, as any solution which starts close to that point remains close to it for all times.

To study other stationary points it is convenient to consider the other two first integrals

$$\begin{aligned} &(\lambda_1 - \lambda_2)X_1^2 - \lambda_2 X_3^2 = c_2, \\ \text{and} \quad &(\lambda_1 - \lambda_2)X_2^2 + \lambda_1 X_3^2 = c_3. \end{aligned}$$

- Let $\lambda_1 > \lambda_2$. Then all the solutions lie in the intersections of S^2 with a family of hyperbolic cylinders $(\lambda_1 - \lambda_2)X_1^2 - \lambda_2 X_3^2 = c_2$ (see Figure 2.2).

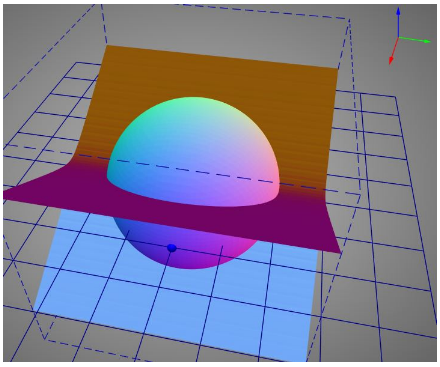


FIGURE 2.2. Intersection of S^2 with the hyperbolic cylinder.

We see that the stationary point $(1, 0, 0)$ shown in Figure 2.2 is unstable, as a solution with a starting point close to it will not remain close; eventually it will be close to its antipodal point. By a similar argument the point $(-1, 0, 0)$ is also unstable.

On the other hand, the stationary point $(0, 1, 0)$ is stable: the solutions lie on the elliptic cylinders $(\lambda_1 - \lambda_2)X_2^2 + \lambda_1 X_3^2 = c_3$ (see Figure 2.3) and so the solution which starts close to the stationary point remains close to it. Similarly the point $(0, -1, 0)$ is stable.

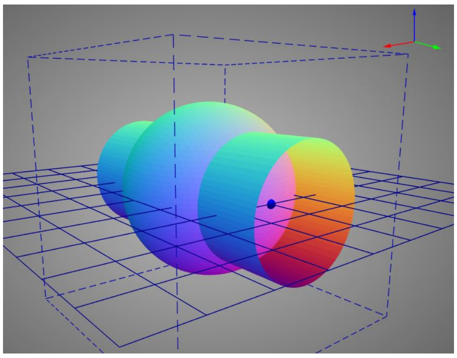


FIGURE 2.3. Intersection of S^2 with the elliptic cylinder.

- If $\lambda_1 < \lambda_2$, a similar argument shows that the stationary points $(\pm 1, 0, 0)$ are stable, and the stationary points $(0, \pm 1, 0)$ are unstable.
- If $\lambda_1 = \lambda_2$, then system (2.2.3) reduces to system (2.2.2), with the stationary points $(0, 0, \pm 1)$ stable, and the stationary points on the circle $X_3 = 0$ unstable.

4. System (2.2.4) is similar to system (2.2.3), but with $\lambda_1 > 0,\ \lambda_2 < 0$. The following equations give three first integrals for the system:

$$\begin{aligned} \lambda_1 X_1^2 + \lambda_2 X_2^2 &= c_1, \\ (\lambda_1 - \lambda_2) X_1^2 - \lambda_2 X_3^2 &= c_2, \\ \text{and} \quad (\lambda_1 - \lambda_2) X_2^2 + \lambda_1 X_3^2 &= c_3. \end{aligned}$$

Note that the first equation defines a family of hyperbolic cylinders, while the last two define families of elliptic cylinders. Arguments similar to the above show that the stationary points $(0, 0, \pm 1)$ are unstable and the stationary points $(0, \pm 1, 0)$ and $(\pm 1, 0, 0)$ are stable.

5. System (2.2.5) gives

$$\begin{cases} \dot{X}_1 = (\lambda_2 - \lambda_3) X_2 X_3 \\ \dot{X}_2 = (\lambda_3 - \lambda_1) X_1 X_3 \\ \dot{X}_3 = (\lambda_1 - \lambda_2) X_1 X_2 \end{cases} \quad \text{, for } \lambda_1, \lambda_2, \lambda_3 > 0,$$

with three first integrals

$$\begin{aligned} (\lambda_3 - \lambda_1) X_1^2 - (\lambda_2 - \lambda_3) X_2^2 &= c_1, \\ (\lambda_1 - \lambda_2) X_2^2 - (\lambda_3 - \lambda_1) X_3^2 &= c_2, \\ \text{and} \quad (\lambda_1 - \lambda_2) X_1^2 - (\lambda_2 - \lambda_3) X_3^2 &= c_3. \end{aligned}$$

- If $\lambda_1 = \lambda_2 = \lambda_3$, then all points are stationary points and are stable.

- If exactly two of $\lambda_1, \lambda_2, \lambda_3$ are equal, then system (2.2.5) reduces to system (2.2.2), in which we have stable stationary points at the poles and unstable stationary points on the equator.
- If $\lambda_1, \lambda_2, \lambda_3$ are pairwise different, we can assume that $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Then $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3 > 0$ and $\lambda_3 - \lambda_1 < 0$. Hence the points $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$ are stable and the points $(0, \pm 1, 0)$ are unstable, by arguments similar to the above.

6. System (2.2.6) is similar to system (2.2.5), with the same stationary points being stable and unstable, respectively.

An examination of similarities between the above cases suggests that the number of cases can be reduced. By relabelling the basis vectors, we can always assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Combining some of the above cases into a single case we get the following classification.

Case	Stationary points	Stability
$\lambda_1 = \lambda_2 = \lambda_3$	Every point of S^2	Stable
$\lambda_1 > \lambda_2 = \lambda_3$	$(\pm 1, 0, 0)$	Stable
	All points on the circle $X_1 = 0$	Unstable
$\lambda_1 = \lambda_2 > \lambda_3$	$(0, 0, \pm 1)$	Stable
	All points on the circle $X_3 = 0$	Unstable
$\lambda_1 > \lambda_2 > \lambda_3$	$(\pm 1, 0, 0)$	Stable
	$(0, \pm 1, 0)$	Unstable
	$(0, 0, \pm 1)$	Stable

TABLE 2.3. Stability of homogeneous geodesics in 3-dimensional unimodular metric Lie algebras.

This table, together with Lemma 2.2, completes the proof of Theorem 0.1.

2.4. Stability analysis: non-unimodular case I. Proof of Theorem 0.2 for a singular matrix M

In this and the next section, we investigate the stability of stationary points in 3-dimensional non-unimodular metric Lie algebras and prove Theorem 0.2. Using the classification in Lemma 2.3 and defining the matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we can reduce the Euler-Arnold equation to the form (2.2.7).

This section deals with the case when the matrix M is singular. By specifying the orthonormal basis, we can take $M = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$, where $\alpha > 0$ (see Section 2.2, page 40) and the system (2.2.7) can be further reduced to the form (2.2.9):

$$\begin{cases} \dot{X}_1 = -(\alpha X_2 + \beta X_3) X_2 \\ \dot{X}_2 = (\alpha X_2 + \beta X_3) X_1 \\ \dot{X}_3 = 0 \end{cases} \implies \begin{cases} \dot{X}_1 = -(\alpha X_2 + \beta X_3) X_2 \\ \dot{X}_2 = (\alpha X_2 + \beta X_3) X_1 \\ X_3 = c \in \mathbb{R}. \end{cases} \quad (2.4.1)$$

It follows that the trajectory of every solution of (2.4.1) on the unit sphere S^2 is either a single (stationary) point $(0, 0, \pm 1)$ if $|c| = 1$, or lies on the circles $X_3 = c,\ X_1^2 + X_2^2 = 1 - c^2$ if $|c| < 1$. We immediately get that the stationary points $(0, 0, \pm 1)$ are stable: if we take an arbitrary starting point $X(0)$ on the sphere, the distance from any point on the trajectory $X(t)$ to either of $(0, 0, \pm 1)$ remains the same (see Figure 2.4–this is the phase portrait in the $X_1 X_2$-plane).

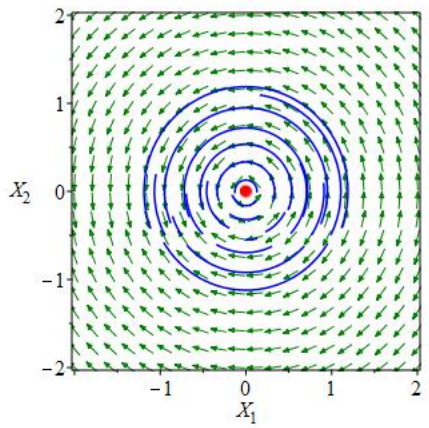


FIGURE 2.4. Stationary points $(0, 0, \pm 1)$ are stable.

To study the nature of the other stationary points we will consider two cases from the Table 2.2, depending on whether $\beta = 0$ or $\beta \neq 0$.

(i) If $\beta = 0$, then from (2.4.1), we have

$$\begin{cases} \dot{X}_1 = -\alpha X_2^2 \\ \dot{X}_2 = \alpha X_1 X_2 \\ X_3 = c \end{cases} . \tag{2.4.2}$$

The set of the stationary points is the circle $X_2 = 0$, and so on every circle $X_3 = c \in (-1, 1)$ we have two stationary points $(\pm\sqrt{1-c^2}, 0, c)$. As $X_1^2 + X_2^2 = 1 - c^2$, the first equation of (2.4.2) gives a separable equation

$$\dot{X}_1 = -\alpha(1 - c^2 - X_1^2),$$

which can be solved explicitly to give

$$X_1 = \sqrt{1-c^2}\left(-1 + \frac{2}{1 + \hat{c}e^{2\alpha t\sqrt{1-c^2}}}\right),$$

where $\hat{c} \geq 0$ depends on the initial condition.
When $t \to \infty$, we have $X_1 \to -\sqrt{1-c^2}$, so the stationary point $\left(-\sqrt{1-c^2}, 0, c\right)$, $c \in (-1, 1)$ is stable, while the stationary point $\left(\sqrt{1-c^2}, 0, c\right)$, $c \in (-1, 1)$ is unstable (see Figure 2.5–this is the directional field between X_1 and t).

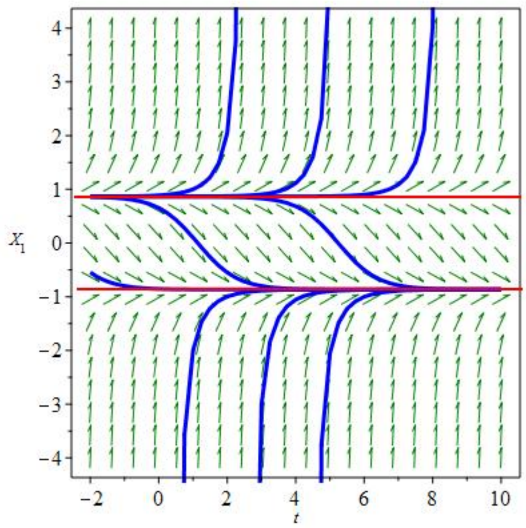


FIGURE 2.5. Stationary points $(\pm\sqrt{1-c^2}, 0, c)$, $c \in (-1, 1)$.

(ii) If $\beta \neq 0$, then as above, we can take $X_3 = c \in (-1, 1)$. The solution lies on the circle $X_1^2 + X_2^2 = 1 - c^2$ in the plane $X_3 = c$, and on this circle, the system (2.4.1) takes the form

$$\begin{cases} \dot{X}_1 = -(\alpha X_2 + \beta c) X_2 \\ \dot{X}_2 = (\alpha X_2 + \beta c) X_1 \end{cases} . \tag{2.4.3}$$

For every $c \in (-1,1)$, the stationary points are given by the intersection of the circle $X_1^2 + X_2^2 = 1 - c^2$ with the line $X_2 = -\frac{\beta c}{\alpha}$. Their number depends on c. We have three possibilities.

(a) $\left|-\frac{\beta c}{\alpha}\right| > \sqrt{1-c^2} \iff |c| > \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. The circle and the line do not meet, and hence there are no stationary points.

(b) $\left|-\frac{\beta c}{\alpha}\right| = \sqrt{1-c^2} \iff |c| = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. The circle and the line meet at the point $(X_1, X_2) = (0, -\frac{\beta c}{\alpha})$, which is the only stationary point. For the system (2.4.3), choose a parametrisation $X_1 = -\frac{|\beta c|}{\alpha}\sin\phi,\ \ X_2 = -\frac{\beta c}{\alpha}\cos\phi$. The stationary point corresponds to $\phi = 2\pi n,\ n \in \mathbb{Z}$. Then the first equation of (2.4.3) becomes

$$\dot{\phi} = |\beta c|(\cos\phi - 1).$$

This is a separable equation which can be solved explicitly to give

$$\phi(t) = 2\,\mathrm{arccot}\left(\cot\frac{\phi(0)}{2} + |\beta c| t\right),$$

where $\phi(0) \in (0, 2\pi)$ is the initial point. So if we start arbitrarily close to the stationary point $\phi = 0$, the function $\phi(t)$ will eventually approach π, so that the corresponding point on the circle will be the antipodal to the stationary point. It follows that the trajectory starting close to the stationary point does not remain close to it, and so the stationary point $\left(0, -\frac{\beta c}{\alpha}\right)$ is unstable (see Figure 2.6–this is the directional field between ϕ and t).

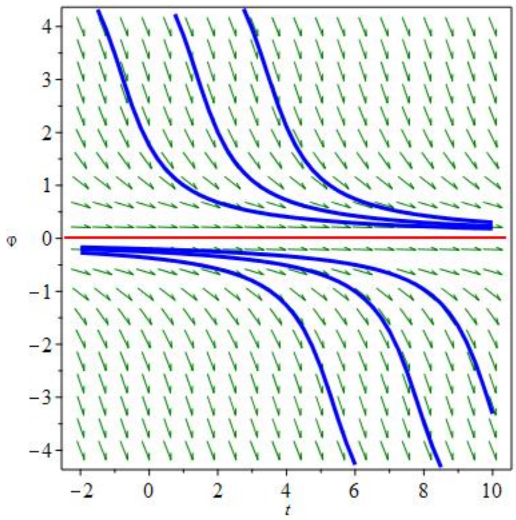


FIGURE 2.6. Stationary points $(0, -\frac{\beta c}{\alpha}, c)$ are unstable.

(c) $\left|-\frac{\beta c}{\alpha}\right| < \sqrt{1-c^2} \iff |c| < \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. The circle and the line meet at two points $(X_1, X_2) = \left(\pm\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}\right)$. For system (2.4.3), we can parameterise $X_1 = \sqrt{1-c^2}\sin\phi$ and $X_2 = \sqrt{1-c^2}\cos\phi$, $\phi \in [-\pi, \pi]$. The two stationary points correspond to $\phi \in [-\pi, \pi]$ where $\cos\phi = -\frac{\beta c}{\alpha\sqrt{1-c^2}} \implies \phi = \pm\arccos\left(-\frac{\beta c}{\alpha\sqrt{1-c^2}}\right)$. Then the first equation of (2.4.3) becomes

$$\dot{\phi} = -\alpha\sqrt{1-c^2}\cos\phi - \beta c. \tag{2.4.4}$$

Although this equation is separable, it is easier to apply Theorem 1.39 to test for stability. In the one-dimensional case, the Jacobian matrix of equation (2.4.4) is given by the derivative of the right-hand side:

$$J(\phi) = \alpha\sqrt{1-c^2}\sin\phi,$$

and hence

$$J\left(\arccos(-\frac{\beta c}{\alpha\sqrt{1-c^2}})\right) = \alpha\sqrt{\alpha^2(1-c^2)-\beta^2c^2} > 0,$$
$$J\left(-\arccos(-\frac{\beta c}{\alpha\sqrt{1-c^2}})\right) = -\alpha\sqrt{\alpha^2(1-c^2)-\beta^2c^2} < 0.$$

So we obtain that the stationary point $(X_1, X_2) = \left(\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}\right)$ is unstable, and the stationary point $(X_1, X_2) = \left(-\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}\right)$ is stable (see Figure 2.7–this is the directional field between ϕ and t).

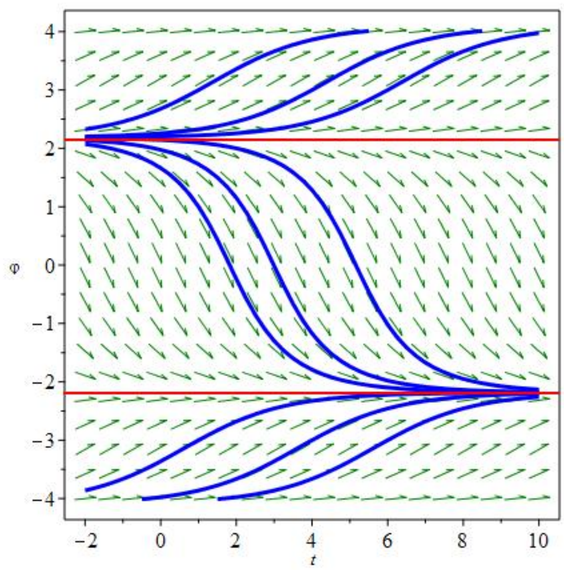


FIGURE 2.7. Stationary points $(\pm\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}, c)$.

Summarising the above we obtain the following dynamics of stationary points on the unit sphere S^2 for the case $\beta \neq 0$.

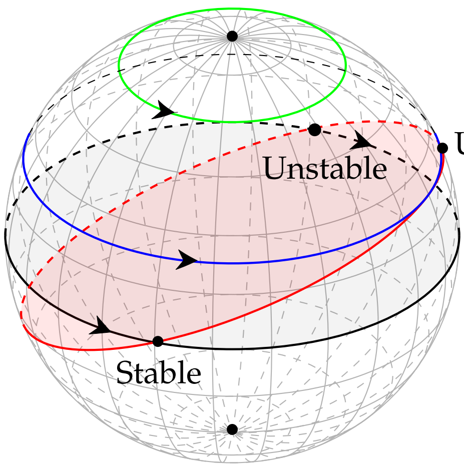


FIGURE 2.8. Stability of stationary points on the unit sphere S^2 in the case $\beta \neq 0$.

This completes the stability analysis of the stationary points in the case when the matrix M is singular. The following table summarises our findings.

β	Stationary points	Stability
	$(0, 0, \pm 1)$	Stable
$\beta = 0$	$\left(-\sqrt{1-c^2}, 0, c\right)$, $\lvert c\rvert < 1$	Stable
	$\left(\sqrt{1-c^2}, 0, c\right)$, $\lvert c\rvert < 1$	Unstable
$\beta \neq 0$	$\left(-\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}, c\right)$, $\lvert c\rvert < \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$	Stable
	$\left(\frac{\sqrt{\alpha^2-c^2(\alpha^2+\beta^2)}}{\alpha}, -\frac{\beta c}{\alpha}, c\right)$, $\lvert c\rvert \leq \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$	Unstable

TABLE 2.4. Stability of homogeneous geodesics in 3-dimensional non-unimodular metric Lie algebras if M is singular.

2.5. Stability analysis: non-unimodular case II. Proof of Theorem 0.2 for a nonsingular matrix M

We now proceed with the study of stability of stationary points when the matrix M is nonsingular. Recall that by a choice of the basis, we can assume that M is given by

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix}, \quad \text{where } \alpha, \alpha+\delta > 0,$$

so that the symmetrisation M^S is diagonal. The Euler-Arnold equation then becomes

$$\begin{cases} \dot{X}_1 = -\alpha X_2^2 - \delta X_3^2 \\ \dot{X}_2 = (\alpha X_2 + \beta X_3) X_1 \\ \dot{X}_3 = (-\beta X_2 + \delta X_3) X_1 \end{cases} \qquad (2.5.1)$$

defined on the unit sphere S^2. According to our classification in Table 2.2, we have three cases depending on the value of δ.

We first consider the stability of the stationary points $(\pm 1, 0, 0)$, which we have in all cases.

(1) Consider the stationary point $(1, 0, 0)$. In a neighbourhood of this point, we have $X_1 = \sqrt{1 - X_2^2 - X_3^2}$. The system (2.5.1) then becomes

$$\begin{cases} \dot{X}_2 = \sqrt{1 - X_2^2 - X_3^2}\,(\alpha X_2 + \beta X_3) \\ \dot{X}_3 = \sqrt{1 - X_2^2 - X_3^2}\,(-\beta X_2 + \delta X_3) \end{cases}.$$

We apply Theorem 1.40. The Jacobian matrix of the above system at $(0, 0)$ is $J(0,0) = \begin{pmatrix} \alpha & \beta \\ -\beta & \delta \end{pmatrix} = M$. As $\alpha + \delta > 0$ (Lemma 2.3), Theorem 1.40 implies that the stationary point $(1, 0, 0)$ is unstable, regardless of the value of $\det(M)$ (see Figure 2.9–this is the phase portrait in the X_2X_3-plane).

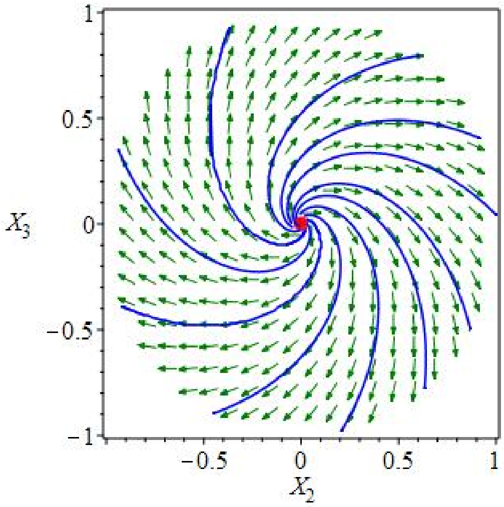


FIGURE 2.9. Stationary points $(1, 0, 0)$ is unstable.

(2) Similarly, in a neighbourhood of the stationary point $(-1, 0, 0)$, we can write $X_1 = -\sqrt{1 - X_2^2 - X_3^2}$. The system (2.5.1) becomes

$$\begin{cases} \dot{X}_2 = -\sqrt{1 - X_2^2 - X_3^2}(\alpha X_2 + \beta X_3) \\ \dot{X}_3 = -\sqrt{1 - X_2^2 - X_3^2}(-\beta X_2 + \delta X_3) \end{cases}.$$

Then the Jacobian matrix is given by $J(0,0) = \begin{pmatrix} -\alpha & -\beta \\ \beta & -\delta \end{pmatrix} = -M$. Thus, if $\delta \geq 0$, we have $\det(-M) = \alpha\delta + \beta^2 > 0$ and $\mathrm{tr}(-M) = -(\alpha + \delta) < 0$, hence the stationary point $(-1, 0, 0)$ is stable by Theorem 1.40 (see Figure 2.10).

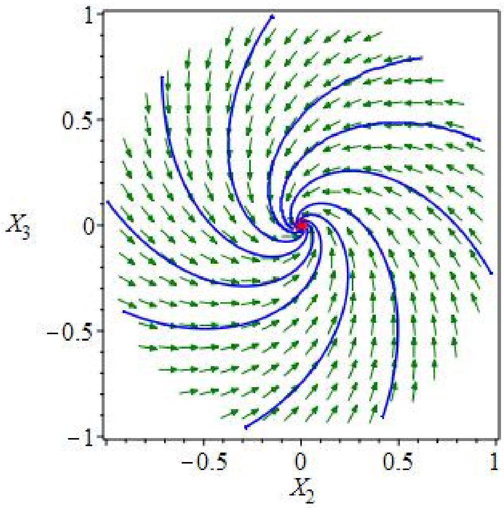
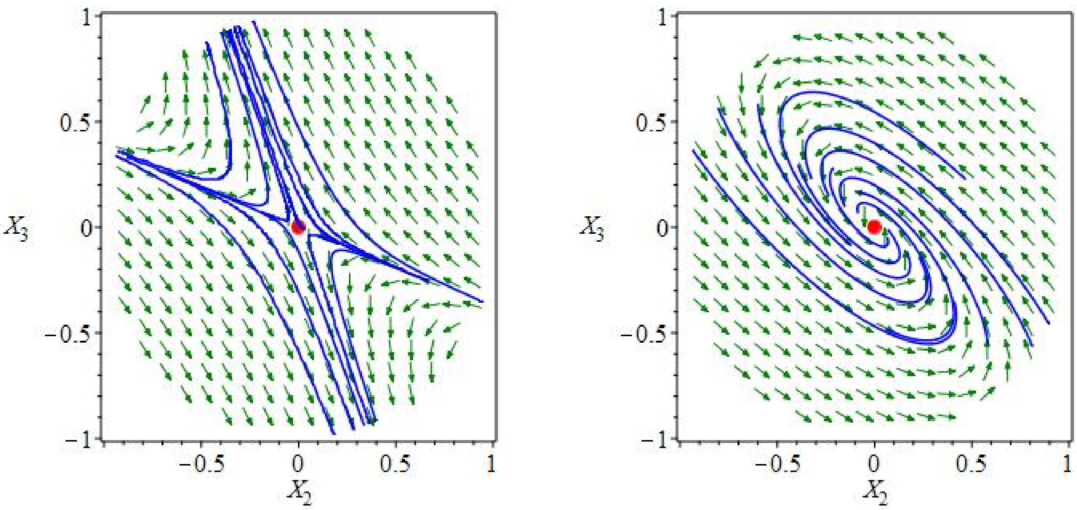


FIGURE 2.10. Stationary points $(-1, 0, 0)$ is stable when $\delta \geq 0$.

However, if $\delta < 0$, then this point is unstable when $\det(M) < 0 \Leftrightarrow \delta < \frac{-\beta^2}{\alpha}$, and is stable when $\det(M) > 0 \Leftrightarrow \delta > \frac{-\beta^2}{\alpha}$ (see Figure 2.11).



(a) Unstable when $\det(M) < 0$

(b) Stable when $\det(M) > 0$

FIGURE 2.11. Stability of $(-1, 0, 0)$ depending on $\det(M)$.

To study the stability of other stationary points of system (2.5.1), we separately consider 3 cases (i), (ii) and (iii) from Section 2.2, page 41 (or see the last three rows of Table 2.2).

(i) $\delta > 0$. We only have one pair of antipodal stationary points $(\pm 1, 0, 0)$, and according to the above analysis, the point $(1, 0, 0)$ is unstable and the point $(-1, 0, 0)$ is stable.

(ii) $\delta = 0$ (note that $\beta \neq 0$, as $\det(M) \neq 0$). We have two pairs of stationary points: $(\pm 1, 0, 0)$ and $(0, 0, \pm 1)$. Again, the aforementioned analysis tells that the point $(1, 0, 0)$ is unstable and the point $(-1, 0, 0)$ is stable. We consider the pair $(0, 0, \pm 1)$. In the neighbourhood of these points, we have $X_3 = \pm\sqrt{1 - X_1^2 - X_2^2}$, so from (2.5.1) we obtain

$$\begin{cases} \dot{X}_1 = -\alpha X_2^2 \\ \dot{X}_2 = \left(\alpha X_2 \pm \beta\sqrt{1 - X_1^2 - X_2^2}\right) X_1 \end{cases} . \tag{2.5.2}$$

We are interested in the stability of the point $(X_1, X_2) = (0, 0)$. The Jacobian matrix at this point has a zero eigenvalue, so Theorem 1.39 is not applicable. Instead we apply Theorem 1.43. First note that by changing the sign of X_2 if necessary, we can reduce the system (2.5.2) to the form

$$\begin{cases} \dot{X}_1 = -\alpha X_2^2 \\ \dot{X}_2 = \left(\alpha X_2 - |\beta|\sqrt{1 - X_1^2 - X_2^2}\right) X_1 \end{cases} . \tag{2.5.3}$$

Following Theorem 1.43, we introduce the Chetaev function $V = -X_1X_2$. The set $\{(X_1, X_2) : V(X_1, X_2) > 0\}$ has two connected components; we take for Q the quadrant $X_1 < 0, X_2 > 0$. Clearly the stationary point (the origin) lies on the boundary of Q. Furthermore, in view of equations (2.5.3), we have

$$\dot{V} = \alpha X_2^3 + \left(|\beta|\sqrt{1 - X_1^2 - X_2^2} - \alpha X_2\right) X_1^2.$$

The first term on the right-hand side is always positive, and the second one is positive provided $\sqrt{1 - X_1^2 - X_2^2} > \alpha|\beta|^{-1}X_2$, which implies $X_1^2 + (1 + \alpha^2\beta^{-2})\, X_2^2 < 1$. The latter inequality defines the interior domain of the ellipse $X_1^2+(1+\alpha^2\beta^{-2})X_2^2 = 1$. Taking that domain for D we obtain that $\dot{V} > 0$ on $D \cap Q$, and so the point $(X_1, X_2) = (0, 0)$ is unstable by Theorem 1.43 (see Figure 2.12–this is the phase portrait in the X_1X_2-plane).

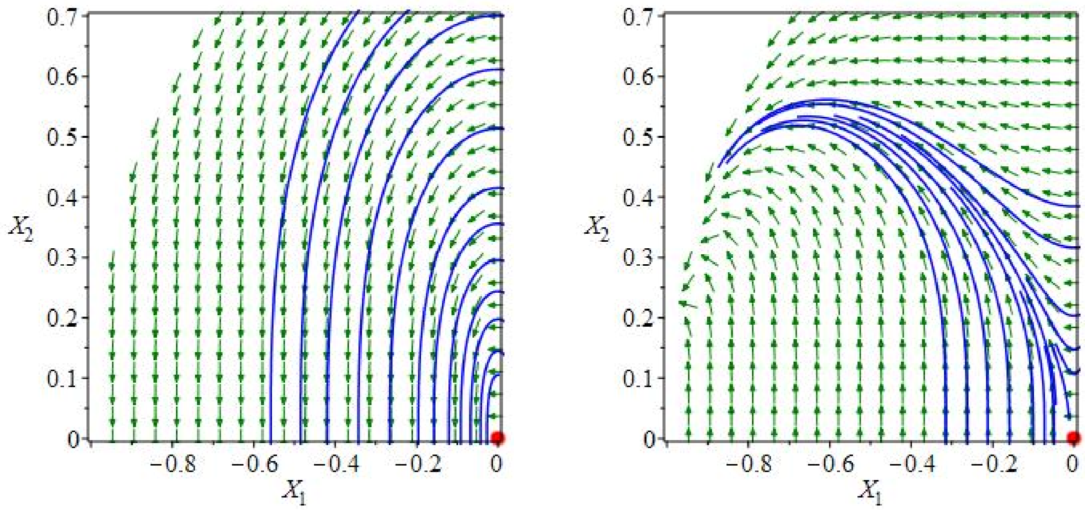


FIGURE 2.12. Stationary points $(0, 0, \pm 1)$ are unstable.

(iii) $\delta < 0$. We have three pairs of antipodal stationary points: $(\pm 1, 0, 0)$ and $\left(\rho^2 + \sigma^2\right)^{-1/2}(0, \pm\sigma, \pm\rho)$. We already know that the point $(1, 0, 0)$ is unstable, and the point $(-1, 0, 0)$ is unstable when $\det(M) < 0$, and stable when $\det(M) > 0$.

Consider the other two pairs $(\rho^2 + \sigma^2)^{-1/2}(0, \pm\sigma, \pm\rho)$, where $\alpha = \rho^2$ and $\delta = -\sigma^2$. To analyse the stability of these points, we will use Theorem 1.46. First we check for the regularity condition of stationary points from Definition 1.45. Let $Y = (Y_1, Y_2, Y_3)$ and $X = (0, X_1, X_2)$ be arbitrary points in $\mathfrak{g}$. We have

$$B(X,Y) = \mathrm{ad}^*_Y X = \begin{pmatrix} 0 & -\alpha Y_2 - \gamma Y_3 & -\beta Y_2 - \delta Y_3 \\ 0 & \alpha Y_1 & \beta Y_1 \\ 0 & \gamma Y_1 & \delta Y_1 \end{pmatrix} \begin{pmatrix} 0 \\ X_1 \\ X_2 \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha X_1 Y_2 - \gamma X_1 Y_3 - \beta X_2 Y_2 - \delta X_2 Y_3 \\ \alpha X_1 Y_1 + \beta X_2 Y_1 \\ \gamma X_1 Y_1 + \delta X_2 Y_1 \end{pmatrix}$$
$$= \begin{pmatrix} -\langle Mu, v\rangle \\ Y_1 M u \end{pmatrix}, \tag{2.5.4}$$

and

$$[Y,X] = \mathrm{ad}_Y X = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha Y_2 - \gamma Y_3 & \alpha Y_1 & \gamma Y_1 \\ -\beta Y_2 - \delta Y_3 & \beta Y_1 & \delta Y_1 \end{pmatrix} \begin{pmatrix} 0 \\ X_1 \\ X_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \alpha X_1 Y_1 + \gamma X_2 Y_1 \\ \beta X_1 Y_1 + \delta X_2 Y_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ Y_1 M^T u \end{pmatrix}, \tag{2.5.5}$$

where $u = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $v = \begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix}$ and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Take X_0 to be one of the four stationary points, that is, $X_0 = \begin{pmatrix} 0 \\ u_0 \end{pmatrix}$, where $u_0 = (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} \pm\sigma \\ \pm\rho \end{pmatrix}$. Then (2.5.4) and (2.5.5) give

$$B(X_0, Y) = \begin{pmatrix} -\langle M u_0, v\rangle \\ Y_1 M u_0 \end{pmatrix} \quad \text{and} \quad [Y, X_0] = \begin{pmatrix} 0 \\ Y_1 M^T u_0 \end{pmatrix},$$

and $B(X_0, Y) = 0$ only when

$$\begin{cases} Y_1 M u_0 = 0 \\ \langle M u_0, v\rangle = 0 \end{cases} \implies \begin{cases} Y_1 = 0 \\ v \text{ is a multiple of } u_0 \\ \qquad \text{(as } M u_0 \perp u_0 \text{ and } \det(M) \neq 0\text{).} \end{cases}$$

X_0 clearly satisfies the above condition and thus is regular. We calculate

$$\begin{aligned} \Phi(Y) &= \|B(X_0, Y)\|^2 + \langle [Y, X_0], B(X_0, Y)\rangle \\ &= \langle M u_0, v\rangle^2 + Y_1^2\big(\|M u_0\|^2 + \langle M u_0, M^T u_0\rangle\big). \end{aligned} \tag{2.5.6}$$

The first term on the right-hand side of 2.5.6 is clearly positive, so Φ is positive definite if the sum in the bracket is positive. From the assumption, we have $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \rho^2 & \beta \\ -\beta & -\sigma^2 \end{pmatrix}$ and $u_0 = (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} s_1\sigma \\ s_2\rho \end{pmatrix}$, where $s_1, s_2 \in \{1, -1\}$. Then

$$\begin{aligned} Mu_0 &= (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} \rho^2 & \beta \\ -\beta & -\sigma^2 \end{pmatrix} \begin{pmatrix} s_1\sigma \\ s_2\rho \end{pmatrix} \\ &= (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} \rho^2 s_1\sigma + \beta\rho s_2 \\ -\beta s_1\sigma - \sigma^2 s_2\rho \end{pmatrix}. \end{aligned}$$

Hence,

$$\|Mu_0\|^2 = \frac{1}{\rho^2 + \sigma^2}\Big(\big(\rho^2 s_1\sigma + \rho s_2\beta\big)^2 + \big(\beta s_1\sigma + \sigma^2 s_2\rho\big)^2 \Big).$$

Similarly, we have

$$\begin{aligned} M^T u_0 &= (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} \rho^2 & -\beta \\ \beta & -\sigma^2 \end{pmatrix} \begin{pmatrix} s_1\sigma \\ s_2\rho \end{pmatrix} \\ &= (\rho^2 + \sigma^2)^{-1/2} \begin{pmatrix} \rho^2 s_1\sigma - \beta\rho s_2 \\ \beta s_1\sigma - \sigma^2 s_2\rho \end{pmatrix}. \end{aligned}$$

Thus,

$$\langle Mu_0, M^T u_0 \rangle = \frac{1}{\rho^2 + \sigma^2}\Big(\big(\rho^2 s_1\sigma\big)^2 - (\rho s_2\beta)^2 + (\beta s_1\sigma)^2 - \big(\sigma^2 s_2\rho\big)^2 \Big).$$

Therefore, the sum in the bracket of (2.5.6) becomes

$$\|Mu_0\|^2 + \langle Mu_0, M^T u_0 \rangle = 2\rho\sigma(\rho\sigma + s_1 s_2\beta).$$

- If $|\beta| < \rho\sigma$, then $\|Mu_0\|^2 + \langle Mu_0, M^T u_0 \rangle > 0$, hence $\Phi(Y)$ is positive definite, and by Theorem 1.46, all four stationary points are stable. Note that this is exactly the case when $\beta^2 < -\alpha\delta \iff \det(M) < 0$.
- If $|\beta| > \rho\sigma$ (which is equivalent to $\det(M) > 0$), then two of the four stationary points are stable by Theorem 1.46; these are the points for which $s_1 s_2 \beta > 0$, as $\Phi(Y)$ is then positive definite. For the other two points, we can apply Theorem 1.40. Choose $s_1, s_2 \in \{-1, 1\}$ in such a way that the sign of β is $-s_1 s_2$. Then we have

$$\rho\sigma + s_1 s_2\beta = \rho\sigma - |\beta| < 0.$$

Denote $r = \sqrt{\sigma^2 + \rho^2}$ and let θ be chosen in such a way that $\rho = s_2 r \sin\theta$, $\sigma = s_1 r \cos\theta$. Introduce spherical coordinates (ϕ, ψ) so that

$$\begin{cases} X_1 = \sin\psi \\ X_2 = \cos\psi\cos(\theta+\phi), \quad \phi \in [0, 2\pi),\ \psi \in \left[\frac{\pi}{2}, \frac{-\pi}{2}\right]. \\ X_3 = \cos\psi\sin(\theta+\phi) \end{cases}$$

Substituting these expressions into system (2.5.1), we find

$$\begin{cases} \dot{\psi} = r^2\cos\psi\sin\phi\sin(2\theta+\phi) \\ \dot{\phi} = \sin\psi(-\beta - \frac{1}{2}r^2\sin(2(\phi+\theta))) \end{cases},$$

and the stationary point of interest is $\psi = \phi = 0$ by the above choice of θ. The Jacobian matrix of this system at $\psi = \phi = 0$ is

$$J(0,0) = \begin{pmatrix} 0 & r^2\sin(2\phi) \\ -\beta - \frac{1}{2}r^2\sin(2\phi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2s_1s_2\rho\sigma \\ -\beta - s_1s_2\rho\sigma & 0 \end{pmatrix},$$

$$\implies \det(J(0,0)) = 2s_1s_2\rho\sigma(\beta + s_1s_2\rho\sigma) = 2\rho\sigma(s_1s_2\beta + \rho\sigma) < 0,$$

since $|\beta| > \rho\sigma$. Hence, by Theorem 1.40, these two points are unstable.

This completes the stability analysis of the stationary points in the case when the matrix M is nonsingular. The following table summarises our findings.

δ		Stationary points	Stability
$\delta > 0$		$(1,0,0)$	Unstable
		$(-1,0,0)$	Stable
$\delta = 0$		$(1,0,0)$	Unstable
		$(-1,0,0)$	Stable
		$(0,0,\pm 1)$	Unstable
$\delta < 0$		$(1,0,0)$	Unstable
	$\det M < 0$	$(-1,0,0)$	Unstable
		$(\rho^2+\sigma^2)^{-1/2}(0,\pm\sigma,\pm\rho)$	Stable
	$\det M > 0$	$(-1,0,0)$	Stable
		$(\rho^2+\sigma^2)^{-1/2}(0,s_1\sigma,s_2\rho)$ $s_1,s_2 \in \{-1,1\},\ s_1s_2\beta > 0.$	Stable
		$(\rho^2+\sigma^2)^{-1/2}(0,s_1\sigma,s_2\rho)$ $s_1,s_2 \in \{-1,1\},\ s_1s_2\beta < 0.$	Unstable

TABLE 2.5. Stability of homogeneous geodesics in 3-dimensional non-unimodular metric Lie algebras if M is nonsingular.

This table, together with Lemma 2.3 and Table 2.4, settles the proof of Theorem 0.2.

CHAPTER 3

Stability of homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras

In this chapter, we continue our stability analysis for 4-dimensional metric Lie algebras. We look at a special class of such algebras, namely the nilpotent Lie algebras. Proceeding in a similar manner to Chapter 2, we begin by introducing nilpotent Lie algebras and obtaining a classification for nilpotent metric Lie algebras of dimension 4 in Section 3.1. Section 3.2 gives the system of differential equations derived from the Euler-Arnold equation that can be used subsequently to find homogeneous geodesics. Then stability analysis for these stationary points is performed in Section 3.3. The references for Section 3.1 are **[8]**, **[13]**, and **[15]**.

3.1. Classification of 4-dimensional nilpotent metric Lie algebras

3.1. DEFINITION. Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{a} \subset \mathfrak{g}$ is said to be a ***Lie subalgebra*** if it is closed under the Lie bracket in $\mathfrak{g}$, that is,

$$[X, Y] \in \mathfrak{a} \quad \text{for } X, Y \in \mathfrak{a}.$$

3.2. DEFINITION. An ***ideal*** in a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{i}$ so that

$$[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}.$$

In other words, an ideal $\mathfrak{i}$ satisfies the condition

$$[X, Y] \in \mathfrak{i} \quad \text{for all } X \in \mathfrak{g}, Y \in \mathfrak{i}.$$

The Lie algebra $\mathfrak{g}$ itself and $\{0\}$ are trivial ideals of $\mathfrak{g}$. From Examples 1.22 and 1.21, it is clear that $\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$. Another important example of an ideal is the following.

3.3. DEFINITION. The ***centre*** $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined by

$$Z(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

We will see how a sequence of ideals of $\mathfrak{g}$ gives rise to the important class of nilpotent Lie algebras.

3.4. DEFINITION. Let $\mathfrak{g}$ be a Lie algebra. The ***descending central series of*** $\mathfrak{g}$ is defined inductively:

$$\mathfrak{g}^0 = \mathfrak{g},$$
$$\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}], \text{ for } k \geq 1.$$

The first term $\mathfrak{g}^1$ is called the ***derived subalgebra*** of $\mathfrak{g}$.

3.5. LEMMA.

(a) $\mathfrak{g}^{k+1} \subseteq \mathfrak{g}^k$.

(b) All members of the descending central series are ideals of $\mathfrak{g}$.

PROOF. We first prove (a) by induction. The base case $\mathfrak{g}^1 \subseteq \mathfrak{g}^0 = \mathfrak{g}$ is obvious because $[X, Y] \in \mathfrak{g}$, for any $X, Y \in \mathfrak{g}$. Now assume that $\mathfrak{g}^{k+1} \subseteq \mathfrak{g}^k$ for some $k \geq 0$. By definition, we have

$$\mathfrak{g}^{k+2} = [\mathfrak{g}, \mathfrak{g}^{k+1}] \subseteq [\mathfrak{g}, \mathfrak{g}^k] = \mathfrak{g}^{k+1},$$

so this completes the induction step. We can obtain (b) readily from (a). □

In light of Lemma 3.5, the descending central series of $\mathfrak{g}$ is a sequence of ideals $\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \ldots$ of $\mathfrak{g}$. Note that for a finite-dimensional Lie algebra, the descending central series cannot decrease for infinitely long, so for some $k \geq 0$, it becomes stable in the sense that $\mathfrak{g}^k = \mathfrak{g}^{k+1}$. And then by definition, $\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k] = [\mathfrak{g}, \mathfrak{g}^{k+1}] = \mathfrak{g}^{k+2}$, and so on, so that $\mathfrak{g}^k = \mathfrak{g}^{k+1} = \mathfrak{g}^{k+2} = \mathfrak{g}^{k+3} = \ldots$. This may happen even at the very first step: it could be that $\mathfrak{g}^0 = \mathfrak{g}^1$ (e.g., for 3-dimensional unimodular Lie algebras such that $\lambda_1, \lambda_2, \lambda_3 \neq 0$, see Lemma 2.2), or at the second step: $\mathfrak{g}^0$ is strictly bigger than $\mathfrak{g}^1$, but then $\mathfrak{g}^1 = \mathfrak{g}^2 = \ldots$ (e.g., for 3-dimensional non-unimodular Lie algebras with a non-singular matrix M, see Lemma 2.3).

3.6. DEFINITION. A Lie algebra $\mathfrak{g}$ is called ***nilpotent*** if $\mathfrak{g}^k = 0$ for some k. If k is the smallest number with this property, that is, if $\mathfrak{g}^{k-1} \neq 0$, then $\mathfrak{g}$ is said to be ***k-step nilpotent***.

3.7. EXAMPLE.

(1) Abelian Lie algebras are one-step nilpotent.
(2) Let $\mathfrak{n}(n)$ be the subalgebra of $\mathfrak{gl}(n)$ consisting of all strictly upper triangular $n \times n$ real matrices. Then $\mathfrak{n}(n)$ is a $(n-1)$-step nilpotent Lie algebra, for the first time we take the commutator, we lose the super diagonal, and then we lose another diagonal for each commutator after that. Consequently, the zero diagonal is expanding to the top right corner of the matrices, hence $\mathfrak{n}^{n-1} = 0$.

(3) Let $\mathfrak{g}$ be the 3-dimensional unimodular Lie algebras discussed in Lemma 2.2. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ relative to which the Lie brackets are given by

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

 (a) Assume that $\lambda_1, \lambda_2, \lambda_3 \neq 0$. Then $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}^0] = \mathfrak{g}^0$. Hence we obtain $\mathfrak{g}^0 = \mathfrak{g}^1 = \mathfrak{g}^2 = \ldots$, so this is *not* a nilpotent algebra.
 (b) Assume that $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$, which gives the 3-dimensional Heisenberg Lie algebra. Then the descending central of $\mathfrak{g}$ is

$$\begin{aligned} \mathfrak{g}^0 = \mathfrak{g} &= \mathrm{Span}(e_1, e_2, e_3), \\ \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}^0] &= \mathrm{Span}(e_3), \\ \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] &= 0 \end{aligned}$$

 Thus $\mathfrak{g}$ is a 2-step nilpotent Lie algebra.

(4) Let $\mathfrak{g}$ be a 4-dimensional metric Lie algebras with an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ relative to which the only nonzero Lie brackets are given by

$$[e_1, e_2] = a e_3 + b e_4, \qquad a, b \neq 0.$$

Then the descending central of $\mathfrak{g}$ is

$$\begin{aligned} \mathfrak{g}^0 = \mathfrak{g} &= \mathrm{Span}(e_1, e_2, e_3, e_4), \\ \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}^0] &= \mathrm{Span}(e_3, e_4), \\ \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] &= 0. \end{aligned}$$

Hence, this algebra is a 2-step nilpotent Lie algebra.

We note several useful properties of nilpotent algebras.

3.8. LEMMA. *Let $\mathfrak{g}$ be a nilpotent algebra of step k, and $Z(\mathfrak{g})$ be the centre of $\mathfrak{g}$. Then*

(a) The descending central series of $\mathfrak{g}$ is strictly decreasing up to $\mathfrak{g}^k$, i.e., $\mathfrak{g} = \mathfrak{g}^0 \supsetneq \mathfrak{g}^1 \supsetneq \mathfrak{g}^2 \supsetneq \ldots \supsetneq \mathfrak{g}^k = 0$.
(b) If $\mathfrak{g} \neq 0$, then $Z(\mathfrak{g}) \neq 0$.
(c) $(\mathrm{ad}_X)^k = 0$ for all $X \in \mathfrak{g}$, i.e., ad_X is a nilpotent linear map (this is the easy direction of Engel's theorem).
(d) Suppose additionally that $\dim \mathfrak{g} > 1$. Then the dimension decreases by at least 2 at the first term of the descending central series, i.e., $\dim \mathfrak{g}^1 \leq \dim \mathfrak{g} - 2$.

PROOF.

(a) This is clear, since if $\mathfrak{g}^l = \mathfrak{g}^{l+1}$ for some $l < k$, then $\mathfrak{g}^l = \mathfrak{g}^{l+1} = \mathfrak{g}^{l+2} = \ldots$, and so $\mathfrak{g}^l = 0$ by definition of nilpotent algebra, and so the step of the algebra is $l < k$, a contradiction.

(b) This follows readily from the fact that the last nonzero term of the descending central series is contained in $Z(\mathfrak{g})$.

(c) Recall that for a Lie algebra $\mathfrak{g}$, $\mathrm{ad}_X(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$. Then the definition of nilpotency can be rephrased as follows: for any $X_1, X_2, \ldots, X_n$, Y in $\mathfrak{g}$, we have

$$[X_1, [X_2, [\ldots [X_k, Y] \ldots]] \in \mathfrak{g}^k = 0,$$

or

$$\mathrm{ad}_{X_1} \mathrm{ad}_{X_2} \cdots \mathrm{ad}_{X_k}(Y) = 0.$$

In particular, $(\mathrm{ad}_X)^k = 0$ for all $X \in \mathfrak{g}$.

(d) Assume that $\dim \mathfrak{g}^1 = \dim \mathfrak{g} - 1$, and choose an arbitrary $X \in \mathfrak{g}$ such that $X \notin \mathfrak{g}^1$. Note that any $Y \in \mathfrak{g}$ can be decomposed as $Y = aX + Z$, where $Z \in \mathfrak{g}^1$ and $a \in \mathbb{R}$. Now taking two arbitrary vectors $Y_1, Y_2 \in \mathfrak{g}$, we have $Y_1 = a_1 X + Z_1, Y_2 = a_2 X + Z_2$, where $Z_1, Z_2 \in \mathfrak{g}^1$ and $a_1, a_2 \in \mathbb{R}$. Then

$$[Y_1, Y_2] = [a_1 X + Z_1, a_2 X + Z_2] = a_1[X, Z_2] - a_2[X, Z_1] + [Z_1, Z_2].$$

But $[X, Z_2], [X, Z_1], [Z_1, Z_2] \in [\mathfrak{g}, \mathfrak{g}^1] = \mathfrak{g}^2$, and so $[Y_1, Y_2] \in \mathfrak{g}^2$. As $Y_1, Y_2 \in \mathfrak{g}$ are arbitrary, this gives $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^1 \subset \mathfrak{g}^2$. But by Lemma 3.5 (a), $\mathfrak{g}^2 \subset \mathfrak{g}^1$, so $\mathfrak{g}^1 = \mathfrak{g}^2$, and then by assertion (a), $\mathfrak{g}^1 = 0$, so that $\mathfrak{g}$ is abelian. By our assumption $\dim \mathfrak{g} \geq 2$, we have $0 = \dim \mathfrak{g}^1 \leq \dim \mathfrak{g} - 2$, as required.

□

We are now ready to give a classification of nilpotent metric Lie algebras of dimension 4. To the best of our knowledge, this classification result did not appear in the literature before.

3.9. LEMMA. *Let $\mathfrak{g}$ be a 4-dimensional nilpotent metric Lie algebra. Then either $\mathfrak{g}$ is abelian, or there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ relative to which the only nonzero Lie brackets (up to skew-symmetry) are given by one of the following:*

$$[e_1, e_2] = ce_4, \qquad c \neq 0, \tag{3.1.1}$$

$$\text{or} \quad [e_1, e_2] = ae_3 + be_4, \qquad [e_1, e_3] = ce_4, \qquad a, c \neq 0. \tag{3.1.2}$$

PROOF. Let $\mathfrak{g}$ be a 4-dimensional nilpotent metric Lie algebra, and consider its derived subalgebra $\mathfrak{g}^1$. Then by Lemma 3.8(d), $\dim \mathfrak{g}^1 \leq \dim \mathfrak{g} - 2 = 4 - 2 = 2$, hence $\dim \mathfrak{g}^1 = 0, 1$ or 2.

1. $\dim \mathfrak{g}^1 = 0$. This implies that $\mathfrak{g}^1 = 0$, and $\mathfrak{g}$ is abelian.
2. $\dim \mathfrak{g}^1 = 1$. Choose an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ such that $e_4 \in \mathfrak{g}^1$. Then we have

$$\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \quad \Longrightarrow [e_i, e_j] = c_{ij} e_4 \quad \text{for} \quad i, j \in \{1, 2, 3, 4\}.$$

By Lemma 3.8 (a) and (b), $\mathfrak{g}^1$ contains a nontrivial vector from the centre $Z(\mathfrak{g})$ of $\mathfrak{g}$, and as $\dim \mathfrak{g}^1 = 1$, e_4 must belong to the centre of $\mathfrak{g}$. Hence

$$[e_i, e_4] = 0 \quad \text{for} \quad i \in \{1, 2, 3, 4\}.$$

So we have

$$\begin{aligned} [e_1, e_2] &= c_{12}e_4, & [e_1, e_3] &= c_{13}e_4, \\ [e_2, e_1] &= -c_{12}e_4, & [e_2, e_3] &= c_{23}e_4, \\ [e_3, e_1] &= -c_{13}e_4, & [e_3, e_2] &= -c_{23}e_4, \end{aligned}$$

while all other brackets are zeros. Hence the matrix

$$C = (c_{ij}) = \begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{12} & 0 & c_{23} \\ -c_{13} & -c_{23} & 0 \end{pmatrix}$$

is skew-symmetric and non-vanishing. Let $u = c_{23}e_1 - c_{13}e_2 + c_{12}e_3$, then $u \neq 0$, and

$$\begin{aligned} [e_1, u] &= [e_1, c_{23}e_1 - c_{13}e_2 + c_{12}e_3] \\ &= c_{23}[e_1, e_1] - c_{13}[e_1, e_2] + c_{12}[e_1, e_3] \\ &= -c_{13}c_{12}e_4 + c_{13}c_{12}e_4 \\ &= 0. \end{aligned}$$

Similarly, $[e_2, u] = [e_3, u] = 0$. We now take a new orthonormal basis $\{e_1', e_2', e_3'\}$ for the subspace spanned by $\{e_1, e_2, e_3\}$ such that e_3' is a unit vector in the direction of u. Then relative to the new orthonormal basis $\{e_1', e_2', e_3', e_4\}$ of $\mathfrak{g}$, the only non-zero Lie bracket (up to skew-symmetry) is

$$[e_1', e_2'] = ce_4, \quad \text{with } c \neq 0 \text{ as } \mathfrak{g}^1 \neq 0.$$

3. $\dim \mathfrak{g}^1 = 2$. Again, as $\mathfrak{g}^1$ contains a nontrivial vector from the centre of $\mathfrak{g}$, choose an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ so that $e_1, e_2 \perp \mathfrak{g}^1$, $e_3, e_4 \in \mathfrak{g}^1$ and e_4 is in the centre of $\mathfrak{g}$. Then by definition, we obtain

$$[e_i, e_4] = 0 \quad \text{and} \quad [e_i, e_j] \in \mathfrak{g}^1 \quad \text{for} \quad i, j \in \{1, 2, 3, 4\}.$$

By listing, the only possible non-zero Lie brackets (up to skew-symmetry) are

$$\begin{aligned} [e_1, e_2] &= a_{12}e_3 + b_{12}e_4, \\ [e_1, e_3] &= a_{13}e_3 + b_{13}e_4, \\ [e_2, e_3] &= a_{23}e_3 + b_{23}e_4. \end{aligned}$$

Assume that $a_{13} \neq 0$, we calculate

$$\begin{aligned}
[e_1,[e_1,e_3]] &= [e_1, a_{13}e_3 + b_{13}e_4] \\
&= a_{13}[e_1,e_3] + b_{13}[e_1,e_4] \\
&= a_{13}^2 e_3 + a_{13}b_{13}e_4,
\end{aligned}$$

and

$$\begin{aligned}
[e_1,[e_1,[e_1,e_3]]] &= \left[e_1, a_{13}^2 e_3 + a_{13}b_{13}e_4\right] \\
&= a_{13}^2[e_1,e_3] + a_{13}b_{13}[e_1,e_4] \\
&= a_{13}^3 e_3 + a_{13}^2 b_{13}e_4,
\end{aligned}$$

and so on. This would never get to 0, contradicting the definition of nilpotency. Hence, we conclude that $a_{13} = 0$. A similar reasoning yields $a_{23} = 0$.
Now assume further that $b_{13} = b_{23} = 0$. Consider $[e_1,e_2] = a_{12}e_3 + b_{12}e_4 \in \mathfrak{g}^1$. Then $\mathfrak{g}^1$ is spanned by a single vector $a_{12}e_3 + b_{12}e_4$, hence $\dim \mathfrak{g}^1 = 1$, contradicting the assumption $\dim \mathfrak{g}^1 = 2$. Thus at least one of b_{13} or b_{23} must be non-zero, and we can now take a new orthonormal basis $\{e_1', e_2', e_3, e_4\}$ for $\mathfrak{g}$ such that

$$\begin{aligned}
e_1' &= \frac{1}{\sqrt{b_{13}^2 + b_{23}^2}} \left(b_{13}e_1 + b_{23}e_2\right), \\
e_2' &= \frac{1}{\sqrt{b_{13}^2 + b_{23}^2}} \left(b_{23}e_1 - b_{13}e_2\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
[e_1', e_2'] &= \left[\frac{1}{\sqrt{b_{13}^2 + b_{23}^2}}(b_{13}e_1 + b_{23}e_2), \frac{1}{\sqrt{b_{13}^2 + b_{23}^2}}(b_{23}e_1 - b_{13}e_2)\right] \\
&= \frac{b_{23}^2}{b_{13}^2 + b_{23}^2}[e_2,e_1] - \frac{b_{13}^2}{b_{13}^2 + b_{23}^2}[e_1,e_2] \\
&= -[e_1,e_2] = ae_3 + be_4,
\end{aligned}$$

and

$$\begin{aligned}
[e_1', e_3] &= \left[\frac{1}{\sqrt{b_{13}^2 + b_{23}^2}}(b_{13}e_1 + b_{23}e_2), e_3\right] \\
&= \frac{b_{13}}{\sqrt{b_{13}^2 + b_{23}^2}}[e_1,e_3] + \frac{b_{23}}{\sqrt{b_{13}^2 + b_{23}^2}}[e_2,e_3] \\
&= \frac{b_{13}^2}{\sqrt{b_{13}^2 + b_{23}^2}}e_4 + \frac{b_{23}^2}{\sqrt{b_{13}^2 + b_{23}^2}}e_4 \\
&= \sqrt{b_{13}^2 + b_{23}^2}e_4 = ce_4,
\end{aligned}$$

and

$$\begin{aligned}
[e_2', e_3] &= \left[\frac{1}{\sqrt{b_{13}^2 + b_{23}^2}} (b_{23}e_1 - b_{13}e_2), e_3 \right] \\
&= \frac{b_{23}}{\sqrt{b_{13}^2 + b_{23}^2}} [e_1, e_3] - \frac{b_{13}}{\sqrt{b_{13}^2 + b_{23}^2}} [e_2, e_3] \\
&= \frac{b_{13}b_{23}}{\sqrt{b_{13}^2 + b_{23}^2}} e_4 - \frac{b_{13}b_{23}}{\sqrt{b_{13}^2 + b_{23}^2}} e_4 \\
&= 0.
\end{aligned}$$

Hence, relative to this basis, the only nonzero Lie bracket (up to skew-symmetry) are

$$\begin{aligned}
&[e_1', e_2'] = ae_3 + be_4, \\
&[e_1', e_3] = ce_4, \qquad\qquad \text{where } a, c \neq 0.
\end{aligned}$$

$\square$

3.2. Homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras

In this section, we apply the Euler-Arnold equation (1.6.1) respectively to the three cases in Lemma 3.9 to obtain equations for geodesics in 4-dimensional nilpotent metric Lie algebras, and proceed to find the homogeneous geodesics subsequently. Again, by Remark 1.6.1, it is sufficient to study the stationary points that are on the unit sphere $S^3 = S^3(1)$ given by $X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1$.

1. $\mathfrak{g}$ is abelian, i.e, $[X, Y] = 0$ for any $X, Y \in \mathfrak{g}$. Hence,

$$\dot{X}(t) = \mathrm{ad}_X^* X = 0. \tag{3.2.1}$$

So all points on the unit sphere S^3 are stationary points.

2. $\mathfrak{g}$ is determined by an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ relative to which the Lie brackets (3.1.1) is defined. We have

$$\begin{aligned}\mathrm{ad}_{e_1}(e_1) &= [e_1, e_1] = 0,\\ \mathrm{ad}_{e_1}(e_2) &= [e_1, e_2] = ce_3,\\ \mathrm{ad}_{e_1}(e_3) &= [e_1, e_3] = 0,\\ \mathrm{ad}_{e_1}(e_4) &= [e_1, e_3] = 0.\end{aligned}$$

We obtain

$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & c & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we find

$$\mathrm{ad}_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -c & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_{e_3} = \mathrm{ad}_{e_4} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned}\mathrm{ad}_{X(t)} &= X_1\mathrm{ad}_{e_1} + X_2\mathrm{ad}_{e_2} + X_3\mathrm{ad}_{e_3} + X_4\mathrm{ad}_{e_4}\\ &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -cX_2 & cX_1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence, the Euler-Arnold equation gives

$$\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\end{pmatrix} = \begin{pmatrix}-cX_2X_3\\ cX_1X_3\\ 0\\ 0\end{pmatrix} \Longrightarrow \begin{cases}\dot{X}_1 = -cX_2X_3\\ \dot{X}_2 = cX_1X_3\\ \dot{X}_3 = 0\\ \dot{X}_4 = 0\end{cases}, \text{for } c \neq 0. \tag{3.2.2}$$

Equating the right-hand side of (3.2.2) to zero, we obtain stationary points given by $X_1 = X_2 = 0$ or $X_3 = 0$. Hence, the set of stationary points of system (3.2.2) is the union of two disjoint subsets, $\mathcal{S}_1$ and $\mathcal{S}_2$ given by the following.

- $\mathcal{S}_1$ is the circle $X_1 = X_2 = 0, X_3^2 + X_4^2 = 1$.
- $\mathcal{S}_2$ is the the sphere $X_3 = 0, X_1^2 + X_2^2 + X_4^2 = 1$ minus the points $(0, 0, 0, \pm 1)$ (so that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint).

3. $\mathfrak{g}$ is determined by an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ relative to which the Lie brackets (3.1.2) is defined. We have

$$\mathrm{ad}_{e_1} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&a&0&0\\0&b&c&0\end{pmatrix}, \qquad \mathrm{ad}_{e_2} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\-a&0&0&0\\-b&0&0&0\end{pmatrix},$$

$$\mathrm{ad}_{e_3} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-c&0&0&0\end{pmatrix}, \qquad \mathrm{ad}_{e_4} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Hence,

$$\mathrm{ad}_{X(t)} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\-aX_2&aX_1&0&0\\-bX_2-cX_3&bX_1&cX_1&0\end{pmatrix}.$$

Thus, the Euler-Arnold equation gives

$$\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\end{pmatrix} = \begin{pmatrix}-aX_2X_3-(bX_2+cX_3)X_4\\ aX_1X_3+bX_1X_4\\ cX_1X_4\\ 0\end{pmatrix}$$

$$\Longrightarrow \begin{cases}\dot{X}_1 = -aX_2X_3-(bX_2+cX_3)X_4\\ \dot{X}_2 = aX_1X_3+bX_1X_4\\ \dot{X}_3 = cX_1X_4\\ \dot{X}_4 = 0\end{cases}, \text{for } a, c \neq 0. \tag{3.2.3}$$

3.2.1. REMARK. Note that system (3.2.3) has an obvious first integral $F_1(X) = X_4$. Moreover, by cross-multiplying the second and third equations, we obtain

$$\begin{aligned} \dot{X}_2 - \frac{a}{cX_4} X_3 \dot{X}_3 - \frac{b}{c} \dot{X}_3 = 0 \\ \Longrightarrow \left(X_2 - \frac{a}{2cX_4} X_3^2 - \frac{b}{c} X_3 \right)^{\cdot} = 0. \end{aligned}$$

It follows that the function $F_2(X) = aX_3^2 + 2bX_3X_4 - 2cX_2X_4$ is another first integral.

For system (3.2.3), if $X_4 = 0$, then the stationary points are

$$X_3 = 0 \quad \text{or} \quad X_1 = X_2 = 0,$$

so we have the circle $X_1^2 + X_2^2 = 1$ and the points $(0, 0, \pm 1, 0)$ as stationary points. If $X_4 \neq 0$, this implies that $X_1 = 0$ and $-aX_2X_3 - (bX_2 + cX_3)X_4 = 0$. Hence the stationary points are the intersection between the unit sphere S^3, the great hypersphere $X_1 = 0 \iff X_2^2 + X_3^2 + X_4^2 = 1$ and the quadratic cone $aX_2X_3 + bX_2X_4 + cX_3X_4 = 0$.

In conclusion, the set of stationary points of system (3.2.3) is the union of two disjoint subsets, $\mathcal{S}_3$ and $\mathcal{S}_4$ given by the following.

- $\mathcal{S}_3$ is the circle $X_3 = X_4 = 0, X_1^2 + X_2^2 = 1$.
- $\mathcal{S}_4$ is the intersection of the great hypersphere $X_1 = 0,\ X_2^2 + X_3^2 + X_4^2 = 1$, with the quadratic cone $P(X_2, X_3, X_4) = 0$, where

$$P(X_2, X_3, X_4) = aX_2X_3 + bX_2X_4 + cX_3X_4,$$

  minus the points $(0, \pm 1, 0, 0)$ (so that $\mathcal{S}_3$ and $\mathcal{S}_4$ are disjoint).

3.3. Stability analysis and proof of Theorem 0.3

In this section, we give the stability analysis of homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras. We look at the form of the systems (3.2.1), (3.2.2) and (3.2.3) and the corresponding stationary point sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ obtained in Section 3.2, and apply various techniques from Section 1.7 to test for their stability.

1. For system (3.2.1), all points are stationary points and are stable.
2. System (3.2.2) gives

$$\begin{cases} \dot{X}_1 = -cX_2X_3 \\ \dot{X}_2 = cX_1X_3 \\ \dot{X}_3 = 0 \\ \dot{X}_4 = 0 \end{cases} \Rightarrow \begin{cases} X_1 = c_1\cos(\alpha ct) + c_2\sin(\alpha ct) \\ X_2 = c_1\sin(\alpha ct) - c_2\cos(\alpha ct) \\ X_3 = \alpha \\ X_4 = \beta \end{cases}.$$

(a) We show that any stationary point in $\mathcal{S}_1$ is stable. Indeed, consider the solution $X(t) = (c_1\cos(\alpha ct) + c_2\sin(\alpha ct), c_1\sin(\alpha ct) - c_2\cos(\alpha ct), \alpha, \beta)$ and fix a stationary point $X_0 = \left(0, 0, \sigma, \pm\sqrt{1-\sigma^2}\right)$. Then we have

$$\begin{aligned} \|X(0) - X_0\| &= \|X(t) - X_0\| \\ &= \sqrt{(\alpha-\sigma)^2 + \left(\beta \mp \sqrt{1-\sigma^2}\right)^2 + c_1^2 + c_2^2} \end{aligned}$$

for all $t \in \mathbb{R}$. So if we choose $\epsilon' = \epsilon$, the stability condition is satisfied. This shows that any stationary point on the circle is stable.

(b) For $X_3 = 0$, fix a stationary point $X_0 = \left(\delta, \mu, 0, \pm\sqrt{1-\delta^2-\mu^2}\right)$. Then

$$\|X(0) - X_0\| = \sqrt{(c_1-\delta)^2 + (c_2+\mu)^2 + \alpha^2 + \left(\beta \mp \sqrt{1-\delta^2-\mu^2}\right)},$$

and

$$\begin{aligned} \|X(t) - X_0\| = \Big(&(c_1\cos(\alpha ct) + c_2\sin(\alpha ct) - \delta)^2 \\ &+ (c_1\sin(\alpha ct) - c_2\cos(\alpha ct) - \mu)^2 + \alpha^2 \\ &+ \left(\beta \mp \sqrt{1-\delta^2-\mu^2}\right)\Big)^{1/2}. \end{aligned}$$

Thus, the solution would not remain close to X_0. Instead, the trajectory would be travelling away from X_0 and coming back to it from the other side. Hence, we conclude that any stationary point in $\mathcal{S}_2$ is unstable.

3. System (3.2.3) is given by

$$\begin{cases} \dot{X}_1 = -aX_2X_3 - (bX_2 + cX_3)X_4 \\ \dot{X}_2 = aX_1X_3 + bX_1X_4 \\ \dot{X}_3 = cX_1X_4 \\ \dot{X}_4 = 0 \end{cases} \qquad \text{, for } a, c \neq 0.$$

(a) We show that any stationary point in $\mathcal{S}_3$ is unstable. Indeed, take $X_0 = (\cos\alpha, \sin\alpha, 0, 0), \alpha \in [0, 2\pi]$ as an arbitrary point from $\mathcal{S}_3$. For the initial condition, choose the point

$$X(0) = \left(\sqrt{1-\epsilon^2}\cos\alpha, \sqrt{1-\epsilon^2}\sin\alpha, \epsilon, 0\right)$$

with a small nonzero ϵ such that $a\epsilon > 0$. We then have

$$\begin{cases} \dot{X}_1 = -a\epsilon X_2 \\ \dot{X}_2 = a\epsilon X_1 \end{cases} \Rightarrow \begin{cases} X_1 = c_1\cos(a\epsilon t) + c_2\sin(a\epsilon t) \\ X_2 = c_1\sin(a\epsilon t) - c_2\cos(a\epsilon t) \end{cases}.$$

Given the initial condition, we obtain

$$\begin{cases} X_1 = \sqrt{1-\epsilon^2}\cos\alpha\cos(a\epsilon t) - \sqrt{1-\epsilon^2}\sin\alpha\sin(a\epsilon t) \\ X_2 = \sqrt{1-\epsilon^2}\cos\alpha\sin(a\epsilon t) + \sqrt{1-\epsilon^2}\sin\alpha\cos(a\epsilon t) \end{cases}.$$

Hence the solution is $\left(\sqrt{1-\epsilon^2}\cos(\alpha + a\epsilon t), \sqrt{1-\epsilon^2}\sin(\alpha + a\epsilon t), \epsilon, 0\right)$. For a small ϵ, the initial condition is arbitrarily close to X_0, but the point on the trajectory corresponding to $t = \frac{\pi}{a\epsilon}$ is almost antipodal to X_0. This shows that any stationary point on the circle $\mathcal{S}_3$ is unstable.

(b) For $\mathcal{S}_4$, it is difficult to determine the stability of the stationary points using the behaviour of the trajectory. Instead, we overcome this difficulty by the aid of Theorem 1.46.

Take a stationary point $X_0 = (0, X_2, X_3, X_4)$ from $\mathcal{S}_4$ and an arbitrary point $Y = (Y_1, Y_2, Y_3, Y_4)$ in $\mathfrak{g}$. We then have

$$B(X_0, Y) = \mathrm{ad}_Y^* X_0 = \begin{pmatrix} 0 & 0 & -aY_2 & -bX_2 - cX_3 \\ 0 & 0 & aY_1 & bY_1 \\ 0 & 0 & 0 & cY_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_2 \\ X_3 \\ X4 \end{pmatrix}$$

$$= \begin{pmatrix} -aX_3Y_2 - (bY_2 + cY_3)X_4 \\ aX_3Y_1 + bX_4Y_1 \\ cX_4Y_1 \\ 0 \end{pmatrix}, \tag{3.3.1}$$

and

$$[Y, X_0] = \mathrm{ad}_Y X_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -aY_2 & aY_1 & 0 & 0 \\ -bY_2 - cY_3 & bY_1 & cY_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ aX_2Y_1 \\ bX_2Y_1 + cX_3Y_1 \end{pmatrix}. \tag{3.3.2}$$

We want to check the regularity condition for X_0, that is, we want to determine the linear subspace

$$L_{X_0} = \{Y \in \mathfrak{g} \mid B(X_0, Y) = 0\},$$

and compute its dimension. From (3.3.1), $B(X_0, Y) = 0$ only when

$$\begin{cases} -aX_3Y_2 - (bY_2 + cY_3)X_4 = 0 \\ aX_3Y_1 + bX_4Y_1 = 0 \\ cX_4Y_1 = 0 \end{cases}.$$

This further divides into two cases.

(1) If $X_4 \neq 0$, then $Y_1 = 0$, and if we let $Y_4 = z_2$ and $Y_2 = -cX_4z_1$, then $L_{X_0} = \{(0, -cX_4z_1, (aX_3 + bX_4)z_1, z_2) \mid z_1, z_2 \in \mathbb{R}\}$, and so $\dim(L_{X_0}) = 2$.

(2) If $X_4 = 0$, then $X_3 \neq 0$, and so $Y_1 = Y_2 = 0$. The subspace L_{X_0} is given by the same formula and hence has the same constant dimension of 2.

Thus by Definition 1.45, every point $X_0 \in \mathcal{S}_4$ is regular, and Theorem 1.46 applies. We find that the orthogonal complement to $L(X_0)$ is the two-dimensional space

$$L(X_0)^\perp = \{(w_1, (ax_3 + bx_4)w_2, cx_4w_2, 0) \mid w_1, w_2 \in \mathbb{R}\}.$$

We compute

$$\|B(X_0, Y)\|^2 = w_2^2\Big((aX_3 + bX_4)^2 + (cX_4)^2\Big)^2 + w_1^2\Big((aX_3 + bX_4)^2 + (cX_4)^2\Big)^2,$$

and

$$\langle [Y, X_0], B(X_0, Y)\rangle = acw_1^2X_2X_4.$$

Then for $Y \in L(X_0)^\perp$, the quadratic form Φ is given by

$$\begin{aligned}\Phi(Y) &= \|B(X_0, Y)\|^2 + \langle [Y, X_0], B(X_0, Y)\rangle \\ &= w_2^2\Big((aX_3 + bX_4)^2 + (cX_4)^2\Big)^2 \\ &\qquad\qquad + w_1^2\Big((aX_3 + bX_4)^2 + (cX_4)^2 + acX_2X_4\Big).\end{aligned}$$

Hence, by Theorem 1.46, if a stationary point $X_0 = (0, X_2, X_3, X_4)$ satisfies the inequality $Q(X_2, X_3, X_4) > 0$, where

$$Q(X_2, X_3, X_4) = (aX_3 + bX_4)^2 + c^2X_4^2 + acX_2X_4, \tag{3.3.3}$$

then X_0 is stable.

To complement the above finding, we compute the Jacobian matrix of system (3.2.3) at $X_0 = (0, X_2, X_3, X_4)$

$$J(X_0) = \begin{pmatrix} 0 & -aX_3 - bX_4 & -aX_2 - cX_4 & -bX_2 - cX_3 \\ aX_3 + bX_4 & 0 & 0 & 0 \\ cX_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the characteristic equation $\det(J(X_0) - \lambda I) = 0$ gives

$$\lambda^2\Big((aX_3 + bX_4)^2 + c^2X_4^2 + acX_2X_4 + \lambda^2\Big) = 0.$$

Hence, two of the eigenvalues of the Jacobian matrix are zeros and the other two are the roots of the equation

$$\lambda^2 = -Q(X_2, X_3, X_4),$$

where the polynomial Q is given by (3.3.3). It now follows from Theorem 1.39 that when $Q(X_2, X_3, X_4) < 0$, the point $X_0 \in \mathcal{S}_4$ is unstable.

In summary, we have proved the following stability theorem.

3.10. THEOREM. *A stationary point $X_0 = (0, X_2, X_3, X_4) \in \mathcal{S}_4$ is stable if $Q > 0$, and is unstable if $Q < 0$, where*

$$Q(X_2, X_3, X_4) = (aX_3 + bX_4)^2 + c^2X_4^2 + acX_2X_4.$$

It remains to determine the stability of the stationary points when $Q = 0$. This set of points (if it is nonempty) is the set of common zeros of the two polynomials P and Q on the unit sphere $X_2^3 + X_3^2 + X_4^2 = 1$, hence can be obtained from the following system

$$\begin{cases} (aX_3 + bX_4)^2 + c^2X_4^2 + acX_2X_4 = 0 \\ aX_2X_3 + bX_2X_4 + cX_3X_4 = 0 \\ X_2^3 + X_3^2 + X_4^2 = 1 \end{cases}.$$

We first determine the set of common zeros of the polynomials $Q(X_2, X_3, X_4)$ and $P(X_2, X_3, X_4)$. Starting with $(aX_3 - bX_4)Q - acX_4P = 0$, we have

$$\begin{aligned} &(aX_3 + bX_4)^3 + c^2bX_4^3 = 0 \\ \Longrightarrow\ & aX_3 + bX_4 = -\sqrt[3]{c^2b}X_4 \end{aligned} \tag{3.3.4}$$

Substituting this into P, we obtain

$$X_4\left(cX_3 - \sqrt[3]{c^2b}X_2\right) = 0.$$

If $X_4 = 0$, then $X_3 = 0$, and the stationary point is $(0, \pm 1, 0, 0)$, which is excluded from $\mathcal{S}_4$. Hence, we can assume that $X_4 \neq 0$, then

$$cX_3 - \sqrt[3]{c^2b}X_2 = 0 \Longrightarrow cX_3 = \sqrt[3]{c^2b}X_2.$$

Now let $b = \beta^3$ and $c = \gamma^3$, we have $\gamma X_3 = \beta X_2 \implies X_2 = \frac{\gamma X_3}{\beta}$. Combining with (3.3.4), we obtain

$$X_3 = \frac{-\beta}{a}(\beta^2 + \gamma^2)X_4 \quad \text{and} \quad X_2 = \frac{-\gamma}{a}(\beta^2 + \gamma^2)X_4.$$

Substituting everything into the unit sphere, we get

$$X_4^2 = \frac{a^2}{(\gamma^2 + \beta^2)^3 + a^2}$$

Choosing z such that $z^2 = \frac{1}{(\gamma^2+\beta^2)^3+a^2}$, we have $X_4 = \pm az$, and the pair of antipodal stationary points is given by

$$X_0 = \left(0, z\gamma(\beta^2 + \gamma^2), z\beta(\beta^2 + \gamma^2), -az\right) \in \mathcal{S}_4. \tag{3.3.5}$$

We prove the following.

3.11. THEOREM. *The points X_0 given by* (3.3.5) *are both stable if $b = 0$, and unstable if $b \neq 0$.*

PROOF. We prove the theorem in two cases.

(1) First suppose that $b = \beta^3 \neq 0$. Consider the set of points W on the unit sphere such that the two first integrals

$$\begin{aligned} &F_1(X) = X_4, \quad \text{and} \\ &F_2(X) = aX_3^2 + 2bX_3X_4 - 2cX_2X_4 \end{aligned}$$

from Remark 3.2.1 take the same values at W as at X_0. That means that

$$F_1(W) = F_1(X_0) \implies W_4 = -az,$$

and

$$F_2(W) = F_2(X_0)$$

$$\implies 2\gamma^3 z W_2 + W_3^2 - 2\beta^3 z W_3 + (\beta^6 - 3\beta^2\gamma^4 - 2\gamma^6)z^2 = 0$$
$$\implies W_2 = \frac{-W_3^2 + 2\beta^3 z W_3 - (\beta^6 - 3\beta^2\gamma^4 - 2\gamma^6)z^2}{2\gamma^3 z}$$

Substituting W_2 and W_4 into the equation $\|W\|^2 = 1$ yields

$$4c^2z^2W_1^2 + \Big(W_3 + z\beta(3\gamma^2 - \beta^2)\Big)\Big(W_3 - z\beta(\beta^2 + \gamma^2)\Big)^3 = 0$$
$$\implies W_1 = \frac{\pm\sqrt{-\Big(W_3 + z\beta(3\gamma^2 - \beta^2)\Big)\Big(W_3 - z\beta(\beta^2 + \gamma^2)\Big)^3}}{2cz}.$$

Now consider a solution $W(t)$ of system (3.2.3) along which the first integrals take the required values. From the third equation, we have

$$\dot{W}_3 = cW_1W_4$$

$$\implies \dot{W}_3 = \pm\frac{1}{2}a\sqrt{-\Big(W_3 + z\beta(3\gamma^2 - \beta^2)\Big)\Big(W_3 - z\beta(\beta^2 + \gamma^2)\Big)^3}.$$

Let $\phi = W_3 - z\beta(\beta^2 + \gamma^2)$; then the differential equation becomes

$$\dot{\phi} = \pm\frac{1}{2}a\sqrt{-(\phi + 4z\beta\gamma^2)\phi^3}.$$

Continuing with the substitution $\psi = \frac{1}{\phi}$ gives a separable differential equation

$$\dot{\psi} = \pm\frac{1}{2}a\sqrt{-1 - 4z\beta\gamma^2\psi}$$
$$\implies \frac{\dot{\psi}}{\sqrt{-1 - 4z\beta\gamma^2\psi}} = \frac{a}{2}$$
$$\implies \sqrt{-1 - 4z\beta\gamma^2\psi} = c - at\beta\gamma^2 z,$$

where $c \in \mathbb{R}$ is an arbitrary constant. Let $\tau = c - at\beta\gamma^2 z$; then ψ and ϕ take the form

$$\psi = \frac{\tau^2 + 1}{-4\beta\gamma^2 z} \implies \phi = \frac{-4\beta\gamma^2 z}{\tau^2 + 1}.$$

Consequently, we obtain a solution

$$W_3(t) = z\beta(\beta^2+\gamma^2) - \frac{4z\beta\gamma^2}{1+\tau^2},$$
$$\text{and} \quad W_1(t) = \frac{\dot{W}_3}{-acz} = \frac{8\beta^2\gamma z\tau}{(\tau^2+1)^2}.$$

Finally, the first equation of system (3.2.3) gives

$$W_2(t) = \frac{\dot{W}_1 + \gamma^3 W_3 W_4}{-aW_3 - \beta^3 W_4} = z\gamma(\beta^2+\gamma^2) + \frac{4z\beta^2\gamma(\tau^2-1)}{(\tau^2+1)^2}.$$

So the whole trajectory of system (3.2.3) having the same values of the first integrals as the stationary point X_0 is given by

$$\begin{aligned} W(t) = \Big(& 8z\gamma\beta^2\tau(\tau^2+1)^{-2}, \\ & z\gamma(\beta^2+\gamma^2) + 4z\beta^2\gamma(\tau^2-1)(\tau^2+1)^{-2}, \\ & z\beta(\beta^2+\gamma^2) - 4z\beta\gamma^2(1+\tau^2)^{-1}, -az \Big), \end{aligned}$$

where $\tau = c - at\beta\gamma^2 z$. Note that the initial point of this trajectory $W(0)$ corresponds to taking $\tau = c$. By taking c in such a way that $|c|$ is very large and that c has the same sign as $z\beta\gamma^2 a$, $W(0)$ can be chosen arbitrarily close to $X_0 = \Big(0, z\gamma(\beta^2+\gamma^2), z\beta(\beta^2+\gamma^2), -az\Big)$. However, for $\tau = 0 \implies t = \frac{c}{a\beta\gamma^2 z} > 0$, we have

$$W(t) = \Big(0, z\gamma(-3\beta^2+\gamma^2), z\beta(\beta^2-3\gamma^2), -az\Big),$$

and

$$\|W(t) - X_0\|^2 = 4|z\gamma\beta|\sqrt{\beta^2+\gamma^2}.$$

Thus the distance from this point $W(t)$ to X_0 is a positive constant regardless of how close we choose the initial condition $W(0)$ from X_0. This proves by definition that X_0 is unstable if $b \neq 0$.

(2) Suppose $b = 0$, $X_0 = \Big(0, \frac{c}{\sqrt{a^2+c^2}}, 0, \frac{-a}{\sqrt{a^2+c^2}}\Big)$. Suppose $X = X(t)$ is a solution to system (3.2.3) on the unit sphere such that $X(0)$ is close to X_0. Recall that we have $F_1(X) = X_4$ and $F_2(X) = aX_3^2 + 2bX_3X_4 - 2cX_2X_4 = aX_3^2 - 2cX_2X_4$ as two first integrals. Consider the function

$$\begin{aligned} G(X) &= X_1^2 + (X_2 + ca^{-1}X_4)^2 \\ &= X_1^2 + X_2^2 + X_3^2 + X_4^2 + \left(\frac{c^2}{a^2} - 1\right) X_4^2 - \frac{aX_3^2 - 2cX_2X_4}{a} \\ &= 1 - (c^2a^{-2} - 1)F_1(X)^2 - a^{-1}F_2(X). \end{aligned}$$

Hence, $G(X)$ is also a first integral of the system. We note that $G(X(0))$ is close to $G(X_0)$ by continuity. But we have

$$G(X_0) = \left(\frac{c}{\sqrt{a^2+c^2}} - \frac{ca^{-1}a}{\sqrt{a^2+c^2}} \right)^2 = 0.$$

Thus, by choosing $X(0)$ sufficiently close to X_0 we get $G(X) = \delta$ for some small $\delta \neq 0$. This gives

$$\begin{aligned}
& X_1^2 + (X_2 + ca^{-1}X_4)^2 = \delta \\
\implies & \left(X_2 + \frac{c}{a} X_4 \right)^2 = \delta - X_1^2 \\
\implies & \left(\left(X_2 - \frac{c}{\sqrt{a^2+c^2}} \right) + \frac{c}{a} \left(X_4 + \frac{a}{\sqrt{a^2+c^2}} \right) \right)^2 \leq \delta \\
\implies & \left| \left(X_2 - \frac{c}{\sqrt{a^2+c^2}} \right) + \frac{c}{a} \left(X_4 + \frac{a}{\sqrt{a^2+c^2}} \right) \right| \leq \sqrt{\delta}.
\end{aligned}$$

Therefore, by the triangle inequality, we have

$$\begin{aligned}
\left| X_2 - \frac{c}{\sqrt{a^2+c^2}} \right| &\leq \left| \left(X_2 - \frac{c}{\sqrt{a^2+c^2}} \right) + \frac{c}{a} \left(X_4 + \frac{a}{\sqrt{a^2+c^2}} \right) \right| \\
&\qquad + \left| \frac{c}{a} \right| \cdot \left| X_4 + \frac{a}{\sqrt{a^2+c^2}} \right| \\
&\leq \sqrt{\delta} + \left| \frac{c}{a} \right| \cdot \left| X_4 + \frac{a}{\sqrt{a^2+c^2}} \right|. \qquad (3.3.6)
\end{aligned}$$

We consider

$$\left| X_4 + \frac{a}{\sqrt{a^2+c^2}} \right| = |F_1(X) - F_1(X_0)| = |F_1(X(0)) - F_1(X_0)|.$$

So by continuity, we can make $|F_1(X(0)) - F_1(X_0)| = \delta'$, for some small positive δ', and (3.3.6) becomes

$$\left| X_2 - \frac{c}{\sqrt{a^2+c^2}} \right| \leq \left| \frac{c}{a} \right| \delta' + \sqrt{\delta}. \qquad (3.3.7)$$

We calculate

$$\begin{aligned}
\|X - X_0\|^2 &= \|X\|^2 + \|X_0\|^2 - 2\langle X, X_0 \rangle \\
&= 2 - 2\left(\frac{c}{\sqrt{a^2+c^2}} X_2 - \frac{a}{\sqrt{a^2+c^2}} X_4 \right) \\
&= \frac{2(a^2+c^2)}{a^2+c^2} - \frac{2cX_2}{\sqrt{a^2+c^2}} + \frac{2aX_4}{\sqrt{a^2+c^2}} \\
&= \frac{2}{\sqrt{a^2+c^2}} \left(a \left(X_4 + \frac{a}{\sqrt{a^2+c^2}} \right) - c \left(X_2 - \frac{c}{\sqrt{a^2+c^2}} \right) \right).
\end{aligned}$$

So

$$
\begin{aligned}
\|X - X_0\|^2 &= \left| \|X - X_0\|^2 \right| \\
&= \frac{2}{\sqrt{a^2+c^2}} \left| a \left(X_4 + \frac{a}{\sqrt{a^2+c^2}} \right) - c \left(X_2 - \frac{c}{\sqrt{a^2+c^2}} \right) \right| \\
&\leq \frac{2}{\sqrt{a^2+c^2}} \left(|a| \left| X_4 + \frac{a}{\sqrt{a^2+c^2}} \right| + |c| \left| X_2 - \frac{c}{\sqrt{a^2+c^2}} \right| \right) \\
&\leq \frac{2}{\sqrt{a^2+c^2}} \left(|a|\delta' + |c| \left(\left| \frac{c}{a} \right| \delta' + \sqrt{\delta} \right) \right).
\end{aligned}
$$

Choosing $X(0)$ close enough to X_0, we can make δ and δ' small enough so that $\|X - X_0\|^2$ is smaller than any given $\epsilon > 0$. This proves by definition that X_0 is stable if $b = 0$.

□

This completes the stability analysis of homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras. The following table summarises our findings on the unit sphere S^3.

<table>
<tr><th>Case</th><th colspan="2">Stationary Point</th><th>Stability</th></tr>
<tr><td>Abelian</td><td colspan="2">Every point of S^3</td><td>Stable</td></tr>
<tr><td rowspan="2">$[e_1, e_2] = ce_4, c \neq 0$</td><td colspan="2">The circle $X_3^2 + X_4^2 = 1$</td><td>Stable</td></tr>
<tr><td colspan="2">The sphere $X_3 = 0$
minus the points $(0, 0, 0, \pm 1)$</td><td>Unstable</td></tr>
<tr><td rowspan="3">$[e_1, e_2] = ae_3 + be_4, [e_1, e_3] = ce_4$,
$a, c \neq 0$</td><td colspan="2">The circle $X_1^2 + X_2^2 = 1$</td><td>Unstable</td></tr>
<tr><td rowspan="2">Intersection of sphere $X_1 = 0$
and $P = 0$</td><td>$Q > 0$ or
$Q = 0$ and $b = 0$</td><td>Stable</td></tr>
<tr><td>$Q < 0$ or
$Q = 0$ and $b \neq 0$</td><td>Unstable</td></tr>
</table>

TABLE 3.1. Stability of homogeneous geodesics in 4-dimensional nilpotent metric Lie algebras.

This table, together with Lemma 3.9, completes the proof of Theorem 0.3.

CHAPTER 4

Stability of homogeneous geodesics in 4-dimensional unimodular metric Lie algebras with a nontrivial centre

The study of stability of homogeneous geodesics in 3-dimensional unimodular metric Lie algebras has been completed in Chapter 2, with Milnor's classification as an important starting point. In this chapter, we look at unimodular metric Lie algebras of the next nontrivial dimension, dimension 4. Again, we begin by obtaining a classification for unimodular algebras in Section 4.1, and we find all homogeneous geodesics for the algebras from Euler-Arnold equation in Section 4.2. Finally, stability analysis for these homogeneous geodesics is conducted in Section 4.3.

4.1. Classification of 4-dimensional unimodular metric Lie algebras with a nontrivial centre

Recall the definition of unimodular Lie algebras from Section 2.1.

4.1. DEFINITION. A Lie algebra $\mathfrak{g}$ is called *unimodular* if $\operatorname{tr}(\operatorname{ad}_X) = 0$ for all $X \in \mathfrak{g}$.

In this section, we consider a classification of 4-dimensional unimodular metric Lie algebras with a nontrivial centre. Note that the nilpotent algebras with which we worked in the last section are of this kind; this follows from Lemma 3.8 (c) and the following resut (see [**14**]).

4.1. PROPOSITION. *Suppose that $T : V \to V$ is a nilpotent linear map and λ is an eigenvalue of T. Then $\lambda = 0$.*

PROOF. By complexifying if necessary, let $Tv = \lambda v$, where $v \neq 0$ is an eigenvector corresponding to the eigenvalue λ. Then by induction, we have

$$T^n v = \lambda^n v$$

for each $n \in \mathbb{N}$. As T is a nilpotent map, $T^k = 0$ for some k, hence $\lambda^k v = 0$, implying $\lambda = 0$. $\square$

By the above proposition, all eigenvalues of a nilpotent map are zeros. As the trace is the sum of the eigenvalues, we obtain that $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all X in a nilpotent Lie algebra, hence the algebra is unimodular. However, note that there are more algebras in this class.

We start by considering all possible cases for $\dim Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. By our assumption, $\dim Z(\mathfrak{g}) > 0$, and so $\dim Z(\mathfrak{g}) \in \{1, 2, 3, 4\}$.

1. If $\dim Z(\mathfrak{g}) = 4$, then every element commutes with each other, hence $\mathfrak{g}$ is abelian. We already considered that case.
2. The case $\dim Z(\mathfrak{g}) = 3$ is not possible by the definition of the centre and the anti-symmetry of the Lie brackets.
3. Suppose $\dim Z(\mathfrak{g}) = 2$, and choose an orthonormal basis e_i for $\mathfrak{g}$ such that $Z(\mathfrak{g}) = \mathrm{Span}(e_3, e_4)$. Then the only nontrivial Lie bracket is $[e_1, e_2] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$. Then we have
$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & c_1 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & c_4 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_{e_2} = \begin{pmatrix} -c_1 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 \\ -c_3 & 0 & 0 & 0 \\ -c_4 & 0 & 0 & 0 \end{pmatrix}.$$
As $\mathfrak{g}$ is unimodular, we have $\mathrm{tr}(\mathrm{ad}_{e_1}) = \mathrm{tr}(\mathrm{ad}_{e_2}) = 0 \implies c_1 = c_2 = 0$. Hence $[e_1, e_2] = c_3e_3 + c_4e_4 \in Z(\mathfrak{g})$, hence the algebra $\mathfrak{g}$ is nilpotent (see Example 3.7 (4)). We already considered this case in the last section.

We can therefore assume that $\dim Z(\mathfrak{g}) = 1$ and choose an orthonormal basis e_i for $\mathfrak{g}$ in such a way that $Z(\mathfrak{g}) = \mathrm{Span}(e_4)$. Define the 3×3 matrix $L_{rk},\ r, k = 1, 2, 3$, as follows: $L_{3k} = \langle [e_1, e_2], e_k \rangle,\ L_{1k} = \langle [e_2, e_3], e_k \rangle,\ L_{2k} = \langle [e_3, e_1], e_k \rangle$, for $k = 1, 2, 3$. From unimodularity, we have

$$\begin{aligned} &\mathrm{tr}(\mathrm{ad}_{e_1}) = 0 \\ \implies &\sum_{i=1}^{4} \langle \mathrm{ad}_{e_1} e_i, e_i \rangle = 0 \\ \implies &\sum_{i=1}^{4} \langle [e_1, e_i], e_i \rangle = 0 \end{aligned}$$

As $e_4 \in Z(\mathfrak{g})$, we have $[e_1, e_4] = 0$, hence this reduces to

$$\begin{aligned} & \langle [e_1, e_2], e_2 \rangle + \langle [e_1, e_3], e_3 \rangle = 0 \\ \implies \quad & \langle [e_1, e_2], e_2 \rangle - \langle [e_3, e_1], e_3 \rangle = 0 \\ \implies \quad & L_{32} - L_{23} = 0 \\ \implies \quad & L_{32} = L_{23}. \end{aligned}$$

Similarly, from $\mathrm{tr}(\mathrm{ad}_{e_2}) = \mathrm{tr}(\mathrm{ad}_{e_3}) = 0$, we get $L_{13} = L_{31}$ and $L_{12} = L_{21}$. Thus L is a symmetric matrix, and so we can choose $\{e_1, e_2, e_3\}$ in such a way that it is diagonal: $L = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then we have

$$\begin{aligned}
& L_{11} = \langle [e_2, e_3], e_1 \rangle = \lambda_1 \\
\Rightarrow \quad & [e_2, e_3] = \lambda_1 e_1 + v_1 e_4, \\
\text{and} \quad & L_{22} = \langle [e_3, e_1], e_2 \rangle = \lambda_2 \\
\Rightarrow \quad & [e_3, e_1] = \lambda_2 e_2 + v_2 e_4, \\
\text{and} \quad & L_{33} = \langle [e_1, e_2], e_3 \rangle = \lambda_3 \\
\Rightarrow \quad & [e_1, e_2] = \lambda_3 e_3 + v_3 e_4.
\end{aligned}$$

Suppose at least two of λ_i are zeros, for example, $\lambda_1 = \lambda_2 = 0$. Then from the above formulas, $\mathfrak{g}^1 = \mathrm{Span}(e_3, e_4)$ and $\mathfrak{g}^2 = \mathrm{Span}(e_4) = Z(\mathfrak{g})$. Then $\mathfrak{g}^3 = 0$, so $\mathfrak{g}$ is nilpotent. Thus we have obtained the following classification of unimodular metric Lie algebras of dimension 4 with a nontrivial centre. To the best of our knowledge, this classification result did not appear in the literature before.

4.2. LEMMA. *Let $\mathfrak{g}$ be a 4-dimensional unimodular metric Lie algebra with a nontrivial centre. Then either $\mathfrak{g}$ is nilpotent, or there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{g}$ relative to which the only nonzero Lie brackets (up to skew-symmetry) are given by*

$$[e_1, e_2] = \lambda_3 e_3 + v_3 e_4, \quad [e_2, e_3] = \lambda_1 e_1 + v_1 e_4, \quad [e_3, e_1] = \lambda_2 e_2 + v_2 e_4, \tag{4.1.1}$$

where $\lambda_i, v_i \in \mathbb{R}$ and no more than one λ_i is 0.

4.2. Homogeneous geodesics in 4-dimensional unimodular metric Lie algebras

We are now able to write down the Euler-Arnold equation for 4-dimensional unimodular metric Lie algebras according to Lemma 4.2. As we already covered the case when $\mathfrak{g}$ is nilpotent in Chapter 3, we only need to deal with the second case, where $\mathfrak{g}$ is defined by an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ relative to which the Lie brackets (4.1.1) is obtained. Take L to be the 3×3 diagonal matrix as previously defined; then (4.1.1) can be rewritten as

$$[x, y] = L(x \times y) + \langle v, x \times y \rangle e_4,$$

where $\times$ is the usual cross-product in $V = \mathbb{R}^3$, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), v = (v_1, v_2, v_3) \in V$. Then for $X = x + x_4 e_4, Y = y + y_4 e_4 \in \mathfrak{g}$, with $x_4, y_4 \in \mathbb{R}$, we have

$$\mathrm{ad}_X Y = [X, Y] = [x, y] = L(x \times y) + \langle v, x \times y \rangle e_4. \tag{4.2.1}$$

Hence, for $Z = z + z_4 e_4 \in \mathfrak{g}$, we have

$$\begin{aligned}
\langle \mathrm{ad}^*_X Y, Z \rangle &= \langle \mathrm{ad}_X Z, Y \rangle && \text{(by definition of } \mathrm{ad}^*\text{)} \\
&= \langle L(x \times z) + \langle v, x \times z \rangle e_4, y + y_4 e_4 \rangle \\
&= \langle L(x \times z), y \rangle + y_4 \langle L(x \times z), e_4 \rangle \\
&\quad + \langle \langle v, x \times z \rangle e_4, y \rangle + y_4 \langle \langle v, x \times z \rangle e_4, e_4 \rangle \\
& && \text{(by linearity of the inner product)} \\
&= \langle L(x \times z), y \rangle + y_4 \langle v, x \times z \rangle \\
& && \text{(by definition of } L \text{ and orthonormality of basis)} \\
&= \langle x \times z, Ly \rangle + y_4 \langle v, x \times z \rangle && \text{(by symmetry of } L\text{)} \\
&= \langle Ly \times x, z \rangle + y_4 \langle v, x \times z \rangle && \text{(by symmetry of } L\text{)} \\
&= \langle (Ly + y_4 v) \times x, z \rangle,
\end{aligned}$$

where we are using the elementary fact that $\langle v, x \times z \rangle = \langle v \times x, z \rangle$. This gives

$$\mathrm{ad}^*_X Y = (Ly + y_4 v) \times x, \tag{4.2.2}$$

for $X = x + x_4 e_4, Y = y + y_4 e_4 \in \mathfrak{g}$.

The Euler-Arnold equation follows readily from 4.2.2

$$\begin{cases} \dot{x} &= (Lx + x_4 v) \times x \\ \dot{x_4} &= 0 \end{cases} \tag{4.2.3}$$

where $X(t) = x(t) + x_4 e_4, x(t) \subset V$.

This can also be written in the expanded form as follows

$$\begin{cases} \dot{x_1} &= v_2x_3x_4 - v_3x_2x_4 + x_2x_3(\lambda_2 - \lambda_3) \\ \dot{x_2} &= v_3x_1x_4 - v_1x_3x_4 + x_1x_3(\lambda_3 - \lambda_1) \\ \dot{x_3} &= v_1x_2x_4 - v_2x_1x_4 + x_1x_2(\lambda_1 - \lambda_2) \\ \dot{x_4} &= 0 \end{cases}. \tag{4.2.4}$$

4.2.1. REMARK. Equations (4.2.4) possess a very remarkable phenomenon. If we add to L any multiple of the identity matrix ("shifting L"), the Euler-Arnold equations (4.2.3) do not change, and so all the dynamics is preserved. This is especially interesting because the algebra $\mathfrak{g}$ itself does change (to a non-isomorphic algebra, in general). One immediate use of this observation is the following. If at least two of the three numbers $\lambda_1, \lambda_2, \lambda_3$ are equal, we can replace L by a diagonal matrix having at least two zeros on the diagonal. The equations (4.2.4) stay the same, but the algebra becomes nilpotent by (4.1.1), and so we know all about stationary points and their stability from Theorem 0.3. We can therefore assume from now on that $\lambda_1, \lambda_2, \lambda_3$ *are pairwise non-equal*. We can also assume that all three are non-zero by shifting L by a small multiple of the identity matrix, if necessary. This will help us to avoid considering too many cases.

4.2.2. REMARK. We have two obvious first integrals for equations (4.2.3)

$$I_1(X) = x_4 \qquad \text{(from } \dot{x_4} = 0), \tag{4.2.5}$$

$$\text{and} \quad I_2(X) = \|x\|^2 \qquad \text{(from Remark 1.6.1 and } I_1). \tag{4.2.6}$$

We obtain the third one $I_3(X)$ by noting that

$$\langle \dot{x}, Lx + x_4v \rangle = \langle (Lx + x_4v) \times x, Lx + x_4v \rangle = 0.$$

Now consider

$$\begin{aligned} \langle Lx + 2x_4v, x \rangle^{\cdot} &= \langle L\dot{x}, x \rangle + \langle Lx, \dot{x} \rangle + 2x_4\langle v, \dot{x} \rangle \\ &\qquad\qquad \text{(by differentiation of inner product)} \\ &= \langle \dot{x}, Lx \rangle + \langle \dot{x}, v \rangle x_4 + \langle Lx, \dot{x} \rangle + x_4\langle v, \dot{x} \rangle \\ &= \langle \dot{x}, Lx + x_4v \rangle + \langle \dot{x}, Lx + x_4v \rangle \\ &\qquad\qquad \text{(by linearity of inner product)} \\ &= 0. \end{aligned}$$

Hence,

$$I_3(X) = \langle Lx, x \rangle + 2x_4\langle v, x \rangle \tag{4.2.7}$$

is another first integral of equations (4.2.3).

We can proceed to find the stationary points of equations (4.2.3). A point $X_0 = x + x_4 e_4 \in \mathfrak{g}$ is stationary when $(Lx + x_4 v) \times x = 0$, which implies that the two vectors $Lx + x_4 v$ and x are linearly dependent. We have several further cases.

1. Assume that $x = 0$. This gives $X_0 = (0, 0, 0, x_4),\ x_4 \in \mathbb{R}$.
2. Assume that $x \neq 0$. Then there exists a unique $s \in \mathbb{R}$ such that

$$Lx + x_4 v = sx. \tag{4.2.8}$$

2.1. For every $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$, equation (4.2.8) gives

$$\begin{aligned} (sI - L)x &= x_4 v \\ \implies x &= x_4 (sI - L)^{-1} v. \end{aligned}$$

Thus, we have stationary points given by

$$X_0 = x_4 \Big((s - \lambda_1)^{-1} v_1, (s - \lambda_2)^{-1} v_2, (s - \lambda_3)^{-1} v_3, 1 \Big),\ x_4 \in \mathbb{R}. \tag{4.2.9}$$

2.2. Assume that $s = \lambda_i$ for some $i = 1, 2, 3$, but $v_i \neq 0$. Then as λ_i are pairwise nonequal by Remark 4.2.1, from equation (4.2.8) we have stationary points given by $X_0 = x_i e_i,\ x_i \in \mathbb{R}$.

2.3. Assume that $v_i = 0$ for some $i = 1, 2, 3$. Suppose $v_1 = 0$, then the cases $s = \lambda_2$ and $s = \lambda_3$ reduce to case 2.2.. Hence taking $s = \lambda_1$ in (4.2.8), we get a 2-plane of stationary points $\{(x_1, \frac{v_2 x_4}{\lambda_1 - \lambda_2}, \frac{v_3 x_4}{\lambda_1 - \lambda_3}, x_4) | x_1, x_4 \in \mathbb{R}\}$. Similarly, we get a corresponding 2-plane of stationary points for $v_2 = 0$ or $v_3 = 0$, or a union of such 2-planes if more than one v_i is zero.

4.3. Stability analysis and proof of Theorem 0.4

In this section, we give the stability analysis of homogeneous geodesics of 4-dimensional unimodular (but not nilpotent) metric Lie algebras obtained from the last section. We note an interesting finding that comes up in this case as well as in other previous cases, namely that the stability status determined by the Jacobian condition (Theorem 1.39) and Arnold's condition (Theorem 1.46) both depend on a similar function and complement each other.

1. Assume that $x = 0$. The stationary point $X_0 = (0, 0, 0, x_4), x_4 \in \mathbb{R}$ is stable; this follows from Remark 4.2.2 that $I_1(X) = x_4$ and $I_2(X) = \|x\|^2$ are first integrals: all the points on any trajectory lie at the same distance from the stationary point X_0.
2. Assume that $x \neq 0$. By (4.2.8), the stationary point $X_0 = x + x_4e_4$ of equations (4.2.3) satisfies $Lx + x_4v = sx$, with some unique $s \in \mathbb{R}$. That means that

$$\begin{cases} v_1x_4 &= (s - \lambda_1)x_1 \\ v_2x_4 &= (s - \lambda_2)x_2 \\ v_3x_4 &= (s - \lambda_3)x_3 \end{cases}.$$

The Jacobian matrix of equations (4.2.4) at X_0 is given by

$$J(X_0) = \begin{pmatrix} 0 & (\lambda_2 - s)x_3 & -(\lambda_3 - s)x_2 & v_2x_3 - v_3x_2 \\ -(\lambda_1 - s)x_3 & 0 & (\lambda_3 - s)x_1 & v_3x_1 - v_1x_3 \\ (\lambda_1 - s)x_2 & -(\lambda_2 - s)x_1 & 0 & v_1x_2 - v_2x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the characteristic equation $\det(J(X_0) - \mu I) = 0$ gives

$$\mu^2\big(\mu^2 + (s - \lambda_3)(s - \lambda_2)x_1^2 + (s - \lambda_3)(s - \lambda_1)x_2^2 + (s - \lambda_2)(s - \lambda_1)x_3^2\big) = 0.$$

Hence, two of the eigenvalues of the Jacobian matrix are zeros and the other two are the roots of the equation

$$\mu^2 = -\sigma(x_1, x_2, x_3),$$

where the polynomial $\sigma(x)$ is given by

$$\sigma(x) = (s - \lambda_3)(s - \lambda_2)x_1^2 + (s - \lambda_3)(s - \lambda_1)x_2^2 + (s - \lambda_2)(s - \lambda_1)x_3^2. \tag{4.3.1}$$

It now follows from Theorem 1.39 that if a stationary point X_0 satisfies the inequality $\sigma(x) < 0$, then X_0 is unstable.

To complement this result, we apply Theorem 1.46. Take a stationary point $X_0 = x + x_4e_4$ and an arbitrary point $Y = y + y_4e_4 \in \mathfrak{g}$. From equations (4.2.2) and (4.2.8), we know that

$$B(X_0, Y) = \mathrm{ad}_Y^* X_0 = (Lx + x_4v) \times y = sx \times y. \tag{4.3.2}$$

We want to find the regularity condition for X_0, that is, we want to determine the linear subspace $L_{X_0} = \{Y = y + y_4 e_4 \mid B(X_0, Y) = 0\}$ and compute its dimension. From (4.3.2) and assuming that $s \neq 0$, we have

$$B(X_0, Y) = 0 \Longleftrightarrow y \parallel x.$$

Hence, $L_{X_0} = \mathrm{Span}(x, e_4)$ and so $\dim(L_{X_0}) = 2$. By continuity, $s \neq 0$ and $x \neq 0$ for nearby stationary points, and so $\dim(L_{X_0})$ is locally constant, hence X_0 is regular. Then the quadratic form Φ is given by

$$\begin{aligned}
\Phi(Y) &= \|B(X_0, Y)\|^2 + \langle [Y, X_0], B(X_0, Y)\rangle \\
&= s^2\|x \times y\|^2 + \langle L(y \times x) + \langle v, y \times x\rangle e_4, sx \times y\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (4.3.2) and (4.2.1))} \\
&= s^2\|x \times y\|^2 + \langle L(y \times x), sx \times y\rangle + \langle\langle v, y \times x\rangle e_4, sx \times y\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by linearity of the inner product)} \\
&= s^2\|x \times y\|^2 + \langle L(y \times x), sx \times y\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by orthonormality)} \\
&= s^2\|x \times y\|^2 - s\langle L(x \times y), x \times y\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by properties of cross product)} \\
&= s\,\langle (sI - L)(x \times y), x \times y\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by linearity of inner product).}
\end{aligned}$$

Note that multiplication by $s \neq 0$ doesn't affect the fact that Φ is definite, and so it suffices to consider the quadratic form

$$\phi(Y) = \langle (sI - L)(x \times y), x \times y\rangle$$

on $V = \mathbb{R}^3$, which we want to be definite when restricted to the subspace $x^\perp$. We calculate

$$\begin{aligned}
\phi(Y) = \Big((s-\lambda_3)x_2^2 + (s-\lambda_2)x_3^2\Big)y_1^2 &+ \Big((s-\lambda_3)x_1^2 + (s-\lambda_1)x_3^2\Big)y_2^2 \\
&+ \Big((s-\lambda_1)x_2^2 + (s-\lambda_2)x_1^2\Big)y_3^2 + 2(\lambda_3 - s)x_1x_2y_1y_2 \\
&\qquad + 2(\lambda_2 - s)x_1x_3y_1y_3 + 2(\lambda_1 - s)x_2x_3y_2y_3. \qquad (4.3.3)
\end{aligned}$$

Let Q be a symmetric matrix given by

$$Q = \begin{pmatrix} (s-\lambda_3)x_2^2 + (s-\lambda_2)x_3^2 & (\lambda_3 - s)x_1x_2 & (\lambda_2 - s)x_1x_3 \\ (\lambda_3 - s)x_1x_2 & (s-\lambda_3)x_1^2 + (s-\lambda_1)x_3^2 & (\lambda_1 - s)x_2x_3 \\ (\lambda_2 - s)x_1x_3 & (\lambda_1 - s)x_2x_3 & (s-\lambda_1)x_2^2 + (s-\lambda_2)x_1^2 \end{pmatrix},$$

then we have $\phi(Y) = \langle Qy, y \rangle$. Note that $Qx = 0$, so x is an eigenvector of Q with corresponding eigenvalue $\eta_1 = 0$. Hence, for ϕ to be definite on its orthogonal complement, the other two eigenvalues η_2 and η_3 of Q must have the same sign. Consider the characteristic polynomial $\chi(\eta)$ of Q

$$\chi(\eta) = \eta(\eta - \eta_2)(\eta - \eta_3) = \eta^3 - (\eta_2 + \eta_3)\eta^2 + \eta_2\eta_3\eta.$$

η_2 and η_3 are of the same sign if and only if the coefficient of η in $\chi(\eta)$ is positive. Computing $\chi(\eta)$ we find that this coefficient equals $\|x\|^2\sigma(x)$, where $\sigma(x)$ is given by (4.3.1). Hence, by Theorem 1.46, if a stationary point X_0 satisfies the inequality $\sigma(x) > 0$, then X_0 is stable.

To complete this case, we remove the assumption $s \neq 0$. Suppose we have a stationary point $X_0 = x + x_4e_4$ such that $s = 0$. Then by Remark 4.2.1, we can replace L by $L + cI, c \neq 0$, so that the Euler-Arnold equations (4.2.3), together with the corresponding stationary points, stay the same. The function $\sigma(x)$ also remains unchanged: we shift both λ_i and s by the same c. Then by repeating the argument, we find that X_0 is stable provided $\sigma(x) > 0$.

In summary, we have proved the following stability theorem.

4.3. THEOREM. *A stationary point $X_0 = x + x_4e_4$ is stable if $\sigma(x) > 0$, and is unstable if $\sigma(x) < 0$, where*

$$\sigma(x) = (s - \lambda_3)(s - \lambda_2)x_1^2 + (s - \lambda_3)(s - \lambda_1)x_2^2 + (s - \lambda_2)(s - \lambda_1)x_3^2.$$

It therefore remains to study the case $\sigma(x) = 0$. Let $X_0 = x + x_4e_4,\ x \neq 0$, be a stationary point with $\sigma(x) = 0$.

2.1. We first deal with the case when $s \in \mathbb{R}$ determined by (4.2.8) satisfies the condition $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$. Consider the set Γ_{X_0} of points $Y = y + y_4e_4 \in \mathfrak{g}$ for which all three first integrals found in Remark 4.2.2 take the same values as at X_0. As first integrals are constant along trajectories, any trajectory which starts at a point of Γ_{X_0} remains in Γ_{X_0} for all $t \in \mathbb{R}$. In other words, Γ_{X_0} is a union of trajectories. Clearly, $X_0 \in \Gamma_{X_0}$ and is a constant trajectory, because X_0 is a stationary point. We prove the following lemma.

4.4. LEMMA. *Let $X_0 = x + x_4e_4,\ x \neq 0$, be a stationary point of* (4.2.3) *with $\sigma(x) = 0$ and $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$. Then*

(a) Γ_{X_0} is compact.

(b) $\Gamma_{X_0} \setminus \{X_0\}$ is homeomorphic to $\mathbb{R}$ (*to an open interval*).

(c) Γ_{X_0} contains no stationary points of (4.2.3) *other than X_0.*

PROOF. We first prove assertion (a). Note that Γ_{X_0} is the intersection of zero sets of a finite number of polynomial functions. As polynomial functions are continuous, the inverse image of a closed set is closed. Since $\{0\}$

is a closed set, the zero set of a polynomial equation is closed, and hence Γ_{X_0} is closed. Moreover, it is bounded as $y_4 = x_4$ and $\|y\|^2 = \|x\|^2$, for any $Y = y + y_4 e_4 \in \Gamma_{X_0}$. It then follows that Γ_{X_0} is compact.
To prove assertion (b), take $Y = y + y_4 e_4 \in \Gamma_{X_0}$. We have $I_i(Y) = I_i(X_0),\ i = 1, 2, 3$, which gives $y_4 = x_4,\ \|y\|^2 = \|x\|^2$ and $\langle Ly, y\rangle + 2\langle v, y\rangle y_4 = \langle Lx, x\rangle + 2\langle v, x\rangle x_4$. Denote $z = y - x \in \mathbb{R}^3$. Then

$$\begin{aligned}
\|z\|^2 + 2\langle z, x\rangle &= \|y - x\|^2 + 2\langle y - x, x\rangle \\
&= \|x\|^2 + \|y\|^2 - 2\langle y, x\rangle + 2\langle y, x\rangle - 2\|x\|^2 \\
&= \|y\|^2 - \|x\|^2 \\
&= 0,
\end{aligned}$$

and from (4.2.8), we have

$$\begin{aligned}
&\langle Lz, z\rangle + 2\langle sx, z\rangle \\
=&\langle Lz, z\rangle + 2\langle Lx + x_4 v, z\rangle \\
=&\langle Lz, z\rangle + 2\langle Lx, z\rangle + 2\langle v, z\rangle x_4 \\
=&\langle L(y - x), y - x\rangle + 2\langle Lx, y - x\rangle + 2\langle v, y - x\rangle x_4 \\
=&\langle L(y - x), y - x\rangle + 2\langle Lx, y - x\rangle + 2\langle v, y - x\rangle x_4 \\
=&\langle Ly, y\rangle - \langle Lx, x\rangle + \langle Lx, y\rangle - \langle Ly, x\rangle + 2\langle v, y\rangle x_4 - 2\langle v, x\rangle x_4 \\
=&\, 0.
\end{aligned}$$

It follows that Γ_{X_0} lies in the hyperplane $y_4 = x_4$, and in that hyperplane, Γ_{X_0} is given by $y = x + z$, where z satisfies the equations

$$\|z\|^2 + 2\langle z, x\rangle = 0 \quad \text{and} \quad \langle (L - sI)z, z\rangle = 0. \tag{4.3.4}$$

Geometrically, the first equation of (4.3.4) gives

$$(z_1 + x_1)^2 + (z_2 + x_2)^2 + (z_3 + x_3)^2 = x_1^2 + x_2^2 + x_3^2,$$

hence it defines a sphere in $\mathbb{R}^3$ of radius $\|x\|$ centered at $-x$. This sphere passes through the origin $z = 0$ and its normal vector at the origin is x. The second equation gives

$$(s - \lambda_1)z_1^2 + (s - \lambda_2)z_2^2 + (s - \lambda_3)z_1^3 = 0,$$

hence it defines a quadratic cone in $\mathbb{R}^3$ with the vertex at the origin. The generatrix l of the cone is given by

$$z = a(L - sI)^{-1}x,\ a \in \mathbb{R}.$$

This is because it passes through the cone vertex and satisfies the cone equation

$$\begin{aligned}\langle (L-sI)z, z\rangle &= a^2\langle (L-sI)(L-sI)^{-1}x, (L-sI)^{-1}x\rangle \\ &= a^2\langle x, (L-sI)^{-1}x\rangle \\ &= \frac{\sigma(x)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} \\ &= 0.\end{aligned}$$

Furthermore, the normal vector to the cone along this line is again x. Thus equations (4.3.4) define the intersection of the cone and the sphere in $\mathbb{R}^3$ such that the vertex of the cone lies on the sphere and the tangent plane to the sphere at this point is tangent to the cone along one of its generatrices l. Moreover, no other generatrix l' of the cone lies in this plane (see Figure 4.1).

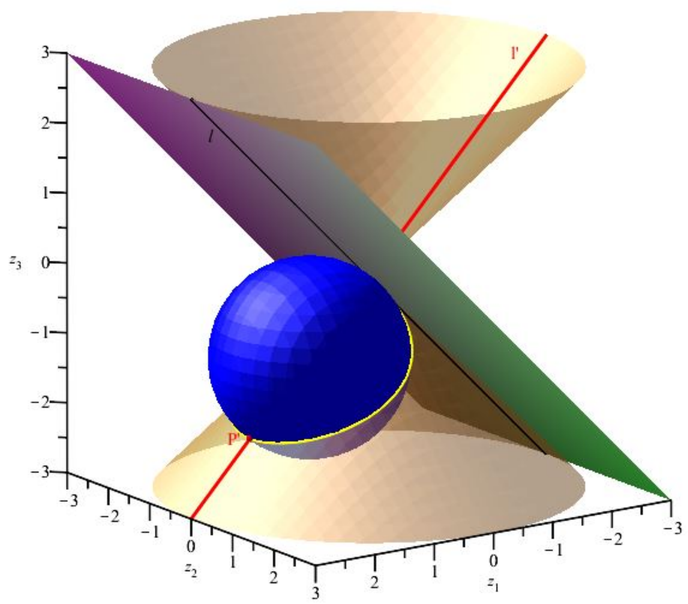
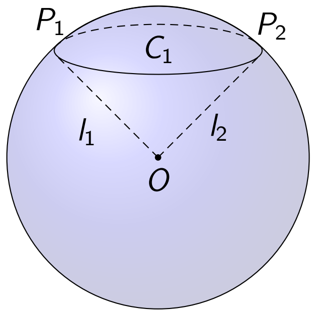


FIGURE 4.1. Intersection between the sphere and the cone given by equations (4.3.4).

We now need a small result here.

4.2. PROPOSITION. *The set of generatrices of the cone is homeomorphic to a circle.*

PROOF. Consider the intersection of the cone with a large sphere centered at the origin. We get a disjoint union of two curves C_1, C_2 which are symmetric to each other about the origin and which are both homeomorphic to a circle. On the other hand, the set of generatrices of the cone is in one-to-one correspondence with the points of the curve, say C_1. Hence, we obtain the result as required. The following diagram shows the homeomorphism.



□

Returning to assertion (b), let $l' \neq l$ be a generatrix of the cone. It passes through the origin and is not tangent to the sphere at the origin, which means that there is a unique point of intersection of l' with the sphere, different from the origin (see Figure 4.1). That point P' continuously depends on l', and so the intersection of the sphere and the cone given by (4.3.4) is the union of the origin and a curve homeomorphic to a circle (from Proposition 4.2) minus one point (see Figure 4.2).

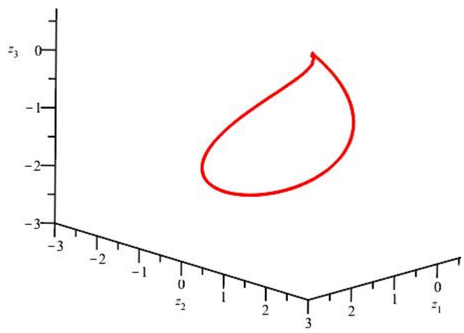


FIGURE 4.2. Intersection curve between the sphere and the cone given by equations (4.3.4).

That curve is homeomorphic to $\Gamma_{X_0} \setminus \{X_0\}$, since it is obtained by a parallel translation in $\mathbb{R}^4 = \mathfrak{g}$. This implies that $\Gamma_{X_0} \setminus \{X_0\}$ is homeomorphic to a circle minus a point, and via stereographic projection, to $\mathbb{R}$. This proves assertion (b).

Finally, we prove assertion (c). Assume that $Y = y + y_4 e_4$ is a stationary point of (4.2.3) in Γ_{X_0}. As $y \neq 0$, by (4.2.8), we have $Ly + y_4 v = s'y$ for some $s' \in \mathbb{R}$. Hence

$$Lz = L(y - x) = s'y - y_4 v - sx + x_4 v = s'y - sz = s'z + (s' - s)x,$$

where $z = y - x$ satisfies (4.3.4). Now if $s' = s$, we have $Lz = sz$, and so $z = 0$ as $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$. Otherwise, if $s' \neq s$, from the second equation of (4.3.4), we get

$$\begin{aligned} s\|z\|^2 &= \langle Lz, z\rangle \\ &= \langle s'z + (s' - s)x, z\rangle \\ &= s'\|z\|^2 + (s' - s)\langle x, z\rangle. \end{aligned}$$

This implies

$$(s - s')\|z\|^2 = -(s - s')\langle x, z\rangle \implies \|z\|^2 + \langle x, z\rangle = 0.$$

By the first equation of (4.3.4), we get $\|z\|^2 = 0 \Leftrightarrow z = 0$. So in either case, we have $z = 0$, and then $y = x$ which implies $Y = X_0$. Thus X_0 is the only stationary point of (4.2.3) in Γ_{X_0}. □

We now take a point $Y \in \Gamma_{X_0} \setminus \{X_0\}$ and consider the trajectory $\gamma = \gamma(t)$ of (4.2.3) passing through Y. Note that $\gamma \subset \Gamma_{X_0}$. By assertion (c) of Lemma 4.4, Y is not a stationary point, hence γ is not a single point, and γ doesn't contain X_0, as the trajectory of X_0 is $\{X_0\}$. It follows that $\gamma \subset \Gamma_{X_0} \setminus \{X_0\}$. Again, assertion (c) gives $\dot\gamma \neq 0$, hence the point $\gamma(t)$ moves monotonically, in the same direction along the open interval $\Gamma_{X_0} \setminus \{X_0\}$ when $t \in (-\infty, \infty)$ ($\Gamma_{X_0} \setminus \{X_0\}$ is an open interval by assertion (b)). As Γ_{X_0} is compact by assertion (a), the limit points

$$Y_\pm = \lim_{t \to \pm\infty} \gamma(t)$$

exist and lie on Γ_{X_0}. But then Y_+ must be a stationary point of (4.2.3), for otherwise a small piece of the trajectory of the solution passing through Y_+ overlaps with the trajectory γ of Y, and so Y_+ lies in the interior of γ. Similar argument applies to Y_-, thus it follows from assertion (c) that

$$Y_+ = Y_- = X_0,$$

and so for the trajectory $\gamma = \gamma(t)$ with $\gamma(0) = Y$, we have $\lim_{t\to\pm\infty} \gamma(t) = X_0$. This means that whenever $t_0 < 0$, we can take $Z = \gamma(t_0)$ such that $\|Z - X_0\| < \delta$ for a given $\delta > 0$. Then the solution of (4.2.3) with Z at t_0 can be as close to the stationary point X_0 as possible, but the future trajectory $t \mapsto \gamma(t_0 + t)$ will pass through the point Y at $t = -t_0$ at a fixed distance $\epsilon = \|Y - X_0\|$, meaning it is not arbitrarily close to X_0 anymore. Thus we have proved the following theorem.

4.5. THEOREM. *A stationary point $X_0 = x + x_4e_4$ satisfying $x \neq 0, \sigma(x) = 0$ and $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$ is unstable.*

2.2. Lastly, we consider a stationary point $X_0 = x + x_4 e_4$ satisfying $x \neq 0, \sigma(x) = 0$ and $s \in \{\lambda_1, \lambda_2, \lambda_3\}$. Recall that λ_i are pairwise nonequal, and so up to relabelling we can assume that $s = \lambda_1 \neq \lambda_2, \lambda_3$. Then from $\sigma(x) = 0$, we get $x_1 = 0$. Then from (4.2.8), we either get $x = 0$, which is invalid; or $v_1 = 0$. So this case reduces to one stationary point

$$X_0 = (0, \frac{v_2 x_4}{\lambda_1 - \lambda_2}, \frac{v_3 x_4}{\lambda_1 - \lambda_3}, x_4),\ x_4 \neq 0. \tag{4.3.5}$$

We prove the following.

4.6. THEOREM. *Suppose that $\sigma(x) = 0$ and $s = \lambda_1$, then the stationary point X_0 given by* (4.3.5) *is stable if $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have the same sign, and unstable if $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have opposite signs.*

PROOF.

(a) We first consider the case when the numbers $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have the same sign $c = \pm 1$. Then $\lambda_i - \lambda_1 = c|\lambda_i - \lambda_1|$ for $i = 2, 3$. Let $Y = y + y_4 e_4 \in \mathfrak{g}$ be a point such that $\|Y - X_0\| < \delta$ for a small positive δ. Note that this means $|y_i - x_i| < \delta$, for all $i = 1, 2, 3, 4$. By Remark 4.2.2, we have $I_1(Y) = y_4$ and $I_2(Y) = \|y\|^2$. Instead of the first integral I_3, it will be more convenient to consider the first integral

$$I_4 = c\left(I_3 - \lambda_1 I_2 + \left(\frac{v_2^2}{\lambda_2 - \lambda_1} + \frac{v_3^2}{\lambda_3 - \lambda_1}\right) I_1^2\right)$$

at Y, that is,

$$\begin{aligned}
I_4(Y) &= c(\lambda_2 - \lambda_1)\left(y_2 + \frac{v_2 y_4}{\lambda_2 - \lambda_1}\right)^2 + c(\lambda_3 - \lambda_1)\left(y_3 + \frac{v_3 y_4}{\lambda_3 - \lambda_1}\right)^2 \\
&= |\lambda_2 - \lambda_1|\left((y_2 - x_2) + \frac{v_2(y_4 - x_4)}{\lambda_2 - \lambda_1}\right)^2 \\
&\qquad + |\lambda_3 - \lambda_1|\left((y_3 - x_3) + \frac{v_3(y_4 - x_4)}{\lambda_3 - \lambda_1}\right)^2 \\
&\leq |\lambda_2 - \lambda_1|\left(\delta + \frac{|v_2|\delta}{|\lambda_2 - \lambda_1|}\right)^2 + |\lambda_3 - \lambda_1|\left(\delta + \frac{|v_3|\delta}{|\lambda_3 - \lambda_1|}\right)^2 \\
&= M\delta^2,
\end{aligned}$$

for some constant $M > 0$ which only depends on λ_i, v_i, but not on X_0 or Y. Let $Z = z + z_4 e_4$ be an arbitrary point on the trajectory of (4.2.3) starting at Y. Then $I_4(Z) = I_4(Y) \leq M\delta^2$. This gives

$$\begin{aligned}
&|\lambda_2 - \lambda_1|\Big((z_2 - x_2) + \frac{v_2(z_4 - x_4)}{\lambda_2 - \lambda_1}\Big)^2 \\
&\qquad\qquad + |\lambda_3 - \lambda_1|\Big((z_3 - x_3) + \frac{v_3(z_4 - x_4)}{\lambda_3 - \lambda_1}\Big)^2 \leq M\delta^2.
\end{aligned}$$

Consequently, for $i = 2, 3$,

$$\left|(z_i - x_i) + \frac{v_i(z_4 - x_4)}{\lambda_i - \lambda_1}\right| \le M_i \delta,$$

where $M_i = \sqrt{\frac{M}{|\lambda_i - \lambda_1|}} > 0$. But $z_4 = y_4$, so for $i = 2, 3$,

$$|z_i - x_i| \le M_i \delta + \frac{|v_i||y_4 - x_4|}{|\lambda_i - \lambda_1|} \le M_i' \delta,$$

where $M_i' = M_i + \frac{|v_i|}{|\lambda_i - \lambda_1|} > 0$. Let $m = \max(1, M_2', M_3')$. Then for $i = 2, 3, 4$

$$|z_i - x_i| \le m\delta,$$

Consider

$$\begin{aligned}
\|z - x\|^2 &= \|z\|^2 + \|x\|^2 - 2\langle z, x\rangle \\
&= \|y\|^2 + \|x\|^2 - 2\langle z, x\rangle \\
&\qquad\qquad \text{(as } I_2(Y) = I_2(Z) \Leftrightarrow \|z\|^2 = \|y\|^2) \\
&= \|y\|^2 + \|x\|^2 - 2\langle y, x\rangle + 2\langle y, x\rangle - 2\langle z, x\rangle \\
&= \|y - x\|^2 + 2\langle y - x, x\rangle - 2\langle z - x, x\rangle. \qquad (4.3.6)
\end{aligned}$$

By Cauchy-Schwartz inequality, we have

$$|\langle y - x, x\rangle| \le \|y - x\| \, \|x\| \le \delta\|x\|,$$

and

$$\begin{aligned}
|\langle z - x, x\rangle| &= |(z_2 - x_2)x_2 + (z_3 - x_3)x_3| \\
&\le |z_2 - x_2| \, \|x\| + |z_3 - x_3| \, \|x\| \\
&\le 2m\delta\|x\|.
\end{aligned}$$

Hence, equation (4.3.6) gives

$$\|z - x\|^2 \le \delta^2 + 2\delta\|x\| + 4m\delta\|x\|. \qquad (4.3.7)$$

Now consider

$$\begin{aligned}
\|Z - X_0\|^2 &= \|z - x\|^2 + (z_4 - x_4)^2 \\
&= \|z - x\|^2 + (y_4 - x_4)^2 && \text{(as } z_4 = y_4) \\
&\le \|z - x\|^2 + \delta^2 \\
&\le \delta^2 + 2\delta\|x\| + 4m\delta\|x\| + \delta^2 && \text{(by (4.3.7))}.
\end{aligned}$$

Therefore

$$\|Z - X_0\| \le \sqrt{2\delta\left(\delta + \|x\|(1 + 2m)\right)}.$$

Thus, the distance between Z and X_0 tends to zero when δ approaches zero. So for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $Y \in \mathfrak{g}$,

$$\|Y - X_0\| < \delta \implies \|Z - X_0\| < \epsilon,$$

for all points Z on the trajectory of (4.2.3) starting at Y.
Hence given the above assumptions, if $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have the same sign, then the stationary point X_0 is stable.

(b) It remains to consider the case when the numbers $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have opposite signs. Relabelling if necessary we can assume that $\lambda_2 - \lambda_1 > 0 > \lambda_3 - \lambda_1$. Denote $\lambda_2 - \lambda_1 = a^2$ and $\lambda_3 - \lambda_1 = -b^2, a, b > 0$. Similar to the above, we consider the set Γ_{X_0} of points $Y = y + y_4 e_4 \in \mathfrak{g}$ for which all three first integrals I_1, I_2, I_3 take the same values as they take at X_0. We know that $y_4 = x_4$, and moreover, $z = y - x$ satisfies equations (4.3.4). The second equation of (4.3.4) gives

$$a^2 z_1^2 - b^2 z_2^2 = 0.$$

We can then let $z_2 = bw$ and $z_3 = caw$, for $w \in \mathbb{R}, c = \pm 1$ and substitute into the first equation of (4.3.4) to get

$$z_1^2 + (a^2 + b^2) w^2 + 2w(bx_2 + cax_3) = 0.$$

As $x \neq 0$, we can choose $c = \pm 1$ in such a way that $(bx_2 + cax_3) \neq 0$. Thus dividing the previous equation by $(a^2 + b^2)^2$ yields

$$\frac{z_1^2}{(a^2+b^2)^2} + \frac{1}{a^2+b^2}\left(w^2 + \frac{2w(bx_2 + cax_3)}{a^2+b^2}\right) = 0.$$

By completing the square, we get

$$\frac{z_1^2}{(a^2+b^2)^2} + \frac{1}{a^2+b^2}\left(w + \frac{bx_2 + cax_3}{a^2+b^2}\right)^2 = \frac{(bx_2 + cax_3)^2}{(a^2+b^2)^3}.$$

Multiplying by $(bx_2 + cax_3)^2$ and denote $A = -\frac{bx_2 + cax_3}{a^2+b^2}$ and $B = \frac{bx_2 + cax_3}{\sqrt{a^2+b^2}}$, we get

$$A^2 z_1^2 + B^2 (w - A)^2 = A^2 B^2$$
$$\Rightarrow \qquad \frac{z_1^2}{B^2} + \frac{(w-A)^2}{A^2} = 1.$$

So we can let $z_1 = B \sin\theta, w = A(1 - \cos\theta)$ for $\theta \in [0, 2\pi)$. Substituting into equations (4.2.3), we get the equation

$$\dot{\theta} = -abcB(1 - \cos\theta)$$
$$\Rightarrow \quad \theta(t) = 2\arccos(abcB(t - t_0)), \quad \text{for some } t_0 \in \mathbb{R}.$$

This gives $z_1(t) = \frac{2abcB^2(t-t_0)}{1+abB^2(t-t_0)^2}$, $w(t) = \frac{2A}{1+abB^2(t-t_0)^2}$, and so

$$\begin{aligned} Y(t) &= X_0 + z(t) \\ &= X_0 + \frac{1}{1+abB^2(t-t_0)^2}\left(2abcB^2(t-t_0), 2Ab, 2Aac\right). \end{aligned}$$

Now by choosing a large positive t_0, we can make $Y(0)$ arbitrarily close to X_0, but the point $Y(t_0)$ on the trajectory lies at a fixed distance from X_0. This proves that given the above assumptions, if $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have opposite signs, then the stationary point X_0 is unstable.

□

This completes the stability analysis of homogeneous geodesics in 4-dimensional unimodular metric Lie algebras with a nontrivial centre. The following table summarises our findings on the sphere S^3, where we define

$$\sigma(x) = (s-\lambda_3)(s-\lambda_2)x_1^2 + (s-\lambda_3)(s-\lambda_1)x_2^2 + (s-\lambda_2)(s-\lambda_1)x_3^2.$$

<table>
<tr><th>Case</th><th colspan="2">Stationary Point</th><th>Stability</th></tr>
<tr><td>$x = 0$</td><td colspan="2">$(0,0,0,x_4), x_4 \in \mathbb{R}$</td><td>Stable</td></tr>
<tr><td rowspan="5">$x \neq 0$</td><td rowspan="5">$X_0 = x + x_4 e_4$ satisfying $Lx + x_4 v = sx$ where $L = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, λ_i are pairwise nonequal and $x_4, s \in \mathbb{R}$</td><td>$\sigma(x) > 0$</td><td>Stable</td></tr>
<tr><td>$\sigma(x) < 0$</td><td>Unstable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s \notin \{\lambda_1, \lambda_2, \lambda_3\}$</td><td>Unstable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s = \lambda_1$ and $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have the same sign</td><td>Stable</td></tr>
<tr><td>$\sigma(x) = 0$ and $s = \lambda_1$ and $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ have opposite signs</td><td>Unstable</td></tr>
</table>

TABLE 4.1. Stability of homogeneous geodesics in 4-dimensional unimodular metric Lie algebras.

This table, together with Lemma 4.2 in Section 4.1, completes the overall proof of Theorem 0.4.

Bibliography

[1] V. ARNOLD, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Annales de l'Institut Fourier, 16 (1966), pp. 319–361.

[2] A. ARVANITOYEORGOS, *An introduction to Lie groups and the geometry of homogeneous spaces*, Student Mathematical Library, American Mathematical Society, Providence, 2003.

[3] L. BIANCHI, *On the three-dimensional spaces which admit a continuous group of motions*, Gen. Relativity Gravitation, 33 (2001), pp. 2171–2253 (2002).

[4] W. E. BOYCE AND R. C. DIPRIMA, *Elementary differential equations and boundary value problems*, John Wiley & Sons Inc., New York, 2012.

[5] D. BUMP, *Lie groups*, Graduate Texts in Mathematics, Springer, New York, 2004.

[6] N. G. CHETAYEV, *The stability of motion*, Pergamon Press, Oxford, 1961.

[7] J. J. DUISTERMAAT AND J. A. C. KOLK, *Lie groups*, Universitext, Springer, Berlin, 2000.

[8] K. ERDMANN AND M. J. WILDON, *Introduction to Lie algebras*, Springer Undergraduate Mathematics, Springer, London, 2006.

[9] L. EULER, *Theoria motus corporum solidorum seu rigidorum ex primis nostræ cognitionis principiis stabilita et ad omnes motus qui in hujusmodi corpora cadere possunt accommodata*, Roser, 1765.

[10] J. GALLIER AND J. QUAINTANCE, *Differential Geometry and Lie Groups: A Computational Perspective*, Geometry and Computing, Springer, 2020.

[11] R. GRIMSHAW, *Nonlinear ordinary differential equations*, Applied Mathematics and Engineering Science Texts, Blackwell Scientific Publications Ltd., Oxford, 1990.

[12] B. C. HALL, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, Springer, New York, 2003.

[13] M. HAZEWINKEL, N. GUBARENI, AND V. V. KIRICHENKO, *Algebras, rings and modules: Lie algebras and Hopf algebras*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2010.

[14] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, Cambridge, 2013.

[15] J. E. HUMPHREYS, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Springer, New York, 1972.

[16] D. W. JORDAN AND P. SMITH, *Nonlinear ordinary differential equations*, Oxford University Press, Oxford, 2007.

[17] V. V. KAĬZER, *Conjugate points of left-invariant metrics on Lie groups*, Izv. Vyssh. Uchebn. Zaved. Mat., (1990), pp. 27–37.

[18] J. M. LEE, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, Springer, New York, 2013.

[19] ——, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, Springer, New York, 2018.

[20] R. A. Marinosci, *Homogeneous geodesics in a three-dimensional Lie group*, Comment. Math. Univ. Carolin., 43 (2002), pp. 261–270.

[21] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math., 21 (1976), pp. 293–329.

[22] H. Naitoh and Y. Sakane, *On conjugate points of a nilpotent Lie group*, Tsukuba Journal of Mathematics, 5 (1981), pp. 143–152.

[23] L. Perko, *Differential equations and dynamical systems*, Texts in Applied Mathematics, Springer, New York, 2001.

[24] M. Spivak, *A comprehensive introduction to differential geometry. Vol. II*, Publish or Perish Inc., Wilmington, Del., 1979.

[25] J. Stillwell, *Naive Lie theory*, Undergraduate Texts in Mathematics, Springer, New York, 2008.