Representability and Probability in Relation Algebra

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Summary

Relation-type algebras are abstract algebraic generalisations of algebras of binary relations on a given set, which are usually called proper relation algebras. Initially, much of the research into these algebras focused on determining whether or not all relation algebras are representable, i.e., embeddedable into proper relation algebras. Once this was answered in the negative, research into these algebras branched out. This thesis focuses two weaker notions of representability that were introduced recently, the subvariety lattices of tense algebras and various relation-type algebras, and probabilistic topics.

After introducing the two weaker notions of representability, namely feeble and qualitative representability, we investigate the representability of chromatic algebras. In particular, we show that every Ramsey algebra has a qualitative representation. By modifying a known graph, we show that a variety of tense algebras has continuum many covers in the subvariety lattice of the variety generated by total tense algebras. Using a previously known relationship between the subvariety lattice of this variety and the subvariety lattice of a certain variety of semiassociative relation algebras, we complete the description of all varieties of height at most two, up to cardinality, in a number of subvariety lattices of varieties of nonassociative relation algebras. Next, by constructing qualitative representations of the generators of these varieties, we extend this result to the subvariety lattices of the varieties generated by feeble and qualitatively representable algebras, and then look at other varieties of low height. Using some known results on counting relation-type algebras, we show that almost all nonassociative relation algebras are symmetric and have the identity as an atom. Consequently, we obtain a simple single variable asymptotic formula that counts the number of isomorphism classes finite relation-type algebras that are in various classes. Lastly, we obtain a 0–1 law for the atom structures of nonassociative relation algebras.

Statement of authorship

This thesis includes work by the author that has been published or accepted for publication as described in the text. Except where reference is made in the text of the thesis, this thesis contains no other material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgement in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

James M. Koussas 9/8/2021

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Introduction

Relation algebras were introduced as an abstract algebraic counterpart to the calculus of relations in the nineteen fourties by Alfred Tarski [**86**]. In its early years, research into relation algebras was dominated by Tarski and his doctoral students, and was chiefly focused on Tarski's representability problem for relation algebras, i.e., the problem of determining whether or not every relation algebra is isomorphic to an algebra with binary relations as elements and with the usual set theoretic operations. A negative solution was discovered within a decade by Lyndon [**65**]. Following this, research into relation algebras branched out more. For example, subvariety lattices, conditions guaranteeing representability, the existence and properties of axiomatisations, the representability of known algebras or families of algebras, probability, computability, and varieties containing relation algebras became more widely studied; see Maddux [**68**]. This thesis focuses on two weaker notions of representability, the representability of chromatic algebras, subvariety lattices, and probabilistic results. The original goal of this thesis was to solve a problem on the subvariety lattice of the variety of relation algebras (Problem 4 here), which lead to the contents of Chapter 3. The remainder of the contents were inspired by conversations with my supervisors, Robin Hirsch, and Roger Maddux, and by other problems that I saw in the literature.

The assumed knowledge for this thesis is roughly equivalent to that of a honours graduate in the general (universal) algebra research group at La Trobe University; more specifically, the lattice theory, general algebra, and model theory subjects at La Trobe University, which are roughly equivalent to Chapters 1–5 of Davey and Priestley [**22**], Chapters 1–5 of Bergman [**13**], and Chapters 1, 3, 5, and 6 of Hodges [**41**]. Chapter 1 contains a relatively brief introduction to Boolean algebras with operators and relation-type algebras that is intended for someone with this background knowledge, and reviews some useful tools from universal algebra and model theory.

Chapter 2 begins with a brief introduction to representations, networks, and two more recent notions of representability, namely feeble and qualitative representability. In Section 2.3, we introduce chromatic algebras, summarise the currently known representability results for these algebras, show that every Ramsey algebra is representable, and show that algebras whose consistent cycles are 3-cycles are feebly representable, but qualitatively representable if and only if the number of diversity atoms is odd. Theorem 2.8 and the results in Section 2.3 are set to appear in Koussas, Kowalski, and Al-Juaid [**39**]. The idea to use the Walecki construction in Theorem 2.19 and the ‘only if’ portion of Theorem 2.21 are due to Tomasz Kowalski.

Chapter 3 begins with a brief summary of the research into the subvariety lattices of various varieties of relation-type algebras and tense algebras. In Section 3.2, we show that the variety generated by the algebra $\mathbf{T}_0$ has $2^{\aleph_0}$ covers in the subvariety lattice of the variety generated by the class of total tense algebras. In Section 3.3, we use this result to show that the variety generated by $\mathbf{A}_3$ has $2^{\aleph_0}$ covers in the subvariety lattices of the varieties of nonassociative and semiassociative relation algebras. Then, by constructing qualitative representations of the generators of these covers, we extend this result to the subvariety lattices of the varieties generated by the classes of all feebly representable relation algebras and qualitatively representable algebras. Additionally, we find all join-irreducible varieties above the varieties generated by $\mathbf{A}_1$ or $\mathbf{A}_2$, but not $\mathbf{A}_3$. The results on tense algebras, semiassociative relation algebras, nonassociative relation algebras have been published in Koussas and Kowalski [**58**], while the results on feeble and qualitatively representable algebras are set to appear in Hirsch, Jackson, Koussas, and Kowalski [**39**]. The graph illustrated in Figure 3.1 is due to Tomasz Kowalski and Lemma 3.42 is partially due to Tomasz Kowalski.

Chapter 4 begins with a summary of the history of probability in model theory, and a summary of the results of this nature in the study of relation-type algebras. In Section 4.2, we show that almost all finite nonassociative relation algebras are symmetric integral relation algebras, and consequently improve an existing two-variable formula that counts the number of isomorphism classes of integral relation algebras to a single-variable one. In Section 4.3, we show that the class of atom structures of nonassociative relation algebras has a 0–1 law using a Fraïssé limit construction. In Section 4.4, we discuss some possible approaches to the problem of showing that almost all nonassociative relation algebras are (some kind of) representable, and show that they cannot lead to a solution. The results from Section 4.2 and Section 4.3 are set to appear in Koussas [**57**]. The idea of proving that almost all nonassociative relation algebras are integral and symmetric was mentioned by Roger Maddux in a private communication discussing a proof that almost all of these algebras are symmetric, which was discovered independently.

The title of the conclusion is due to Andrzej Sapkowski.

A note on notation

??

We reserve 'Lemma', 'Theorem', and 'Corollary' for results where we claim (some) originality, and use 'Proposition' for known results.

We use $\subseteq$ for inclusion and $\subset$ for proper inclusion.

We usually use script symbols, such as $\mathscr{A}$, for sets of sets.

We will assume that the natural numbers $\mathbb{N}$ do not include 0. We will use ω to denote the set $\mathbb{N} \cup \{0\}$ of non-negative integers. The cardinality of $\mathbb{N}$ (or ω) will be denoted by $\aleph_0$. The powerset of a set X is defined to be the set of all subsets of X, and is denoted by $\wp(X)$. The cardinality of $\wp(\mathbb{N})$ (or $\wp(\omega)$) will be denoted by $2^{\aleph_0}$. It is well known that $\mathbb{R}$ has cardinality $2^{\aleph_0}$, hence it is fairly common to refer to $2^{\aleph_0}$ as the cardinality of the continuum.

We will usually omit superscripts from the interpretation of an operation symbol, provided that no ambiguity arises; for example, when working with a single structure.

We generally use boldface symbols for algebras and standard letters for their universe. For example, A will be used for the universe of an algebras $\mathbf{A}$.

We will usually use a, b, and c for atoms; i, j, k, and ℓ for elements of index sets; m and n for natural numbers; p, q, r, and s for integers; u, v, and w for nodes; x, y, and z for elements of algebras; and μ and ν for homomorphisms.

If $\bar{x}$ is a tuple, we denote the i^{th} coordinate of $\bar{x}$ by x_i. We start indexing with 1, so $\bar{x} = (x_1, x_2, \dots)$, for example.

To avoid confusion with lattice symbols and to match the use of $\approx$ in algebra, we will use $\curlyvee$ and $\curlywedge$ for (first-order) logical disjunction and conjunction, respectively. To avoid reusing symbols, we use $\neg$ for logical negation, $'$ for Boolean complement, and ^{c} for relative set complement. Respectively, $\neg^0$ and $\neg^1$ mean no symbol and $\neg$.

Let $\mathcal{K}$ be an arbitrary class of similar algebras. The classes of all isomorphic copies, subalgebras, homomorphic images, direct products, and ultraproducts of members of $\mathcal{K}$ will be denoted by $\mathbb{I}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$, $\mathbb{H}(\mathcal{K})$, $\mathbb{P}(\mathcal{K})$, and $\mathbb{U}(\mathcal{K})$, respectively. The variety generated by $\mathcal{K}$ will be denoted by $\mathrm{Var}(\mathcal{K})$. The class of all subdirectly irreducible members of $\mathcal{K}$ will be denoted by $\mathrm{Si}(\mathcal{K})$. For an example of this notation, a well known result of Tarski says $\mathrm{Var}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$, for any class $\mathcal{K}$ of similar algebras.

For each $n \in \mathbb{N}$, the set $\{0, \ldots, n-1\}$ of remainders modulo n is denoted by $\mathbb{Z}_n$, the usual modular addition and multiplication operations are denoted by $\oplus_n$ and $\otimes_n$, and the remainder of $p \in \mathbb{Z}$ modulo n is denoted by $R_n(p)$.

To match the notation used in Davey and Priestley [**22**], we will use $\vee$, $\wedge$, $\cdot$, $'$, and e rather than $+$, $\cdot$, $;$, ($-$ or $^-$), and $1'$ as operation symbols for relation-type algebras. For the same reason, we use $\bigvee$ and $\bigwedge$ for arbitrary joins (suprema) and meets (infima).

The set and ordered set of join-irreducible elements of a lattice $\mathbf{L}$ will be denoted by $J(\mathbf{L})$ and $\mathbf{J}(\mathbf{L})$, respectively.

The polynomial time, nondeterministic polynomial time, and complement nondeterministic polynomial time complexity classes will be denoted by $\mathtt{P}$, $\mathtt{NP}$, and co-$\mathtt{NP}$, respectively.

CHAPTER 1

Preliminaries

Our primary objects of study will be examples of Boolean algebras with operators. In this chapter, we will give a general overview of Boolean algebras with operators, introduce the specific classes of Boolean algebras with operators that we will study later, and look at some general algebraic properties of Boolean algebras with operators.

1.1. Boolean algebras with operators

Before discussing Boolean algebras with operators, we will revise Boolean algebra. See Chapter 1 of Bell and Slomson [**11**], Chapter 4 of Burris and Sankappanavar [**16**], Chapter 4 of Davey and Priestley [**22**], Chapters 1–8 of Givant and Halmos [**33**], Chapter 2 of Hirsch and Hodkinson [**36**], or Chapter 4 of Maddux [**70**] for a more comprehensive introduction to the topic.

Definition 1.1 (Boolean algebra)**.** An algebra $\mathbf{A} = \langle A; \vee, \wedge, ', 0, 1\rangle$ of the signature (or type) $(2, 2, 1, 0, 0)$ is called a *Boolean algebra* if:

(B1) $\mathbf{A} \models x \vee y \approx y \vee x$ and $\mathbf{A} \models x \wedge y \approx y \wedge x$;
(B2) $\mathbf{A} \models x \vee (y \vee z) \approx (x \vee y) \vee z$ and $\mathbf{A} \models x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$;
(B3) $\mathbf{A} \models x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ and $\mathbf{A} \models x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
(B4) $\mathbf{A} \models 0 \vee x \approx x$ and $\mathbf{A} \models 1 \wedge x \approx x$;
(B5) $\mathbf{A} \models x \vee x' \approx 1$ and $\mathbf{A} \models x \wedge x' \approx 0$.

Notation 1.2. We will assume that $'$ is applied first; $x \vee y'$ means $x \vee (y')$, for example.

The most common examples of Boolean algebras come from logic and set theory. Indeed, the study of Boolean algebra was motivated by Boolean logic.

Example 1.3 (Boolean logic)**.** Let $L := \{F, T\}$, where $F \neq T$. Define $\curlyvee \colon L^2 \to L$, $\curlywedge \colon L^2 \to L$, and $\neg \colon L \to L$ as follows.

x	y	$x \curlyvee y$
F	F	F
F	T	T
T	F	T
T	T	T

x	y	$x \curlywedge y$
F	F	F
F	T	F
T	F	F
T	T	T

x	$\neg x$
F	T
T	F

FIGURE 1.1. Operation tables for $\curlyvee$, $\curlywedge$, and $\neg$.

It is easy to check that $\mathbf{L} := \langle L; \curlyvee, \curlywedge, \neg, F, T\rangle$ is a Boolean algebra. If p is a statement, i.e., a sentence that is either true or false, let $v(p) := F$ if p is false and let $v(p) := T$ if

p is true. By construction, we have $v(p \text{ or } q) = v(p) \curlyvee v(q)$, $v(p \text{ and } q) = v(p) \curlywedge v(q)$, and $v(\text{not } p) = \neg v(p)$, for all statements p and q. Thus, $\mathbf{L}$ encodes Boolean logic.

Example 1.4 (Powerset algebra)**.** If X is a set, then the algebra $\langle \wp(X); \cup, \cap, {}^c, \varnothing, X\rangle$ is a Boolean algebra, where ^{c} is complementation relative to X.

Some basic properties of Boolean algebras are summarised in the following Lemma; a proof can be found in Chapter 4 of [**22**], for example.

Proposition 1.5. *Let* $\mathbf{A}$ *be a Boolean algebra.*

(1) *If* $x, y \in A$*, then* $x \wedge y = x$ *if and only if* $x \vee y = y$*.*
(2) *The binary relation* $\leqslant$ *on* A *defined by*

$$x \leqslant y \iff x \wedge y = x$$

is an order relation. Further, $x \vee y$ *and* $x \wedge y$ *are the least upper bound and greatest lower bound of* $\{x, y\}$*, respectively, for all* $x, y \in A$*.*

Notation 1.6. We define $\leqslant$ as above on any algebra with a Boolean algebra reduct. Note that $\leqslant$ coincides with inclusion in powerset algebras; we write $\subseteq$ rather than $\leqslant$.

Proposition 1.7. *Let* $\mathbf{A}$ *be a Boolean algebra.*

(1) *If* $x, y \in A$ *with* $x \wedge y = 0$ *and* $x \vee y = 1$*, then* $x = y'$ *and* $y = x'$*.*
(2) $\mathbf{A} \models 0' \approx 1$ *and* $\mathbf{A} \models 1' \approx 0$*.*
(3) $\mathbf{A} \models x'' \approx x$*.*
(4) $\mathbf{A} \models (x \vee y)' \approx x' \wedge y'$ *and* $\mathbf{A} \models (x \wedge y)' \approx x' \vee y'$*.*
(5) *If* $x, y \in A$*, then* $x \leqslant y$ *if and only if* $y' \leqslant x'$*.*
(6) *If* $x, y \in A$*, then* $x \leqslant y$ *if and only if* $x \wedge y' = 0$*.*

We will usually use (B1)–(B5) and the preceding pair of propositions silently.

Next, we state the definition of a Boolean algebra with operators (as presented by Jipsen in [**43**]). For more details, see Chapter 1 of Givant [**31**], Chapter 2 of [**36**], Jónsson [**53**], Jónsson and Tarski [**54**] or Chapter 4 of Maddux [**70**], for example. To motivate this abstract definition, we mention that relation algebras, tense algebras, cylindric algebras, modal algebras, closure algebras, projective algebras, polyadic algebras, and monadic algebras are all examples of Boolean algebras with operators; this observation is stated in [**53**].

Definition 1.8 (Operator)**.** Let $\mathbf{A}$ be a Boolean algebra, let $n \in \omega$, and let f be an n-ary operation on A.

(1) Let $\bar{x} \in A^n$. The operation on A given by $x \mapsto f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$ will be denoted by $f_{\bar{x},i}$. We call $f_{\bar{x},i}$ the $(\bar{x}, i)$*-translate* of f.
(2) We call f *normal* if $f_{\bar{x},i}(0) = 0$, for all $\bar{x} \in A^n$ and $1 \leqslant i \leqslant n$.
(3) We call f *additive* if $f_{\bar{x},i}(x \vee y) = f_{\bar{x},i}(x) \vee f_{\bar{x},i}(y)$, for all $x, y \in A$, $\bar{x} \in A^n$, and $1 \leqslant i \leqslant n$.

(4) We call f an *operator* (on $\mathbf{A}$) if f is normal and additive.

The term 'additive' is commonly used in the literature because many authors use $+$ and $\cdot$ rather than $\vee$ and $\wedge$, respectively; see [**54**], for example.

Notation 1.9. Let $\mathbf{A}$ be an algebra of a signature that contains every Boolean algebra operation symbol. The reduct of $\mathbf{A}$ to $\{\vee, \wedge, ', 0, 1\}$ will be denoted by $\mathbf{A}^\flat$.

Definition 1.10 (Boolean algebra with operators)**.** Let $\mathbf{A}$ be an expansion of a Boolean algebra. We call $\mathbf{A}$ a *Boolean algebra with operators* if $\mathbf{A}^\flat$ is a Boolean algebra and each non-Boolean operation of $\mathbf{A}$ is an operator on $\mathbf{A}^\flat$.

Some general (but rather boring) examples of operators are given below.

Example 1.11. Let $\mathbf{A}$ be a Boolean algebra.

(1) Every nullary operation on A is an operator.
(2) The identity map is an operator.
(3) The unary operation u on A given by

$$u(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is an operator.

Conjugates, which were defined in [**54**], are more interesting examples of operators.

Definition 1.12 (Conjugate)**.** Let $\mathbf{A}$ be a Boolean algebra and let f and g be unary operations on A. We call g a *conjugate* of f (with respect to $\mathbf{A}$) if

$$f(x) \wedge y = 0 \iff x \wedge g(y) = 0,$$

for all $x, y \in A$. We call f *selfconjugate* (with respect to $\mathbf{A}$) if f is a conjugate of f.

By Proposition 1.7(6), g is a conjugate of f with respect to a Boolean algebra $\mathbf{A}$ if and only if $({}' \circ f, {}' \circ g)$ is a Galois connection with respect to the ordered set $\langle A; \leqslant \rangle$, i.e.,

$$x \leqslant g(y)' \iff y \leqslant f(x)',$$

for all $x, y \in A$.

Some basic examples of conjugates are given below.

Example 1.13. Let $\mathbf{A}$ be a Boolean algebra.

(1) For each $x \in A$, the unary operation on A given by $y \mapsto x \wedge y$ is selfconjugate.
(2) The functions in Example 1.11(2) and Example 1.11(3) are selfconjugate.

The following result from [**54**] summarises some useful properties of conjugates. Note that the observation above on the relation between Galois connections and conjugates can be used to prove the first statement.

Proposition 1.14. *Let* $\mathbf{A}$ *be a Boolean algebra and let* f *and* g *be unary operations on* A.

(1) *If* g *is a conjugate of* f, *then* f *and* g *preserve existing joins. In particular, both* f *and* g *are operators on* $\mathbf{A}$.
(2) g *is a conjugate of* f *if and only if* $f(x \wedge g(y)') \vee (f(x) \wedge y') = f(x) \wedge y'$ *and* $g(y \wedge f(x)') \vee (f(x) \wedge y') = f(x) \wedge y'$, *for all* $x, y \in A$. *Thus, the conjugacy of a pair of operation symbols is definable by equations.*

Now we can introduce our first concrete class of Boolean algebras with operators, namely tense algebras.

Definition 1.15 (Tense algebra)**.** We call an algebra $\mathbf{A} = \langle A; \vee, \wedge, ', f, g, 0, 1\rangle$ of the signature $(2, 2, 1, 1, 1, 0, 0)$ a *tense algebra* if $\mathbf{A}^\flat$ is a Boolean algebra and g is a conjugate of f with respect to $\mathbf{A}^\flat$. The class of all tense algebras will be denoted by TA.

Based on Proposition 1.14(2), TA is an equational class, and therefore a variety.

Concrete examples of tense algebras can be constructed from (directed) graphs.

Example 1.16 (Complex algebra)**.** Let R be a binary relation on a set V, and let f_R and g_R be the image and preimage operations of R, respectively, i.e., define f_R and g_R by

$$X \mapsto \{v \in V \mid (x, v) \in R, \text{ for some } x \in X\}$$

and

$$X \mapsto \{v \in V \mid (v, x) \in R, \text{ for some } x \in X\},$$

respectively. Then the algebra $\mathbf{Cm}(\langle V; R\rangle) := \langle \wp(V); \cup, \cap, ^c, f_R, g_R, \varnothing, V\rangle$ is a tense algebra, called the *complex algebra* of the directed graph $\langle V; R\rangle$.

In Example 1.3, we saw that Boolean algebras are connected to Boolean logic. Tense algebras are also related to a form of logic, namely *tense* (or *temporal*) logic. In this setting, one views the elements of V in Example 1.16 as moments of time and thinks of $(x, y) \in R$ as meaning 'x is earlier than y'. So, $f_R(X)$ is the set of moments preceding a moment in X and $g_R(X)$ is the set of moments following a moment in X. For general references on modal logic, we direct the reader to Blackburn, de Rijke, and Venema [**14**] and Kracht [**61**].

1.2. Relation-type algebras

In this section, we aim to introduce three classes of Boolean algebras with operators, namely the classes of nonassociative relation algebras, semiassociative relation algebras, and relation algebras. These algebras will be the focus of most of remainder of this thesis, and will therefore be covered in more depth than the algebras that we defined in Section 1.1. For general references, we recommend Givant [**30**], Givant [**32**], Hirsch and Hodkinson [**36**], and Maddux [**70**].

Definition 1.17 (Relation-type algebra)**.** A *relation-type algebra* is an algebra of the signature $\{\vee, \wedge, \cdot, ', \breve{}, 0, 1, e\}$, where $\vee$, $\wedge$, and $\cdot$ are binary operation symbols, $'$ and $\breve{}$ are unary operation symbols, and 0, 1, and e are nullary operation symbols.

Notation 1.18. Throughout, we always assume that $'$ and $\breve{}$ are applied first, followed by $\cdot$. We will use multiplicative notation for $\cdot$, i.e., we will write xy rather than $x \cdot y$. For example, $x\breve{}z \wedge y'$ means $((x\breve{}) \cdot z) \wedge (y')$.

Definition 1.19 (Relation algebras)**.** A relation-type algebra $\mathbf{A}$ is called a *nonassociative relation algebra* if $\mathbf{A}^\flat$ is a Boolean algebra, e is an identity element for $\cdot$, and the *triangle laws* (or *Peircean laws*) hold, i.e., we have

$$xy \wedge z = 0 \iff x\breve{}z \wedge y = 0 \iff zy\breve{} \wedge x = 0,$$

for all $x, y, z \in A$. The class of all nonassociative relation algebras will be denoted by NA. We call $\mathbf{A} \in \mathsf{NA}$ a *semiassociative relation algebra* if $\mathbf{A}$ satisfies the *semiassociative law*, i.e., $\mathbf{A} \models (x1)1 \approx x(11)$. An algebra $\mathbf{A} \in \mathsf{NA}$ is called a *relation algebra* if $\cdot$ is associative. The classes of all semiassociative relation algebras and relations algebras will be denoted by SA and RA, respectively. We call (the value of the term) $d := e'$ the *diversity element* of $\mathbf{A}$. An algebra $\mathbf{A} \in \mathsf{NA}$ is said to be *symmetric* if $\mathbf{A} \models x\breve{} \approx x$.

It is easy to check that a relation-type algebra $\mathbf{A}$ satisfies the triangle laws if and only if $x \mapsto xy\breve{}$ is a conjugate of $x \mapsto xy$, for all $y \in A$, and $y \mapsto x\breve{}y$ is a conjugate of $y \mapsto xy$, for all $x \in A$. Further, since e is an identity for $\cdot$, $\breve{}$ is selfconjugate. By Proposition 1.14(2), the triangle laws are equivalent to equations, so NA, SA, and RA are all varieties.

As one might expect, relation algebras can be constructed from sets of (binary) relations.

Example 1.20 (Full relation algebra)**.** If D is a set, $\langle \wp(D^2); \cup, \cap, \circ, {}^c, {}^{-1}, \varnothing, D^2, \mathrm{id}_D \rangle$ is a relation algebra, where $\circ$ is (relational) composition, ${}^{-1}$ is relational converse, id_D is the identity (or diagonal) relation, i.e., we have

$$\mathrm{id}_D = \{(x, y) \in D^2 \mid x = y\},$$

$$R \circ S = \{(x, y) \in D^2 \mid (x, z) \in R \text{ and } (z, y) \in S, \text{ for some } z \in D\},$$

and

$$R^{-1} = \{(x, y) \in D^2 \mid (y, x) \in R\},$$

for all $R, S \subseteq D^2$. This algebra is referred to as the *full relation algebra* on D, and is denoted by $\mathbf{Re}(D)$. If $D = \{1, \dots, n\}$, for some $n \in \mathbb{N}$, it is common to write $\mathbf{Re}(n)$ rather than $\mathbf{Re}(D)$. The diversity element of $\mathbf{Re}(D)$ is called the *diversity relation* on D, and is denoted by di_D.

Like Boolean algebras and tense algebras, relation algebras are connected to logic. Binary relations are a useful tool for talking about relationships between objects formally. For example, the phrase 'is a parent of' defines a binary relation on the set of all people. The relational converse of the relation defined by this phrase corresponds to the phrase 'is a child of'. Similarly, the relational composition of the relation defined by 'is a parent of' relation with itself corresponds to 'is a grandparent of'. Such ideas motivated the study of the calculus of relations in the late nineteenth century, which was chiefly advanced by De Morgan, Peirce, and Schröder; see De Morgan [**23**], Peirce [**80**], and Schröder [**82**]. Full relation algebras were originally defined in Tarski's work on the calculus of relations, and the study of relation algebras grew out of Tarski's attempt to axiomatise them; see Tarski [**86**]. Nonassociative and semiassociative relation algebras were first defined by Maddux in [**72**] as a natural generalisation of relation algebras.

Concrete examples of relation algebras can also be constructed from groups.

Example 1.21 (Group relation algebra)**.** If $\mathbf{G}$ is a group, $\langle \wp(G); \cup, \cap, \cdot, ^c, ^{-1}, \varnothing, G, \{e\}\rangle$ is a relation algebra, where $\cdot$ is complex multiplication and $^{-1}$ is complex inverse, i.e., $XY = \{xy \mid x \in X, y \in Y\}$ and $X^{-1} = \{x^{-1} \mid x \in X\}$, for all $X, Y \subseteq G$.

Some basic properties of these algebras are stated below; see Chapter 2 of [**32**], Chapter 3 of [**36**], or Chapter 6 of [**70**]. We will usually use these results silently. As is customary, we will extend concepts from Boolean algebras to Boolean algebras with operators, and therefore to relation-type algebras in particular. For example, we will call a nonassociative relation algebra $\mathbf{A}$ *complete* when $\mathbf{A}^\flat$ is complete, and we will call a an *atom* of $\mathbf{A}$ when a is an atom of $\mathbf{A}^\flat$.

Proposition 1.22. *Let* $\mathbf{A} \in \mathsf{NA}$*.*

(1) *If* $x \in A$ *and* $S \subseteq A$ *are such that* $\bigvee S$ *exists, then both* $\bigvee\{xs \mid s \in S\}$ *and* $\bigvee\{sx \mid s \in S\}$ *exist and equal* $x(\bigvee S)$ *and* $(\bigvee S)x$*, respectively. In particular,* $\mathbf{A} \models x(y \vee z) \approx xy \vee xz$ *and* $\mathbf{A} \models (x \vee y)z \approx xz \vee yz$*.*
(2) *If* $S \subseteq A$ *such that* $\bigvee S$ *exists, then* $\bigvee\{s^\smallsmile \mid s \in S\}$ *exists and is equal to* $\bigvee S^\smallsmile$*. In particular,* $\mathbf{A} \models (x \vee y)^\smallsmile \approx x^\smallsmile \vee y^\smallsmile$*.*
(3) $\mathbf{A} \models 0^\smallsmile = 0$*,* $\mathbf{A} \models 1^\smallsmile = 1$*,* $\mathbf{A} \models e^\smallsmile = e$*, and* $\mathbf{A} \models d^\smallsmile = d$*.*
(4) $\mathbf{A} \models (xy)^\smallsmile \approx y^\smallsmile x^\smallsmile$*.*
(5) *If* a *is an atom, then* $a^\smallsmile$ *is an atom.*
(6) *If* $\mathbf{A} \models e \approx 0$*, then* $\mathbf{A}$ *is trivial.*
(7) *If* $1 < m \leqslant n$*, there exists* $\mathbf{B} \in \mathsf{NA}$ *with* n *atoms and exactly* m *atoms below* e*.*

We often work with algebras in which e is an atom, so it will be useful to introduce notation for them. This definition abuses language slightly; a non-trivial $\mathbf{A} \in \mathsf{NA}$ is called *integral* if $\mathbf{A} \models \forall x, y\colon xy = 0 \to x = 0 \curlyvee y = 0$, which is equivalent to e being an atom when $\mathbf{A} \in \mathsf{SA}$, but not in general. See [**70**] or Section 6.15 of Maddux [**72**] for further details.

Notation 1.23 (INA, ISA, and IRA)**.** The classes of all members of NA, SA, and RA in which e is an atom will be denoted by INA, ISA, and IRA, respectively.

By Proposition 1.22, the operations of a complete atomic (and, in particular, a finite) nonassociative relation algebra are completely determined by their values on its atoms. The table of $\cdot$ restricted to the atoms of a finite $\mathbf{A} \in \mathsf{NA}$ is often called the *atom table* of $\mathbf{A}$; since $\mathbf{A}$ has $\log_2(|A|)$ atoms, such a table is a very compact and convenient description of $\mathbf{A}$. These ideas motivate the following (and the study of duality and atom structures in general; for more details, see Chapter 1 of Givant [**31**], or Chapter 2 of [**36**].

Definition 1.24 (Atom structure)**.** Let $\mathbf{A}$ be a relation-type algebra with $\mathbf{A}^\flat$ a complete atomic Boolean algebra. We call $\mathbf{At}(\mathbf{A}) := \langle \mathrm{At}(\mathbf{A}); f_{\mathbf{A}}, I_{\mathbf{A}}, T_{\mathbf{A}} \rangle$ the *atom structure* of $\mathbf{A}$, where $\mathrm{At}(\mathbf{A})$ is the set of atoms of $\mathbf{A}$, $f_{\mathbf{A}}$ is the unary operation on $\mathrm{At}(\mathbf{A})$ defined by $x \mapsto x^{\smile}$, $I_{\mathbf{A}} := \{a \in \mathrm{At}(\mathbf{A}) \mid a \leqslant e^{\mathbf{A}}\}$, and $T_{\mathbf{A}} := \{(a, b, c) \in \mathrm{At}(\mathbf{A})^3 \mid ab \geqslant c\}$. If $e^{\mathbf{A}}$ is an atom, we also call $\mathbf{At}_e(\mathbf{A}) := \langle \mathrm{At}(\mathbf{A}); f_{\mathbf{A}}, e^{\mathbf{A}}, T_{\mathbf{A}} \rangle$ the *atom structure* of $\mathbf{A}$.

It turns out that the classes of these structures can be first-order axiomatised. Firstly, we introduce the appropriate signatures.

Definition 1.25 (Atom-type structure)**.** An *atom-type structure* is a structure of the signature $\{f, T, I\}$, where f is a unary operation symbol, T is a ternary relation symbol, and I is a unary relation symbol. An *integral atom-type structure* is a structure of the signature $\{f, e, T\}$, where f is a unary operation symbol, e is a nullary operation symbol (i.e., constant), and T is a ternary relation symbol.

Definition 1.26 (FAS, FSIAS, and FSIAS_e)**.** Let FAS denote the class of all finite atom-type structures such that:

(P) for all $a, b, c \in U$, we have $(f(a), c, b), (c, f(b), a) \in T$ whenever $(a, b, c) \in T$;
(I) for all $a, b \in U$, we have $a = b$ if and only if there is some $i \in I$ with $(a, i, b) \in T$.

An atom-type structure $\mathbf{U}$ is said to be *symmetric* and *integral* if $\mathbf{U} \models f(x) \approx x$ and $|I| = 1$, respectively. The class of all symmetric integral members of FAS will be denoted by FSIAS. Let FSIAS_e denote the class of all finite integral atom-type structures where $f(x) \approx x$ and:

(IP) for all $a, b, c \in U$, we have $(f(a), c, b), (c, f(b), a) \in T$ whenever $(a, b, c) \in T$;
(II) for all $a, b \in U$, we have $a = b$ if and only if $(a, e, b) \in T$.

The following construction generalises how a complete atomic nonassociative relation algebra is recovered from its atom structure.

Definition 1.27 (Complex algebra)**.** Let $\mathbf{U}$ be an atom-type structure. We call the relation-type algebra $\mathbf{Cm}(\mathbf{U}) := \langle \wp(U); \cup, \cap, \cdot_{\mathbf{U}}, {}^c, {}^{\smile}{}_{\mathbf{U}}, \varnothing, U, I \rangle$ the *complex algebra* of

$\mathbf{U}$, where $\cdot_{\mathbf{U}}$ and $\breve{}_{\mathbf{U}}$ are defined by

$$X \cdot_{\mathbf{U}} Y = \{z \in U \mid (x, y, z) \in T, \text{ for some } x \in X, y \in Y\}$$

and

$$X^{\smile_{\mathbf{U}}} = \{f(x) \mid x \in X\},$$

for all $X, Y \subseteq U$. In the case where $\mathbf{U}$ is an integral atom-type structure, we call $\mathbf{Cm}_e(\mathbf{U}) := \langle \wp(U); \cup, \cap, \cdot_{\mathbf{U}}, {}^c, {}^{\smile_{\mathbf{U}}}, \varnothing, U, \{e^{\mathbf{U}}\}\rangle$ the *complex algebra* of $\mathbf{U}$.

The connections between these classes is summarised below; see [**72**], for example. The third result is unsurprising, since FSIAS and FSIAS_e are essentially the same structures in different signatures.

Proposition 1.28. FAS *is precisely the class of all atom structures of finite members of* NA*,* FSIAS *is precisely the class of all atom structures of finite and symmetric members of* INA*. Every finite member of* NA *isomorphic to the complex algebra of a member of* FAS*. Every finite symmetric member of* INA *is isomorphic to the complex algebra of a member of* FSIAS*. Thus, there are bijective correspondences between the sets of isomorphism classes from:*

(1) *the class of finite members of* NA *and* FAS*;*
(2) *the class of finite and symmetric members of* INA *and* FSIAS*;*
(3) FSIAS *and* FSIAS_e*.*

Next, we introduce the important notion of a cycle from Maddux [**67**].

Definition 1.29 (Cycles)**.** Let $\mathbf{U}$ be an atom-type structure and let $a, b, c \in U$. We call (a, b, c), $(f(a), c, b)$, $(b, f(c), f(a))$, $(f(b), f(a), f(c))$, $(f(c), a, f(b))$, and $(c, f(b), a)$ the *Peircean transforms* of (a, b, c). The set of Peircean transforms of (a, b, c) is called a *cycle* and is denoted by $[a, b, c]$. We call (a, b, c) an *identity triple* if $I \cap \{a, b, c\} \neq \varnothing$ and a *diversity triple* otherwise. We call $[a, b, c]$ an *identity cycle* if it contains an identity triple and a *diversity cycle* otherwise. We call (a, b, c) *consistent* if $(a, b, c) \in T$ and *forbidden* otherwise. Lastly, we call $[a, b, c]$ *consistent* if $[a, b, c] \subseteq T$ and *forbidden* if $[a, b, c] \cap T = \varnothing$. These concepts will be extended to integral atom-type structures in the obvious way.

The following result from [**67**] illustrates the importance of cycles as a tool for axiomatising and understanding atom structures.

Proposition 1.30. (1) *Let* $\mathbf{U}$ *be an atom-type structure.*

- (a) *The following are equivalent:*
  - (i) $\mathbf{U}$ *satisfies* (P)*;*
  - (ii) *for all* $a, b, c \in U$*, the cycle* $[a, b, c]$ *is either consistent or forbidden.*
- (b) *The following are equivalent:*
  - (i) $\mathbf{U}$ *satisfies* (I)*;*

(ii) *for all* $a, b \in U$*, we have* $a = b$ *if and only if* $[a, i, b]$ *is consistent, for some* $i \in I$.

(c) *If* $\mathbf{U}$ *is integral, then the following are equivalent:*

(i) $\mathbf{U}$ *satisfies* (I)*;*

(ii) $f(e) = e$ *and* $\{[a, e, a] \mid a \in U\}$ *is the set of consistent identity cycles, where* e *is the unique element of* I.

(2) *Let* $\mathbf{U}$ *be an integral atom type structure.*

(a) *The following are equivalent:*

(i) $\mathbf{U}$ *satisfies* (IP)*;*

(ii) *for all* $a, b, c \in U$*, the cycle* $[a, b, c]$ *is either consistent or forbidden.*

(b) *The following are equivalent:*

(i) $\mathbf{U}$ *satisfies* (II)*;*

(ii) $f(e) = e$ *and* $\{[a, e, a] \mid a \in U\}$ *is the set of consistent identity cycles.*

1.3. Tools from general algebra and model theory

In this section, we aim to give a brief overview of some of the concepts and results from universal algebra and model theory that we will need later. We refer the reader to Bergman [**13**], Burris and Sankappanavar [**16**], and Werner [**88**] for more on universal algebra; and to Bell and Slomson [**11**], Chang and Keisler [**18**], and Hodges [**41**] for more on model theory. Firstly, we recall two different notions of discriminator terms.

Definition 1.31 (Discriminator terms)**.** Let $\mathbf{A}$ be an algebra. A *ternary discriminator term* for $\mathbf{A}$ is a term t whose interpretation is the ternary discriminator function on A, i.e.,

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y. \end{cases}$$

A variety $\mathcal{V}$ is called a *discriminator variety* if there is a term that is a ternary discriminator term for all subdirectly irreducible members of $\mathcal{V}$. If $\mathbf{A}$ is an expansion of a Boolean algebra, then a term u whose interpretation satisfies

$$u(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is called a *unary discriminator* term for $\mathbf{A}$.

A simple calculation gives the following result, which implies that these two notions are equivalent for Boolean algebras with operators. See Chapter 2 of [**43**], for example.

Proposition 1.32. *Let* $\mathbf{A}$ *be an expansion of a Boolean algebra.*

(1) *If* t *is a ternary discriminator term for* $\mathbf{A}$*, then* $t(0, x, 1)'$ *is a unary discriminator term for* $\mathbf{A}$.

(2) *If u is a unary discriminator term for $\mathbf{A}$, then* $(x \wedge u(x \oplus y)) \vee (z \wedge u(x \oplus y))'$ *is a ternary discriminator term for $\mathbf{A}$, where* $x \oplus y := (x \wedge y') \vee (x' \wedge y)$.

Example 1.33. The term $1x1$ is a unary discriminator for every subdirectly irreducible member of RA, so RA is a discriminator variety.

The study of discriminator terms and discriminator varieties was motivated by attempts to capture some useful properties of Boolean algebras, some of which are summarised below. We refer to Chapter 6 of [**13**], Chapter 4 of [**16**], Jónsson [**50**], Pixley [**81**], or [**88**] for further details.

Proposition 1.34. *Let $\mathcal{K}$ be a class of similar non-trivial algebras with a common discriminator term. Then*

(1) *every element of $\mathcal{K}$ is simple;*
(2) *all directly indecomposable and subdirectly irreducible elements of* $\mathrm{Var}(\mathcal{K})$ *are simple;*
(3) *the class of simple (and therefore the class of subdirectly irreducible) elements of* $\mathrm{Var}(\mathcal{K})$ *is precisely* $\mathbb{ISU}(\mathcal{K})$*;*
(4) $\mathrm{Var}(\mathcal{K})$ *is congruence permutable, congruence distributive, congruence extensile, congruence regular, congruence uniform, and semisimple;*
(5) *the subvariety lattice of* $\mathrm{Var}(\mathcal{K})$ *is distributive.*

Based on Proposition 1.34(5), the varieties of finite height in the subvariety lattice of a discriminator variety are completely determined by join-irreducible elements of finite height. The following result is a consequence of Jónsson's Theorem that characterises join-irreducible covers in the subvariety lattices of discriminator varieties; see Lemma 3.30 from Jipsen [**43**].

Proposition 1.35. *Let $\mathcal{V}$ be a discriminator variety, let $\mathcal{U}$ be a proper subvariety of $\mathcal{V}$, and let $\mathbf{A} \in \mathcal{V}$ be simple.*

(1) *If $\mathbf{A}$ is infinite, then* $\mathrm{Var}(\mathbf{A})$ *is a join-irreducible cover of $\mathcal{U}$ if and only if* $\mathbf{A} \notin \mathcal{U}$, $\mathcal{U} \subseteq \mathrm{Var}(\mathbf{A})$, *and we have* $\mathbf{A} \in \mathbb{ISU}(\mathbf{B})$, *for all* $\mathbf{B} \in \mathbb{ISU}(\mathbf{A}) \setminus \mathcal{U}$.
(2) *If $\mathbf{A}$ is finite, then* $\mathrm{Var}(\mathbf{A})$ *is a join-irreducible cover of $\mathcal{U}$ if and only if* $\mathcal{U} \subseteq \mathrm{Var}(\mathbf{A})$ *and* $\mathbb{S}(\mathbf{A}) \setminus \mathcal{U} = \{\mathbf{A}\}$.

Next, we state a pair of results on ultraproducts. Their proofs are relatively routine applications of techniques in Chapter 5 of [**11**], Chapter 4 of [**18**], or Chapter 8 of [**41**].

Proposition 1.36. *Let I be a non-empty set and let $\mathscr{U}$ be an ultrafilter over I.*

(1) *Let* $\{\mathbf{A}_i \mid i \in I\}$ *and* $\{\mathbf{B}_i \mid i \in I\}$ *be indexed sets of similar algebras, let* $\bar{a} \in \prod_{i\in I} A_i$*, let* $\bar{b} \in \prod_{i\in I} B_i$*, let* $\mu_i \colon \mathbf{A}_i \to \mathbf{B}_i$ *be an isomorphism that satisfies* $\mu_i(\bar{a}(i)) = \bar{b}(i)$*, for each* $i \in I$*, and define* $\ell \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ *by*

$\ell(a)(i) = \mu_i(a(i))$, for all $i \in I$. Then the map $\bar{x}/\mathscr{U} \mapsto \ell(\bar{x})/\mathscr{U}$ is a well-defined isomorphism from the subalgebra of $\prod_{i\in I} \mathbf{A}_i/\mathscr{U}$ generated by $\bar{a}/\mathscr{U}$ to the subalgebra of $\prod_{i\in I} \mathbf{B}_i/\mathscr{U}$ generated by $\bar{b}/\mathscr{U}$.

(2) *Let $\mathbf{A}$ be an algebra and let $a \in A$. For each $x \in A$, define $\bar{x}\colon I \to A$ by $i \mapsto x$. Then, for each $x \in A$, the subalgebra of $\mathbf{A}$ generated by a is isomorphic to the subalgebra of $\mathbf{A}^I/\mathscr{U}$ generated by $\bar{a}/\mathscr{U}$ via the canonical map $x \mapsto \bar{x}/\mathscr{U}$.*

To conclude this section, we will summarise the presentation of Fraïssé limits from Chapter 6 of [**41**]. The construction of these structures generalises the construction of the ordered set of rational numbers from the class of finite chains and the random graph (also known as the Rado graph or Erdős-Rényi graph) from the class of finite graphs.

Definition 1.37 (Age)**.** Let $\mathbf{A}$ be a structure. The *age of* $\mathbf{A}$ is the class of all finitely generated structures that embed into $\mathbf{A}$.

Definition 1.38 (HP, JEP, and AP)**.** Let $\mathcal{K}$ be a class of similar structures. We say that $\mathcal{K}$ has the *hereditary property* (HP) if $\mathcal{K}$ is closed under forming finitely generated structures. We say that $\mathcal{K}$ has the *joint embedding property* (JEP) if, for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is some $\mathbf{C} \in K$ that both $\mathbf{A}$ and $\mathbf{B}$ embed into. We say that $\mathcal{K}$ has the *amalgamation property* (AP) if, for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\mu\colon \mathbf{A} \to \mathbf{B}$ and $\nu\colon \mathbf{A} \to \mathbf{C}$, there is some $\mathbf{D} \in \mathcal{K}$ and embeddings $\mu'\colon \mathbf{B} \to \mathbf{D}$ and $\nu'\colon \mathbf{C} \to \mathbf{D}$ such that $\mu' \circ \mu = \nu' \circ \nu$.

Definition 1.39 (Homogeneity)**.** Let $\mathbf{A}$ be a structure. We call $\mathbf{A}$ *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{A}$. We call $\mathbf{A}$ *weakly homogeneous* if, for all finitely generated structures $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}$ with $\mathbf{B} \leqslant \mathbf{C}$ and all embeddings $\mu\colon \mathbf{B} \to \mathbf{A}$, there is an embedding $\nu\colon \mathbf{C} \to \mathbf{A}$ extending μ.

Proposition 1.40. *A finite or countable structure is ultrahomogeneous if and only if it is weakly homogeneous.*

Proposition 1.41 (Fraïssé's Theorem)**.** *Let S be a countable signature and let $\mathcal{K}$ be a class of at most countable S-structures, that has the HP, JEP, and AP. Then there is an S-structure $\mathbf{F}$ (called a* Fraïssé limit of $\mathcal{K}$*), unique up to isomorphism, such that (the universe of) $\mathbf{F}$ is at most countable, $\mathcal{K}$ is the age of $\mathbf{F}$, and $\mathbf{F}$ is ultrahomogeneous.*

To state the last result of this section, we need to recall some definitions.

Definition 1.42 (Uniform local finiteness)**.** Let $\mathcal{K}$ be a class of similar structures. We say that $\mathcal{K}$ is *uniformly locally finite* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$, such that, for all $\mathbf{A} \in \mathcal{K}$, each $n \in \mathbb{N}$, and every subset S of A with $|S| \leqslant n$, the substructure of $\mathbf{A}$ generated by S has cardinality at most $f(n)$.

Definition 1.43 (κ-categorical)**.** Let κ be an infinite cardinal and let T be a first-order theory. We say that T is *κ-categorical* if T has a unique model of cardinality κ up to isomorphism.

The following result is Theorem 6.4.1 of [**41**].

Proposition 1.44. *Let S be a finite signature, let $\mathcal{K}$ be a uniformly locally finite class of S-structures with the HP, JEP, and AP, and at most countably many isomorphism types of finitely generated S-structures, and let $\mathbf{F}$ be a Fraïssé limit of $\mathcal{K}$. Then the first-order theory of $\mathbf{F}$ is $\aleph_0$-categorical and has quantifier elimination.*

CHAPTER 2

Representability

In this chapter, we will revise the classical notion of representability for relation algebras, and then look at two notions of representability that appeared in the literature more recently, namely feeble and qualitative representability, which will be central in the following chapters. We will also cover networks and some of their applications.

2.1. Background

Before we can give the classical definition of a representation, we will need to generalise the structures from Example 1.20.

Definition 2.1 (Proper relation algebra)**.** Let E be an equivalence relation on a set D. We call $\mathbf{Re}(E) := \langle \wp(E); \cup, \cap, \circ, {}^c, {}^{-1}, \varnothing, E, \mathrm{id}_D \rangle$ the *proper relation algebra on E*.

To help distinguish between the different notions of representability we will be using, we use the term strong representability for the classical notion of representability.

Definition 2.2 (Strong representation)**.** Let $\mathbf{A} \in \mathsf{RA}$. We call $\mathbf{A}$ *strongly representable* if there exists an embedding $\phi\colon \mathbf{A} \to \mathbf{Re}(E)$, for some equivalence relation E over a set D. An embedding of this form will be called a *strong representation* of $\mathbf{A}$. In the case where $E = D^2$, a representation is called a *square representation* of $\mathbf{A}$. The classes of all strongly representable and square representable members of RA will be denoted by RRA and SqRRA, respectively.

Some basic properties of strongly representable relation algebras are summarised in the proceeding proposition. For further details, see Chapter 3 of Hirsch and Hodkinson [**36**], Hirsch and Hodkinson [**38**], Jónsson and Tarski [**55**], Lyndon [**65**], Monk [**74**], and Tarski [**84**], for example.

Proposition 2.3.
(1) *If E is an equivalence relation, then $\mathbf{Re}(E) \in \mathsf{RA}$.*
(2) RRA *is a variety, and hence an equational class. In fact,* RRA *is canonical.*
(3) *Every simple member of* RRA *embeds into $\mathbf{Re}(D)$, for some set D.*
(4) $1(x1)$ *is a unary discriminator term for* RRA*.*
(5) RRA *is a proper subvariety of* RA*. Further,* RRA *is not finitely first-order axiomatisable relative to* RA*, i.e., there is no finite set F of first-order sentences with $\mathsf{RRA} = \{\mathbf{A} \in \mathsf{RA} \mid \mathbf{A} \models F\}$, or equivalently, a first-order sentence σ with $\mathsf{RRA} = \{\mathbf{A} \in \mathsf{RA} \mid \mathbf{A} \models \sigma\}$.*
(6) RRA *has no finite first-order axiomatisation.*

(7) *The problem of deciding whether a finite* $\mathbf{A} \in \mathsf{RA}$ *is in* RRA *is undecidable.*
(8) *The problem of deciding whether an equation holds over* RRA *is undecidable.*

As we saw in Section 1.2, the study of algebras of relations arose fairly naturally from the study of the calculus of relations, so the study of strongly representable relation algebras is understandably one of the central focuses of research into relation-type algebras. However, strong representability has limitations in some contexts, particularly in applied ones. For example, relation-type algebras often appear in the study of constraint satisfaction problems (CSPs), where strong representability is too restrictive to model some problems. Loosely speaking, strong representations require certain properties to hold everywhere they can possibly hold, when it is sometimes more natural to require that they only hold somewhere, or that only allowable properties hold. (Respectively, these conditions correspond to qualitative and feeble representability.) To an algebraist, strong representability could be seen as too restrictive, as it only applies to relation algebras. For general introductions to CSPs and applications of relation-type algebras, see Allen [**6**], Lecoutre [**62**], Maddux [**69**], and Tsang [**87**].

Weaker notions of representability have appeared in articles since the early nineteen fifties. For example, Jónsson and Tarski defined what was called weak representability in [**55**]. Here a weak representation is an embedding of a relation algebra into an algebra of binary relations where every operation other than meet and complement is the usual set theoretic one. In [**55**], Jónsson and Tarski show that weak representations exist for all relation algebras. This was intended to be a first step to showing that all relation algebras are strongly representable, which was later shown to be impossible by Lyndon, who constructed a nonrepresentable relation algebra in [**65**]. We will focus on feeble and qualitative representability, which were defined by Hirsch, Jackson, and Kowalski in [**40**]. Firstly, we will need to introduce what they call a herd.

Definition 2.4 (Herd)**.** Let D be a set. We call a subset $\mathscr{H}$ of $\wp(D^2)$ a *herd* (over D) if $\mathscr{H}$ is a subuniverse of $\langle \wp(D^2); \cup, \cap, {}^c, {}^{-1}, \varnothing, D^2, \mathrm{id}_U \rangle$. We call D the *base* of $\mathscr{H}$.

Definition 2.5 (Feeble and qualitative representations)**.** Let $\mathbf{A} \in \mathsf{NA}$, let $\mathscr{H}$ be a herd, and let $\phi\colon A \to \mathscr{H}$ be an injective map that preserves $\vee$, $\wedge$, $'$, $\breve{}$, 0, 1, and e.

(F) ϕ is called a *feeble representation* (of $\mathbf{A}$ in $\mathscr{H}$) if we have $\phi(x) \mid \phi(y) \subseteq \phi(xy)$, for all $x, y \in A$.
(Q) ϕ is called a *qualitative representation* (of $\mathbf{A}$ in $\mathscr{H}$) if, for all $x, y, z \in A$, we have that
$$\phi(x) \mid \phi(y) \subseteq \phi(z) \iff xy \leqslant z.$$

If $\mathbf{A}$ has a feeble or qualitative representation, we call $\mathbf{A}$ *feebly* or *qualitatively representable*, respectively. The classes of all feebly and qualitatively representable members of NA will be denoted by FRA and QRA, respectively. Define $\mathsf{GFRA} := \mathrm{Var}(\mathsf{FRA})$ and $\mathsf{GQRA} := \mathrm{Var}(\mathsf{QRA})$. (This notation follows Hirsch, Jackson, Koussas, and Kowalski [**39**]. The 'G' refers to 'generated', as FRA and QRA are not varieties.)

As in [**40**], we restrict these definitions to the equivalents of square representations, i.e., strong representations that map an algebra into the full relation algebra on a set. In [**86**], Tarski only considers square representations. In the literature, it is common to define RRA as the variety generated by the class of all square representable relation algebras. Thus, our definitions are in line with conventional definitions if we view FRA and QRA as equivalents of square representable algebras, and GFRA and GQRA as equivalents of RRA. Further, almost all of the algebras we consider in this thesis are simple, which implies that membership in (FRA and GFRA) or (QRA and GQRA) are equivalent for these algebras.

The following gives an equivalent definition of qualitative representability.

Proposition 2.6. *Let $\mathbf{A} \in \mathsf{NA}$, let $\mathscr{H}$ be a herd over a base D, and let $\phi\colon A \to \mathscr{H}$ be an injective map preserving $\vee$, $\wedge$, $'$, $\breve{}$, 0, 1, and e. Then* (1) *and* (2) *are equivalent, and the three statements are equivalent when $\mathbf{A}$ is complete and atomic:*

(1) *ϕ is a feeble representation of $\mathbf{A}$ in $\mathscr{H}$;*
(2) *for all $x, y, z \in A$, we have $xy \wedge z \neq 0$ if there are $u, v, w \in D$ with $(u, v) \in \phi(x)$, $(v, w) \in \phi(y)$, and $(u, w) \in \phi(z)$;*
(3) *for all $a, b, c \in \mathrm{At}(\mathbf{A})$, we have $ab \wedge c \neq 0$ if there are $u, v, w \in D$ with $(u, v) \in \phi(a)$, $(v, w) \in \phi(b)$, and $(u, w) \in \phi(c)$.*

Similarly, (1) *and* (2) *below are equivalent, and the three statements are equivalent when $\mathbf{A}$ is complete and atomic:*

(4) *ϕ is a qualilitive representation of $\mathbf{A}$ in $\mathscr{H}$;*
(5) *for all $x, y, z \in A$, we have $xy \wedge z \neq 0$ if and only if there are $u, v, w \in D$ with $(u, v) \in \phi(x)$, $(v, w) \in \phi(y)$, and $(u, w) \in \phi(z)$;*
(6) *for all $a, b, c \in \mathrm{At}(\mathbf{A})$, we have $ab \wedge c \neq 0$ if and only if there are $u, v, w \in D$ with $(u, v) \in \phi(a)$, $(v, w) \in \phi(b)$, and $(u, w) \in \phi(c)$.*

The following summarises the main properties of GFRA and GQRA from [**39**] and [**40**].

Proposition 2.7. (1) $\mathsf{RRA} \subset \mathsf{GQRA} \subset \mathsf{GFRA}$.
(2) $1(x1)$ *is a unary discriminator term for* GFRA *and* GQRA.
(3) *Every non-trivial element of* FRA *(and therefore* QRA*) is simple.*
(4) *Every simple element of* GFRA *and* GQRA *is in* FRA *and* QRA*, respectively.*
(5) *Every class $\mathcal{K}$ with $\mathsf{RRA} \subseteq \mathcal{K} \subseteq \mathsf{GFRA}$ has no finite first-order axiomatisation. In particular,* GFRA *and* GQRA *have no finite first-order axiomatisations.*
(6) *If $\mathbf{A} \in \mathsf{FRA}$ or $\mathbf{A} \in \mathsf{QRA}$, then $\mathbf{A}$ has a feeble representation on a base with at most $2|\log_2(A)|$ or a qualitative representation on a base with at most $3|\log_2(A)|^3$ points, respectively. So, the problems of deciding whether a finite $\mathbf{A} \in \mathsf{NA}$ is a member of FRA or QRA are both in* $\mathtt{NP}$*, and hence decidable. Further, both problems are* $\mathtt{NP}$*-complete (using the number of atoms as the input size, not the size of the algebra).*

(7) *The problems of deciding whether a finite* $\mathbf{A} \in \mathsf{NA}$ *is a member of* GFRA *or* GQRA *are both* $\mathtt{NP}$*-complete (using the number of atoms as the input size, not the size of the algebra). In particular, both problems are decidable.*
(8) GQRA *and* RRA *are not finitely first-order axiomatisable relative to* GFRA*.*
(9) *The problem of deciding whether an equation holds over* QRA *is* co-$\mathtt{NP}$*-complete.*

We can use a brief decidability argument to obtain a similar result to Proposition 2.7(7). This result is due to the author and is set to appear in [**39**].

Theorem 2.8. RRA *is not finitely first-order axiomatisable relative to* GQRA*.*

Proof. For a contradiction, say RRA is finitely first-order axiomatisable relative to GQRA. Then there is a first-order sentence σ such that $\mathsf{RRA} = \{\mathbf{A} \in \mathsf{GQRA} \mid \mathbf{A} \models \sigma\}$. By Proposition 2.7(7), we can decide whether a finite $\mathbf{A} \in \mathsf{RA}$ is a member of RRA: firstly, check whether we have $\mathbf{A} \in \mathsf{GQRA}$; then check whether we have $\mathbf{A} \models \sigma$; then accept if and only if both conditions hold. This contradicts Proposition 2.3(7), so we are done. □

2.2. Networks

A strong representation of an atomic algebra can be viewed as a labelled directed graph; take the base as the set of vertices and label edges by the atom whose image contains it. For square representations, this graph is complete. By looking at labelled complete graphs, one can study feeble, qualitative, and strong representations from a different perspective, and look at various ways of approximating or constructing these representations. Further, these graphs also relate to network problems that arise in applications. For further details, we refer to Chapter 7 of Hirsch and Hodkinson [**36**] and Hirsch, Jackson, and Kowalski [**40**]; we will mostly follow the conventions in [**40**].

Definition 2.9 (Network)**.** Let $\mathbf{A} \in \mathsf{NA}$, let N be a set, and let $\lambda\colon N^2 \to A$. We call $\langle N; \lambda\rangle$ a *network* (over $\mathbf{A}$). We call $\langle N; \lambda\rangle$ *atomic* if $\lambda[N^2] \subseteq \mathrm{At}(\mathbf{A})$. We call $\langle N; \lambda\rangle$ *consistent* if:

(N1) $\lambda(u,u) \leqslant e$, for all $u \in N$;
(N2) $\lambda(u,v)\lambda(v,w) \wedge \lambda(u,w) \neq 0$, for all $u,v,w \in N$.

The elements of N and values of λ are usually called *nodes* and *labels*, respectively.

Some basic properties of networks from [**40**] are summarised below.

Proposition 2.10. *Let* $\mathbf{A} \in \mathsf{NA}$ *and let* $\langle N; \lambda\rangle$ *be a network (over* $\mathbf{A}$*).*

(1) *If* $\langle N; \lambda\rangle$ *is consistent, then* $\lambda(u,v) \wedge \lambda(v,u)^{\smile} \neq 0$*, for all* $u,v \in N$*.*
(2) *If* $\langle N; \lambda\rangle$ *is consistent, then* $\lambda(u,v) \neq 0$*, for all* $u,v \in N$*.*
(3) *If* $\langle N; \lambda\rangle$ *is consistent and atomic, then* $\lambda(v,u) = \lambda(u,v)^{\smile}$*, for all* $u,v \in N$*.*
(4) *If* $\langle N; \lambda\rangle$ *is atomic, then* $\langle N; \lambda\rangle$ *satisfies* (N2) *if and only if* $(\lambda(u,v), \lambda(v,w), \lambda(u,w))$ *is consistent in* $\mathbf{A}$*, for all* $u,v,w \in N$*.*

Networks can be used to define a sequence of subclasses of NA that approximate RRA in the sense that they are strictly decreasing (in the inclusion order) and intersect to RRA. These classes were originally defined by Maddux in [**66**] using matrices rather than networks, but these formulations are essentially equivalent; see [**36**].

Definition 2.11 (Bases)**.** Let $\mathbf{A} \in \mathsf{NA}$ and let $n \geqslant 2$. A *n-dimensional relational basis* for $\mathbf{A}$ is a set $\mathcal{R}$ of consistent atomic networks over $\mathbf{A}$ with underlying set $\{1, \dots, n\}$ such that:

(RB1) for all $a \in \mathrm{At}(\mathbf{A})$, there exists $\langle \{1, \dots, n\}; \lambda \rangle \in \mathcal{R}$ with $\lambda(1, 2) = a$;

(RB2) for all $\langle \{1, \dots, n\}; \lambda \rangle \in \mathcal{R}$, $u, v, w \in \{1, \dots, n\}$ such that $w \neq u$ and $w \neq v$, and $a, b \in \mathrm{At}(\mathbf{A})$ with $\lambda(u, v) \leqslant ab$, there exists $\langle \{1, \dots, n\}; \lambda' \rangle \in \mathcal{R}$ with $\lambda'(u, w) = a$, $\lambda'(w, v) = b$, and $\lambda'(u', v') = \lambda(u', v')$, for all $u', v' \in \{1, \dots, n\}$ with $(u', v') \notin \{(u, w), (v, w), (w, u), (w, v)\}$.

Definition 2.12 (RA_n)**.** For each $n \geqslant 2$, let RA_n denote the class of all relation-type algebras that embed into an atomic member of NA with an n-dimensional relational basis.

A Hasse diagram of inclusions for the classes we have defined is shown below.

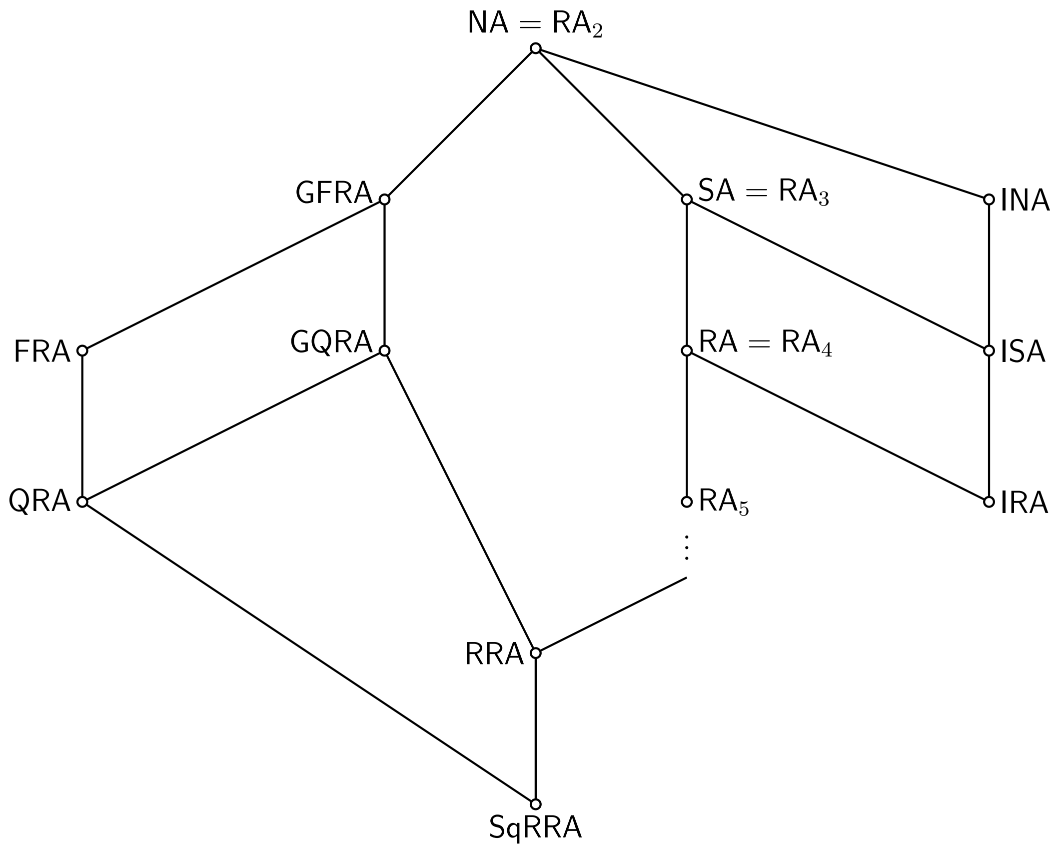

Figure 2.1. A Hasse diagram of classes.

Like the majority of the classes of Boolean algebras with operators we have discussed, these classes are connected to logic. The class RA_n is the class of models of the equational forms of statements in the calculus of relations that can be proved using at most n variables; for further details, see Maddux [**66**].

Some basic properties of these classes are summarised below; we refer to [**36**], Hirsch and Hodkinson [**37**], [**66**], Maddux [**67**], and Maddux [**70**] for further details. Here canonical means closed under canonical extensions; see Chapter 2 of [**36**].

Proposition 2.13. (1) $\mathsf{RA}_2 = \mathsf{NA}$, $\mathsf{RA}_3 = \mathsf{SA}$, $\mathsf{RA}_4 = \mathsf{RA}$, *and* $\bigcap_{n=1}^{\infty} \mathsf{RA}_n = \mathsf{RRA}$.
(2) RA_n *is a canonical variety, for all* $n \geqslant 3$.
(3) *If* $m > n \geqslant 2$, *then* $\mathsf{RA}_n \subset \mathsf{RA}_m$. *Further, if* $m > n \geqslant 4$, *then* RA_m *is not finitely first-order axiomatisable relative to* RA_n.
(4) *If* $n \geqslant 2$ *and* $\mathbf{A} \in \mathsf{NA}$ *is finite, then* $\mathbf{A} \in \mathsf{RA}_n$ *if and only if* $\mathbf{A}$ *has a* n*-dimensional relational basis.*
(5) *Any finite set of equations that hold in all members of* RRA *is included in the equational theory of* RA_n, *for some* $n \geqslant 2$.
(6) *For each* $n \geqslant 2$, *the problem of deciding whether a finite* $\mathbf{A} \in \mathsf{NA}$ *is in* RA_n *is in* P. *In particular, these problems are decidable.*

The connections between networks and feeble and qualitative representability are summarised in the following pair of results based on results from [**40**].

Proposition 2.14. *Let* $\mathbf{A} \in \mathsf{NA}$ *be atomic. If there is a consistent atomic network* $\langle N; \lambda \rangle$ *such that, for each atom* a, *there exist* $u, v \in N$ *with* $\lambda(u, v) = a$, *then* $\mathbf{A} \in \mathsf{FRA}$. *Further, the converse holds if* $\mathbf{A}$ *is finite.*

Proposition 2.15. *Let* $\mathbf{A} \in \mathsf{NA}$ *be atomic. If there is a consistent atomic network* $\langle N; \lambda \rangle$ *such that, for all consistent triples* (a, b, c) *of* $\mathbf{A}$, *there exist* $u, v, w \in N$ *with* $\lambda(u, v) = a$, $\lambda(v, w) = b$, *and* $\lambda(u, w) = c$, *then* $\mathbf{A} \in \mathsf{QRA}$. *Further, the converse holds if* $\mathbf{A}$ *is finite.*

For contrast, we state the equivalent result for strong representations; see Alm, Maddux, and Manske [**4**].

Proposition 2.16. *Let* $\mathbf{A} \in \mathsf{NA}$ *be atomic. If there is a consistent atomic network* $\langle N; \lambda \rangle$ *such that: for all* $u \in N$ *and* $a \in A$, *there exists some* $v \in N$ *satisfying* $\lambda(u, v) = a$; *and, for all consistent triples* (a, b, c) *of* $\mathbf{A}$ *and* $u, v \in N$ *with* $\lambda(u, v) = c$, *there exists* $w \in N$ *that satisfy* $\lambda(u, w) = a$ *and* $\lambda(w, v) = b$, *then* $\mathbf{A}$ *has a square representation. Further, the converse holds if* $\mathbf{A}$ *is finite.*

The three preceding results effectively justify the claims about the requirements representations have on networks in Section 2.1; all triangles that appear must be consistent, feeble representations require every atom to appear, qualitative representations require every consistent triangle to appear, and strong representations require every consistent triangle to appear everywhere it can.

2.3. Representability of chromatic algebras

To conclude this chapter, we will study the algebras $\mathfrak{E}_n(X)$ defined in Maddux [**68**]. The results in this section will appear in Al Juaid, Jackson, Koussas, and Kowalski [**1**]. The idea to use the Walecki construction in Theorem 2.19, Lemma 2.20, and the 'only if' portion of Theorem 2.21 are due to Tomasz Kowalski. Unless stated otherwise, the remaining results are due to the author.

Following the conventions in [**1**], we will call these algebras chromatic algebras. As well as being interesting in their own right, studying the representability of these algebras is quite illustrative and highlights the difference between the three notions of representability we will be focusing on. We begin by defining these algebras formally.

Definition 2.17 (Chromatic algebras)**.** For every $n \in \mathbb{N}$ and $X \subseteq \{1, 2, 3\}$, we define $A_n := \{e, a_1, \ldots, a_{n-1}\}$, $f_n = \mathrm{id}_{A_n}$, and

$$T_{X,n} := \Big(\bigcup\{[a, e, a] \mid a \in A_n\}\Big) \cup \Big(\bigcup\{[a, b, c] \mid |\{a, b, c\}| \in X \text{ and } a, b, c \in A_n \backslash \{e\}\}\Big).$$

We call $\mathfrak{E}_n(X) := \mathbf{Cm}(\langle A_n; f_n, T_{X,n}, \{e\}\rangle)$ the *chromatic algebra* (on X and n).

We will drop set brackets whenever possible. For example, we will write $\mathfrak{E}_n(1, 3)$ rather than $\mathfrak{E}_n(\{1, 3\})$, and a_i rather than $\{a_i\}$.

Two important classes of relation-type algebras can be defined as chromatic algebras. Chromatic algebras of the form $\mathfrak{E}_n(1, 3)$ are often called *Lyndon algebras* in the literature. These algebras were first studied by Jónsson in [**49**]. Later, in [**64**], Lyndon showed that, $\mathfrak{E}_n(1, 3)$ is strongly representable if and only if there is a projective plane of order $n - 2$, for all $n \geqslant 5$. Using the Bruck-Ryser-Chowla Theorem on the nonexistence of projective planes, in [**74**], Monk constructs a strongly representable ultraproduct of nonrepresentable Lyndon algebras; Monk's original proof of Proposition 2.3(5) used this construction. In [**52**], Jónsson used these algebras to show that there is no equational axiomatisation of RRA that uses only finitely many variables. Since so many important results in the theory of relation algebras use these algebras, it should not be surprising that the problem of determining whether they are strongly representable has attracted some interest; Maddux mentions it in [**68**], for example. Both the feeble and qualitative representability of Lyndon algebras is studied in [**1**]. We have that $\mathfrak{E}_1(1, 3), \mathfrak{E}_2(1, 3) \in \mathsf{RRA}$, $\mathfrak{E}_3(1, 3) \notin \mathsf{FRA}$, and $\mathfrak{E}_4(1, 3) \in \mathsf{FRA} \setminus \mathsf{QRA}$. When $n \geqslant 5$, we always have $\mathfrak{E}_n(1, 3) \in \mathsf{FRA}$, and the membership of $\mathfrak{E}_n(1, 3)$ in QRA is equivalent to the existence of a certain type of geometry; we refer to [**1**] for details.

Another important class is the class of all chromatic algebras of the form $\mathfrak{E}_n(2, 3)$, which are often called *Ramsey algebras*, due to their connection to Ramsey theory. By Ramsey's Theorem (see Chapter 5 of Jukna [**56**]), a complete graph (edge) coloured by a finite set of colours always has a triangle that has a monochromatic triangle, provided its vertex set is larger than a constant that depends on the number of colours; by Lemma 2.16, representable Ramsey algebras must be representable on a finite set, as every triangle in a network is consistent and all one-label triangles are forbidden. Other common names for these algebras include *Monk algebras* and *Maddux algebras*; see Hirsch and Hodkinson [**36**], for example. Despite being less popular than Lyndon algebras, the representability problem for these algebras has attracted some interest. In [**19**], Comer defines these algebras and constructs strong representations of $\mathfrak{E}_n(2,3)$, for all $n \leqslant 6$. In [**59**], Kowalski gives strong representations for all $7 \leqslant n \leqslant 119$, except 9 and 14, and in [**5**], Alm and Manske give strong representations for all $7 \leqslant n \leqslant 399$, except 9 and 14; their approaches were developed concurrently and independently. In Alm [**3**], the latter upper bound was improved to 1999. Using Walecki's cycle decomposition (as in Alspach [**7**]), we will solve the qualitative (and hence feeble) representability problem by defining networks that give representations of $\mathfrak{E}_n(2,3)$, for all $n \in \mathbb{N}$. The edges that are coloured by the first colour are illustrated below. Subsequent colours are obtained by rotating the previous pattern one step clockwise.

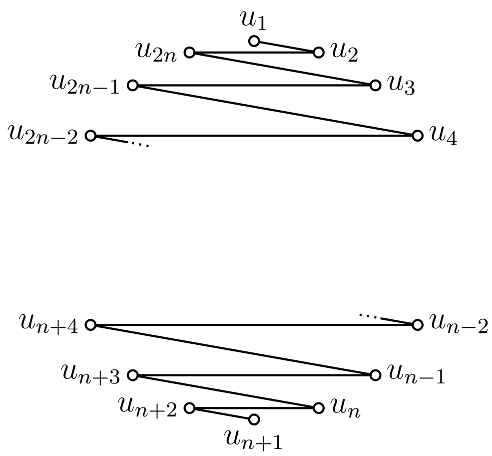

FIGURE 2.2. The edges coloured by the first colour.

Example 2.18. Some small concrete examples of these colourings are shown below.

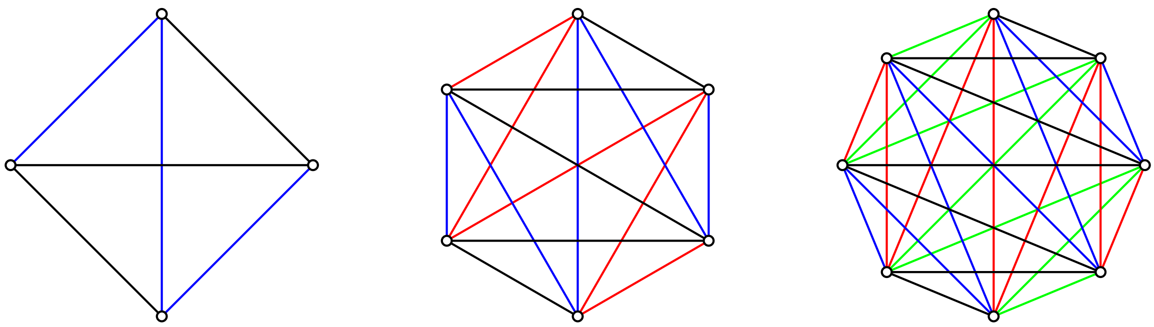

FIGURE 2.3. Three small examples of colourings.

Theorem 2.19. $\mathfrak{E}_n(2,3) \in \mathsf{QRA}$, *for all* $n \in \mathbb{N}$.

PROOF. It is clear that $\mathfrak{E}_1(2,3)$ is strongly and therefore qualitatively representable. For notational convenience, we will construct qualitative representations of $\mathfrak{E}_{n+1}(2,3)$, for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and, for each $m \in \mathbb{N}$, define $s_m \colon \mathbb{Z} \to \mathbb{Z}$ by $x \mapsto R_m(x-1)+1$. Using some basic modular arithmetic, it is easy to check that

$$s_n(s_{2n}(p) + s_{2n}(q)) = s_n(p+q),$$

for all $p, q \in \mathbb{Z}$. We will use this observation silently to omit s_{2n} within brackets. Let $N := \{u_1, \dots, u_{2n}\}$ with $u_i \neq u_j$ if $i \neq j$, and define $\lambda \colon A_n^2 \to N$ using Figure 2.3. More formally, we first define $\lambda(u_i, u_i) = e$, for all $1 \leqslant i \leqslant 2n$. Next, by construction, the index of $\lambda(u_1, u_{1+s})$ is $\lceil (s+1)/2 \rceil$, for all $1 \leqslant s \leqslant 2n-1$; u_{2n} moves to u_1 for a_2, hence we have $\lambda(u_1, u_2) = a_2 = \lambda(u_1, u_3)$, and similarly for the remaining values of s. So, based on the clockwise rotation and periodicity involved in the construction above, the index of $\lambda(u_i, u_{i+s})$ is given by

$$s_n\left(\left\lceil \frac{s+1}{2} \right\rceil + i - 1\right),$$

for all $1 \leqslant i \leqslant 2n$ and $s \in \mathbb{Z} \setminus 2n\mathbb{Z}$; the edges from u_2 have the same pattern as u_1, but shifted by 1, and similarly for the remaining values. This formally describes λ. By construction, the triangles that appear will never have the same label on all edges, since the edges of a given label form a Hamiltonian path. So, based on Lemma 2.15, it remains to show that $\langle N; \lambda \rangle$ has all two-label or three-label triangles.

Let $1 \leqslant i, j, k \leqslant n$ with $i < j$. Firstly, assume that $k \neq i$. Define $\ell := s_{2n}(i+j-k)$, $s := 2k - 2j + 1$, and $t := 2k - 2i$. Observe that $s, t \in \mathbb{Z} \setminus 2n\mathbb{Z}$. Based on the above, the index of $\lambda(u_\ell, u_{s_{2n}(\ell+s)})$ is

$$\begin{aligned} s_n\left(\left\lceil \frac{(2k-2j+1)+1}{2} \right\rceil + (i+j-k) - 1\right) &= s_n(k - j + 1 + i + j - k - 1) \\ &= i. \end{aligned}$$

Similarly, the index of $\lambda(u_\ell, u_{s_{2n}(\ell+t)})$ is

$$\begin{aligned} s_n\left(\left\lceil \frac{(2k-2i)+1}{2} \right\rceil + (i+j-k) - 1\right) &= s_n(k - i + 1 + i + j - k - 1) \\ &= j. \end{aligned}$$

Since $i < j$, we have $s < t$. Now, we have

$$\begin{aligned} t - s &= (2k - 2i) - (2k - 2j + 1) \\ &= 2(j-i) - 1, \end{aligned}$$

which is in $\mathbb{Z} \setminus 2n\mathbb{Z}$, hence the index of $\lambda(u_{s_{2n}(\ell+s)}, u_{s_{2n}(\ell+t)})$ is

$$\begin{aligned} & s_n\left(\left\lceil \frac{(2(j-i)-1)+1}{2} \right\rceil + ((2k-2j+1) + (i+j-k)) - 1\right) \\ &= s_n(j - i + 2k - 2j + 1 + i + j - k - 1) \\ &= k. \end{aligned}$$

Thus, u_ℓ, $u_{s_{2n}(\ell+s)}$, and $u_{s_{2n}(\ell+t)}$ form a triangle with edges coloured by a_i, a_j, and a_k.

Next, assume that $k \neq j$. Define $\ell := s_{2n}(i+j-k)$, $s := 2k-2j$, and $t := 2k-2i+1$. Similarly to the above, the index of $\lambda(u_\ell, u_{s_{2n}(\ell+s)})$ and $\lambda(u_\ell, u_{s_{2n}(\ell+s)})$ are given by

$$\begin{aligned} s_n\left(\left\lceil \frac{(2k-2j)+1}{2} \right\rceil + (i+j-k) - 1\right) &= s_n(k-j+1+i+j-k-1) \\ &= i \end{aligned}$$

and

$$\begin{aligned} s_n\left(\left\lceil \frac{(2k-2i+1)+1}{2} \right\rceil + (i+j-k) - 1\right) &= s_n(k-i+1+i+j-k-1) \\ &= j. \end{aligned}$$

Similarly to the previous case, we have

$$\begin{aligned} t-s &= (2k-2i+1)-(2k-2j) \\ &= 2(j-1)+1, \end{aligned}$$

so the index of $\lambda(u_{s_{2n}(\ell+s)}, u_{s_{2n}(\ell+t)})$ is

$$\begin{aligned} &s_n\left(\left\lceil \frac{(2(j-i)+1)+1}{2} \right\rceil + ((2k-2j)+(i+j-k)) - 1\right) \\ &= s_n(j-i+1+2k-2j+i+j-k-1) \\ &= k. \end{aligned}$$

Thus, u_ℓ, $u_{s_{2n}(\ell+s)}$, and $u_{s_{2n}(\ell+t)}$ form a triangle with edges coloured by a_i, a_j, and a_k.

Combining the observations from the two cases above, by taking ($k = i$ and $k \neq j$), ($k = j$ and $k \neq i$), or $k \neq i, j$, we can construct every two-label or three-label triangle, which is what we wanted to show. $\square$

While the remaining cases are not as well-studied as Lyndon or Maddux algebras, their feeble, qualitative, and strong representation problems are completely solved. By Lemma 2.14 and Lemma 2.16, we have $\mathfrak{E}_1(\varnothing) \in \mathsf{RRA}$ and $\mathfrak{E}_n(\varnothing) \notin \mathsf{FRA}$ if $n > 1$, as there is no suitable network. Similarly, $\mathfrak{E}_1(1), \mathfrak{E}_2(1) \in \mathsf{RRA}$ and $\mathfrak{E}_n(1) \notin \mathsf{FRA}$, for all $n > 2$. Using Lemma 2.16, it is easy to check that $\mathfrak{E}_1(2), \mathfrak{E}_2(2), \mathfrak{E}_3(2) \in \mathsf{RRA}$. In [**1**], it is shown that $\mathfrak{E}_n(2) \in \mathsf{FRA} \setminus \mathsf{QRA}$, for all $n > 3$. We have $\mathfrak{E}_n(1,2) \in \mathsf{RRA}$, for all $n \in \mathbb{N}$; this is stated in [**68**] and follows from Theorem 422 from Maddux [**70**]. It is stated in [**68**] that $\mathfrak{E}_n(1,2)$ is only strongly representable on a finite set if $n < 4$. In [**47**], Jipsen, Maddux, and Tuza show that $\mathfrak{E}_n(1,2,3) \in \mathsf{RRA}$, for all $n \in \mathbb{N}$. Further, the representations constructed in [**47**] are on base sets with finitely many points. Using networks as above, it is easy to verify that $\mathfrak{E}_1(3) \in \mathsf{RRA}$ and $\mathfrak{E}_2(3), \mathfrak{E}_3(3) \notin \mathsf{FRA}$. It is stated in [**68**] that $\mathfrak{E}_4(3)$ is strongly representable on a 4-element set, and that $\mathfrak{E}_n(3) \notin \mathsf{RA}$ when $n > 4$, which implies that $\mathfrak{E}_n(3) \notin \mathsf{RRA}$ when $n > 4$. We will see that the qualitative representability of $\mathfrak{E}_n(3)$, for $n > 4$, is equivalent to the existence of a commutative idempotent quasigroup. Using this observation, we show that $\mathfrak{E}_n(3) \in \mathsf{FRA}$, for all $n > 4$; see Corollary 2.23. Recall that a quasigroup is an algebra with a single binary operation (also known as a binar or magma) with both left and right cancellation properties, and that an idempotent quasigroup is one that is a model of the equation $x^2 \approx x$. We will make use of the following existence result for these quasigroups; $\oplus_n$, $\otimes_n$, and R_n are defined in the note on notation.

Lemma 2.20. *If $n \in \mathbb{N}$ is odd, then 2 has a multiplicative inverse in $\mathbb{Z}_n$ and $\langle \mathbb{Z}_n; \cdot_n \rangle$ is a commutative idempotent quasigroup, where $\cdot_n$ is given by $(p, q) \mapsto (p \oplus_n q) \otimes_n 2^{-1}$.*

Proof. Clearly, $\lceil n/2 \rceil$ is the multiplicative inverse of 2 in $\mathbb{Z}_n$. For all $p, q, r \in \mathbb{Z}_n$, we have

$$\begin{aligned} p \cdot_n q = p \cdot_n r &\implies (p \oplus_n q) \otimes_n 2^{-1} = (p \oplus_n r) \otimes_n 2^{-1} \\ &\implies p \oplus_n q = p \oplus_n r \\ &\implies q = r, \end{aligned}$$

hence $\langle \mathbb{Z}_n; \cdot_n \rangle$ has right cancellation. By symmetry, $\langle \mathbb{Z}_n; \cdot_n \rangle$ also has left cancellation. The commutativity of $\langle \mathbb{Z}_n; \cdot_n \rangle$ follows immediately from the commutativity of $\oplus_n$. Lastly, we have

$$\begin{aligned} p \cdot_n p &= (p \oplus_n p) \otimes_n 2^{-1} \\ &= R_n(R_n(2p) \otimes_n 2^{-1}) \\ &= p, \end{aligned}$$

so $\langle \mathbb{Z}_n; \cdot_n \rangle$ is idempotent. Combining these results, we find that $\langle \mathbb{Z}_n; \cdot_n \rangle$ is a commutative idempotent quasigroup, as required. □

Theorem 2.21. *If $n > 3$, then $\mathfrak{E}_{n+1}(3) \in \mathsf{QRA}$ if and only if n is odd.*

Proof. Firstly, for the reverse implication, assume that n is odd. By Lemma 2.20(1), $\langle \mathbb{Z}_n; \cdot_n \rangle$ is a commutative idempotent quasigroup. Now, define $N := \{u_0, u_1, \dots, u_n\}$, where $u_i \neq u_j$, for all distinct $0 \leqslant i, j \leqslant n$, and define $\lambda\colon N^2 \to A_n$ by

$$\lambda(u, v) = \begin{cases} e & \text{if } u = v \\ a_i & \text{if } (u, v) \in \{(u_i, u_n), (u_n, u_i)\} \\ a_{i \cdot j} & \text{if } (u, v) = (u_i, u_j) \text{ and } i, j \in \mathbb{Z}_n \text{ are distinct;} \end{cases}$$

since $\langle \mathbb{Z}_n; \cdot_n \rangle$ is commutative, λ is a well-defined map. If $u, v, w \in N$ with $u = v$, $u = w$, or $v = w$, then $(\lambda(u, v), \lambda(v, w), \lambda(u, w))$ is clearly a consistent identity triple. If $i, j \in \mathbb{Z}_n$ are distinct, then $i \cdot_n j \neq i, j$, since $\langle \mathbb{Z}_n; \cdot_n \rangle$ is an idempotent quasigroup. From this, it follows that u_i, u_j, and u_n form a triangle with three different labels, hence the corresponding triples are consistent. Let $i, j, k \in \mathbb{Z}_n$ be mutually distinct. As $\langle \mathbb{Z}_n; \cdot_n \rangle$ is a commutative quasigroup, $i \cdot_n j$, $j \cdot_n k$, and $i \cdot_n k$ are mutually distinct, so $(\lambda(u_i, u_j), \lambda(u_j, u_k), \lambda(u_i, u_k))$ is consistent. Thus, $\langle N; \lambda \rangle$ is a consistent network. Clearly, all identity triples are witnessed on $\{u_n\}$ or subsets of the form $\{u_i, u_n\}$. Now, let (a_i, a_j, a_k) be a consistent diversity triple, i.e., let i, j, and k be mutually distinct. Define $p := R_n(i - j + k)$, $q := R_n(i + j - k)$, and $r := R_n(-i + j + k)$. Then

$$\begin{aligned} p \cdot_n q &= (p \oplus_n q) \otimes_n 2^{-1} \\ &= R_n(i - j - k + i + j - k) \otimes_n 2^{-1} \\ &= R_n(R_n(2i) \otimes_n 2^{-1}) = i, \end{aligned}$$

$$\begin{aligned} q \cdot_n r &= (q \oplus_n r) \otimes_n 2^{-1} \\ &= R_n(i + j - k - i + j + k) \otimes_n 2^{-1} \\ &= R_n(R_n(2j) \otimes_n 2^{-1}) = k, \end{aligned}$$

and

$$\begin{aligned} p \cdot_n r &= (q \oplus_n r) \otimes_n 2^{-1} \\ &= R_n(i - j - k - i + j + k) \otimes_n 2^{-1} \\ &= R_n(R_n(2k) \otimes_n 2^{-1}) = j, \end{aligned}$$

so $(\lambda(u_p, u_q), \lambda(u_q, u_r), \lambda(u_p, u_r)) = (a_i, a_j, a_k)$, hence $\langle N; \lambda\rangle$ witnesses (a_i, a_j, a_k). So, by Lemma 2.15, we have $\mathfrak{E}_{n+1}(3) \in \mathsf{QRA}$.

Now, for forward implication, assume that $\mathfrak{E}_{n+1}(3) \in \mathsf{QRA}$. Based on Lemma 2.15, there is a consistent atomic network, say $\langle N; \lambda\rangle$, that witnesses every consistent triple. Let $u \in N$. Since every consistent diversity triple has mutually distinct coordinates, we have that

$$v \neq w \implies \lambda(u, v) \neq \lambda(u, w),$$

for all $v, w \in N \setminus \{u\}$. From this, it follows that $|N \setminus \{u\}| \leqslant n$, so we have $|N| \leqslant n+1$. As $\langle N; \lambda\rangle$ has $\binom{|N|}{3}$ triangles and all $\binom{n}{3}$ diversity cycles appear, we need $\binom{n}{3} \leqslant \binom{|N|}{3}$, so $n \leqslant |N|$. Thus, $|N| = n$ or $|N| = n + 1$.

First, assume that $|N| = n + 1$. Based on the above, for all $1 \leqslant i \leqslant n$ and $u \in N$, there is a unique $v \in N \setminus \{u\}$ with $\lambda(u, v) = a_i$, since

$$v \neq w \implies \lambda(u, v) \neq \lambda(u, w),$$

for all $v, w \in N \setminus \{u\}$. Consistent diversity triples have mutually distinct coordinates, so, for all $u, v, u', v' \in N$, we have that

$$\lambda(u, v) = \lambda(u', v') \implies \{u, v\} = \{u', v'\} \text{ or } \{u, v\} \cap \{u', v'\} = \varnothing.$$

So, for all $1 \leqslant i \leqslant n$, there is an even number of elements u of N such that $\lambda(u, v) = a_i$, for some $v \in N$. Combining these observations, we find that $|N| = n + 1$ is even, hence n is odd.

Next, assume that $|N| = n$. Then $\langle N; \lambda\rangle$ has $\binom{n}{3}$ triangles and must witness $\binom{n}{3}$ consistent cycles, so every cycle is witnessed on exactly one triangle in $\langle N; \lambda\rangle$. Thus, all n atoms are the label of the same number of edges. There are $\binom{n}{2} = n(n-1)/2$

edges, so each atom labels $(n-1)/2$ edges. This implies that $n-1$ must be even, i.e., that n must be odd. So, n is odd in every case and the two statements are equivalent, which is what we wanted to show. $\square$

The following example illustrates how to construct such a representation.

Example 2.22. We aim to construct a network for $\mathfrak{E}_6(3)$. In $\mathbb{Z}_5$, we have $2^{-1} = 3$. Using this, we find the Cayley table for $\langle \mathbb{Z}_5; \cdot_5 \rangle$, as defined in the proof of Lemma 2.21.

$\cdot$	0	1	2	3	4
0	0	3	1	4	2
1	3	1	4	2	0
2	1	4	2	0	3
3	4	2	0	3	1
4	2	0	3	1	4

FIGURE 2.4. The Cayley table for $\langle \mathbb{Z}_5; \cdot_5 \rangle$.

Using black, blue, red, green, and yellow to denote 0, 1, 2, 3, and 4, respectively, with u_5 in the centre and u_0, u_1, u_2, u_3, and u_4 being listed clockwise from the top, we get the following colouring from the proof of Theorem 2.20.

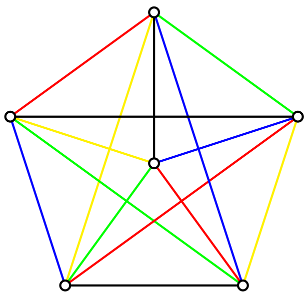

FIGURE 2.5. A colouring for $\mathfrak{E}_6(3)$.

Using Theorem 2.21, we can verify the claims we made earlier about the feeble and qualitative representability of $\mathfrak{E}_n(3)$, for $n > 4$.

Corollary 2.23. *Let $n > 4$.*

(1) *We have $\mathfrak{E}_n(3) \in \mathsf{QRA}$ if and only if n is even.*
(2) *We have $\mathfrak{E}_n(3) \in \mathsf{FRA}$.*

PROOF. The first statement follows immediately from Theorem 2.21. By Proposition 2.7(1), the second result follows from (1) when n is even. Assume that n is odd, Then $n-1 > 3$ is even. By (1) and Lemma 2.15, there is a consistent atomic network, say $\langle N; \lambda \rangle$, that witnessing all consistent triples of $\mathfrak{E}_{n-1}(3)$. Let $u, v \in N$ be distinct, and define $\lambda' \colon N^2 \to A_n$ by

$$\lambda'(u', v') := \begin{cases} \lambda(u', v') & \text{if } \{u', v'\} \neq \{u, v\} \\ a_n & \text{if } \{u', v'\} = \{u, v\}. \end{cases}$$

By construction, $\langle N, \lambda' \rangle$ is a consistent network that witnesses every atom in A_n, hence by Lemma 2.14, we have $\mathfrak{E}_n(3) \in \mathsf{FRA}$, so (2) holds. $\square$

The results from this section are summarised below.

	RRA	QRA \ RRA	FRA \ QRA	NA \ FRA
$\mathfrak{E}_n(\varnothing)$	$n = 1$			$n > 1$
$\mathfrak{E}_n(1)$	$n = 1, 2$			$n > 2$
$\mathfrak{E}_n(2)$	$n = 1, 2, 3$		$n > 3$	
$\mathfrak{E}_n(3)$	$n = 1, 4$	Even $n > 4$	Odd $n > 4$	$n = 2, 3$
$\mathfrak{E}_n(1, 2)$	$n \in \mathbb{N}$			
$\mathfrak{E}_n(1, 3)$	$n = 1, 2$		$n = 4$	$n = 3$
$\mathfrak{E}_n(2, 3)$	$n \leqslant 1999$			
$\mathfrak{E}_n(1, 2, 3)$	$n \in \mathbb{N}$			

	RRA	QRA	FRA	NA \ FRA
$\mathfrak{E}_n(\varnothing)$	$n = 1$	$n = 1$	$n = 1$	$n > 1$
$\mathfrak{E}_n(1)$	$n = 1, 2$	$n = 1, 2$	$n = 1, 2$	$n > 2$
$\mathfrak{E}_n(2)$	$n = 1, 2, 3$	$n = 1, 2, 3$	$n > 3$	
$\mathfrak{E}_n(3)$	$n = 1, 4$	$n = 1, 4$ and even $n > 4$	$n \neq 2, 3$	$n = 2, 3$
$\mathfrak{E}_n(1, 2)$	$n \in \mathbb{N}$	$n \in \mathbb{N}$	$n \in \mathbb{N}$	
$\mathfrak{E}_n(1, 3)$	$n = 1, 2$	$n = 1, 2$	$n \neq 3$	$n = 3$
$\mathfrak{E}_n(2, 3)$	$n \leqslant 1999$	$n \in \mathbb{N}$	$n \in \mathbb{N}$	
$\mathfrak{E}_n(1, 2, 3)$	$n \in \mathbb{N}$	$n \in \mathbb{N}$	$n \in \mathbb{N}$	

FIGURE 2.6. A summary of the known representability results on chromatic algebras.

2.4. Open problems

In Section 2.3, we completed the study of the feeble and qualitative representability of $\mathfrak{E}_n(X)$, except for the case where $X = \{1, 3\}$. This suggests the following problem.

Problem 1. Determine the values of n for which $\mathfrak{E}_n(1, 3) \in \mathsf{GQRA}$.

The strong representability status of some chromatic algebras are still unknown.

Problem 2. Determine the values of n for which $\mathfrak{E}_n(X) \in \mathsf{RRA}$, for $X = \{1, 3\}, \{2, 3\}$.

Recall that the qualitative representability of a Lyndon algebra is equivalent to the existence of a certain geometry, while its strong representability is equivalent to the existence of a certain projective plane. Thus, both Problem 1 and (part of) Problem 2 are equivalent to problems in finite geometry; we refer to Al Juaid, Jackson, Koussas, and Kowalski [**1**] and Lyndon [**65**] for further details.

Earlier, in Section 2.3, we saw that all chromatic algebras of the form $\mathfrak{E}_n(1, 2, 3)$ that are strongly representable must be strongly representable over finite base sets. More broadly, determining whether or not all strongly representable integral relation algebras that have an atom a that satisfies $a \leqslant bc$, for all diversity atoms b and c, must have strong representations over a finite base set is also an interesting problem. Such an atom is called a *flexible atom*. The following problem is often called the *flexible atom conjecture*, and is quite well known; see Maddux [**68**], Alm, Maddux, and Manske [**4**], and Chapter 21 of Hirsch and Hodkinson [**36**], for example.

Problem 3. Determine whether or not every relation algebra with a flexible atom has a strong representation over a finite base set.

CHAPTER 3

Subvariety lattices

The subvarieties of a variety, i.e., the subclasses of a variety that are themselves varieties, form a lattice when ordered by inclusion. (Since varieties are proper classes rather than sets, this is not strictly true. However, there is a correspondence between varieties and equational theories, so treating the collection of subvarieties of a variety as a set causes no harm.) Almost all problems in algebra are centred around classes of algebras or sets of equations, so information about subvariety lattices can be useful. Thus, it should be unsurprising that the the subvariety lattices of many well-known varieties have been studied extensively. For example, the subvariety lattices of the varieties of groups, monoids, semigroups, and lattices are covered in Neumann [**78**], Shevrin, Vernikov, and Volkov [**83**], Gusev, Lee, and Vernikov [**35**], and Rose and Jipsen [**48**], respectively. Boolean algebras with operators are no exception; much of the research into these algebras relates to subvariety lattices. In this chapter, we aim to summarise some of this work and present some new results on subvariety lattices.

3.1. Background

The subvariety lattice of a given variety always has a bottom element and a top element, namely the variety consisting of all trivial algebras and the given variety itself, respectively. Thus, one could attempt to describe a subvariety lattice using a 'bottom up' approach, i.e., by finding all of its atoms, then finding all of the covers of its atoms, and so on. Dually, one could take a 'top down' approach by finding all coatoms, all of their lower covers, and so on. Due to the correspondence between equational theories and (fully invariant) congruences on free algebras, the subvariety lattice of a congruence distributive variety is always distributive, so the problem of finding all covers or lower covers of a given subvariety reduces to the problem of finding all join-irreducible covers or meet-irreducible lower covers, respectively. In particular, this always applies to varieties of Boolean algebras with operators.

Most of the early attempts to study the subvariety lattice of the variety of relation algebras followed one of these two approaches. The earliest result on this subvariety lattice, published by Tarski in [**85**], implies that this lattice has exactly three atoms. These atoms are generated by $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$, which can be characterised as the minimal subalgebras of the full relation algebra on a one-element set, a two-element set, and a three-element set, respectively. From results published by Jónsson and Tarski in [**55**], it follows that $\mathrm{Var}(\mathbf{A}_1)$ does not have any join-irreducible covers.

In [**51**], Jónsson showed that $\mathrm{Var}(\mathbf{A}_2)$ has a unique join-irreducible cover that has no join-irreducible covers, and constructed a countably infinite set of coatoms below RA. The join-irreducible covers of $\mathrm{Var}(\mathbf{A}_3)$ have not been completely classified. Thus far, 21 of these covers have been discovered, of which 20 are generated by finite algebras Most were found and listed by Jipsen in [**43**]; the infinite generator is due to Jónsson. The problem of classifying these covers is posed by Jónsson and Maddux in [**68**], by Hirsch and Hodkinson in Chapter 21 of [**36**], and by Givant in Chapter 18 of [**30**].

Another approach to studying the structure of a subvariety lattice is to study its sublattices, typically by constructing sublattices that are isomorphic to well known lattices. For example, in [**9**], Andréka, Givant, and Németi show that the subvariety lattice of the variety of (representable) relation algebras has a subalgebra isomorphic to the powerset lattice of a countably infinite set, and hence a sublattice isomorphic to the chain of the real numbers. Later, in [**8**], the same authors constructed an embedding from the powerset lattice of the set of natural numbers into the subvariety lattice of the variety of relation algebras with the property that a set is recursive if and only if the equational theory of its image is decidable. Some other approaches to studying subvariety lattices include studying splittings and looking at the centre of the lattice; Jónsson takes both of these approaches in [**51**].

As one might expect, the subvariety lattices of other varieties of relation-type algebras have also attracted some interest. As well as being interesting in their own right, results on these lattices can also provide insight into the classification problem mentioned above. For example, in [**43**], Jipsen solves the corresponding problem for what he calls neat symmetric relation algebras, so the remaining covers in the relation algebra case are generated by algebras that are not both neat and symmetric. Results on varieties containing the variety of relation algebras are usually not directly useful for this classification problem, but can involve useful ideas. In [**10**], Andréka, Jónsson, and Németi observe that the subvariety lattice of the variety of semiassociative relation algebras has the same atoms as the subvariety lattice of the variety of relation algebras, and that $\mathrm{Var}(\mathbf{A}_1)$, $\mathrm{Var}(\mathbf{A}_2)$, and the cover of $\mathrm{Var}(\mathbf{A}_2)$ have the same join-irreducible covers as in the relation algebra case. In [**45**], Jipsen, Kramer, and Maddux show that $\mathrm{Var}(\mathbf{A}_3)$ has a countably infinite set of join-irreducible covers in the subvariety lattice of the variety of a certain variety of semiassociative relation algebras.

The subvariety lattices of other varieties of Boolean algebras with operators have also been investigated, particularly the subvariety lattices of varieties related to modal logics, due to their correspondence with lattices of normal extensions. For example, Kowalski shows in [**60**] that the variety of tense algebras has $2^{\aleph_0}$ atoms (which was already known to Blok). One of these atoms, namely the variety generated by $\mathbf{T}_0$, which can be characterised as the complex algebra of a reflexive point, was shown to have a countable set of covers by Jipsen, Kramer, and Maddux in [**45**]. For further results, we mention Makinson [**73**], Blok [**15**], Kowalski [**60**], and Kracht [**61**].

3.2. Varieties of tense algebras

In this section, we will show that $\mathrm{Var}(\mathbf{T}_0)$ has $2^{\aleph_0}$ covers in two subvariety lattices. The majority of this content was published previously in Koussas and Kowalski [**58**]. Definition 3.5 is due to Tomasz Kowalski, Proposition 3.3 is due to Peter Jipsen, and the remainder of the results in this section are the sole work of the author.

Our construction is based on an infinite set of frames (or graphs) defined by Jipsen, Kramer, and Maddux in [**45**], which were based on the veiled recession frame defined by Blok in [**15**]. We begin by recalling the definition of a total tense algebra from [**45**].

Definition 3.1 (Total tense algebra)**.** We will call $\mathbf{A} \in \mathsf{TA}$ *total* if $f(x) \vee g(x) = 1$, for all $x \in A$ with $x \neq 0$. The class of all total tense algebras will be denoted by TTA.

The class of total tense algebras is not a variety, but it generates a finitely based variety; see [**44**] and [**45**] for further details.

Notation 3.2. The subvariety lattices of TA and $\mathrm{Var}(\mathsf{TTA})$ will be denoted by $\mathbf{\Lambda}_{\mathsf{TA}}$ and $\mathbf{\Lambda}_{\mathsf{TTA}}$, respectively. Let $\mathbf{T}_0 := \mathbf{Cm}(\langle \{0\}; \{(0,0)\}\rangle)$ and denote $\mathrm{Var}(\mathbf{T}_0)$ by T_0.

Recall that a binary relation R on a set U is said to be *total* if we have $(x,y) \in R$ or $(y,x) \in R$, for all $x, y \in U$.

Proposition 3.3.
(1) *If R is a total binary relation on a set U, then we have $\mathbf{Cm}(\langle U; R\rangle) \in \mathsf{TTA}$.*
(2) *If $\mathbf{A} \in \mathsf{TTA}$, then $f(x) \vee g(x)$ is a unary discriminator term for $\mathbf{A}$.*

Before we can present our main construction, we will need to introduce some notation.

Notation 3.4. Let $\mathbb{E} := \{2n \mid n \in \mathbb{N}\}$ and let $\mathbb{O} := \{2n+1 \mid n \in \mathbb{N}\}$. For all $S \subseteq \mathbb{O}$, define $S_{\mathbb{E}} := S \cup \mathbb{E}$.

Definition 3.5. Let $V := \{a_{p,m} \mid p \in \mathbb{Z}, m \in \mathbb{N}\}$, where $a_{p,m} \neq a_{q,n}$ if $p \neq q$ or $m \neq n$. For each $S \subseteq \mathbb{O}$, define $\mathbf{F}_S := \mathbf{Cm}(\langle V; R_S\rangle)$, where R_S is defined by

$$\begin{aligned} R_S := {} & \{(a_{p,m}, a_{q,n}) \mid p > q \text{ or } (p = q \text{ and } m \geqslant n)\} \\ & \cup \{(a_{p,1}, a_{p+1,m}) \mid p \in \mathbb{Z}, m \in S_{\mathbb{E}}\} \\ & \cup \{(a_{p,m}, a_{p,m+1}) \mid p \in \mathbb{Z}, m \in \mathbb{N}\}. \end{aligned}$$

As is usually the case with graphs, drawings are more digestible than written definitions. Thus, we will usually refer to Figure 3.1 (below) rather than referring to Definition 3.5 explicitly.

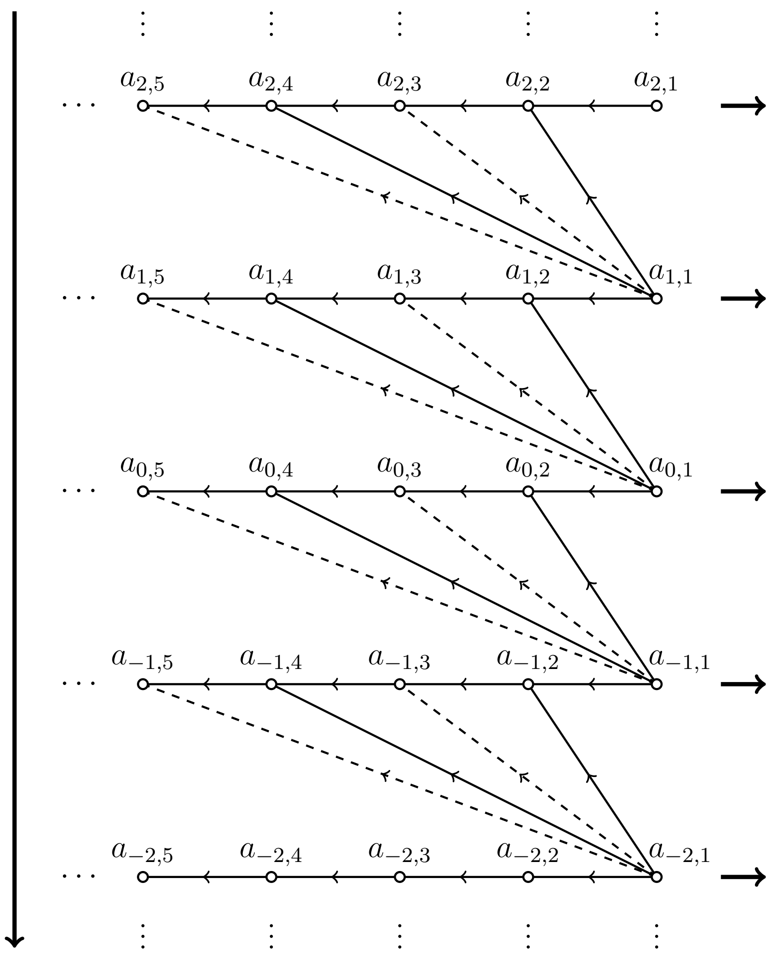

FIGURE 3.1. A (directed) graph drawing of $\langle V; R_S \rangle$.

This graph drawing uses some conventions from [**45**] that warrant some explanation. For the sake of cleanliness, we omit loops in the drawing, as there are loops at every vertex. As usual, $a_{p,m}$ pointing at $a_{q,n}$ means that $(a_{p,m}, a_{q,n})$ is in the relation. Non-bold and non-dashed edges indicate pointing. For example, $(a_{0,1}, a_{1,2})$ is included. The bold vertical arrow indicates that a vertex points at all of the vertices below it. For example, the vertex $a_{1,1}$ points at $a_{0,1}$, $a_{0,2}$, and $a_{-1,1}$. The bold horizontal arrow indicates that a vertex points at every vertex to its right. So, $a_{0,4}$ points at $a_{0,3}$, $a_{0,2}$, and $a_{0,1}$, for example. The dashed arrows indicate edges whose presence depends on the choice of the subset S. For example, in the cases where we have $3 \in S$ and $5 \notin S$, $a_{0,1}$ will point at $a_{1,3}$, but $a_{0,1}$ will not point at $a_{1,5}$. Similarly, $a_{1,1}$ will point at $a_{2,3}$, but $a_{1,1}$ will not point at $a_{2,5}$.

For a second example, we will find every vertex pointed at by $a_{0,1}$. When $m \in \mathbb{N}$ is even, $a_{0,1}$ always points at $a_{1,m}$. If $m \in \mathbb{O}$, then $a_{0,1}$ points at $a_{1,p}$ if $p \in S$. If $p < 0$ and $m \in \mathbb{N}$, then $a_{0,1}$ always points at $a_{p,m}$. Finally, $a_{0,1}$ always points at both $a_{0,1}$ and $a_{0,2}$.

The following is the only proof that uses Definition 3.5 directly rather than referring to Figure 3.1.

Lemma 3.6. *Let $S \subseteq \mathbb{O}$. Then R_S is total.*

Proof. Let $p, q \in \mathbb{Z}$ and let $m, n \in \mathbb{N}$. Since $m \geqslant m$, it follows that $(a_{p,m}, a_{p,m}) \in R_S$, so R_S is reflexive. Assume that $a_{p,m} \neq a_{q,n}$. If $p = q$, then $m \neq n$, so $m > n$ or $m < n$, hence $(a_{p,m}, a_{q,n}) \in R_S$ or $(a_{q,n}, a_{p,m}) \in R_S$. Similarly, if $p \neq q$, we have $p > q$ or $p < q$, which tells us that $(a_{p,m}, a_{q,n}) \in R_S$ or $(a_{q,n}, a_{p,m}) \in R_S$. Combining these results, we find that R is total, which is what we wanted. $\square$

Combining Lemma 3.6 with Proposition 1.34 and Proposition 3.3, we get the following result.

Corollary 3.7. *Let $S \subseteq \mathbb{O}$. Then we have $\mathbb{IS}(\mathbf{F}_S) \subseteq \mathsf{TTA}$ and every element of $\mathbb{IS}(\mathbf{F}_S)$ is simple.*

As in [**45**], we will work with subalgebras generated by atoms, not full complex algebras. These subalgebras will be more difficult to describe explicitly than the subalgebras from [**45**]; it will be convenient to define these algebras in terms of some relatively basic elements rather than defining them as subalgebras generated by atoms.

Definition 3.8. For all $p \in \mathbb{Z}$, $m \in \mathbb{N}$, and $S \subseteq \mathbb{O}$, we define

$$\begin{aligned} V_p &:= \{a_{p,n} \mid n \in \mathbb{N}\}, \\ A_{p,m} &:= \{a_{p,m}\}, \\ D_p &:= \{a_{q,n} \mid q \leqslant p, n \in \mathbb{N}\}, \\ U_p &:= \{a_{q,n} \mid q \geqslant p, n \in \mathbb{N}\}, \\ S_{p,m} &:= \{a_{p,n} \mid n \in S_{\mathbb{E}}, n \geqslant m\}, \\ \bar{S}_{p,m} &:= \{a_{p,n} \mid n \notin S_{\mathbb{E}}, n > 1, n \geqslant m\}. \end{aligned}$$

Now, for each $S \subseteq \mathbb{O}$, let

$$\mathscr{S}_S := \{A_{p,m}, S_{p,m}, \bar{S}_{p,m} \mid p \in \mathbb{Z}, m \in \mathbb{N}\} \cup \{D_p, U_p \mid p \in \mathbb{Z}\}$$

and let $\mathscr{B}_S$ be the set of finite unions of elements of $\mathscr{S}_S$.

The following result describes the action of the operators on all of the sets defined above. To avoid double subscripts, we will write f_S and g_S rather than f_{R_S} and g_{R_S}, respectively.

Lemma 3.9. *Let $S \subseteq \mathbb{O}$, let $p \in \mathbb{Z}$, let $m \in \mathbb{N}$, and define $T := \{n \in \mathbb{N} \setminus S_{\mathbb{E}} \mid n \geqslant m\}$. Then*

(1) $f_S(A_{p,1}) = A_{p,1} \cup A_{p,2} \cup D_{p-1} \cup S_{p+1,1}$,
(2) $f_S(A_{p,m}) = \bigcup\{A_{p,n} \mid n \leqslant m+1\} \cup D_{p-1}$ *when* $m > 1$,
(3) $f_S(D_p) = D_p \cup S_{p+1,1}$,
(4) $f_S(U_p) = V$,
(5) $f_S(S_{p,m}) = D_p$,

(6) $f_S(\bar{S}_{p,m}) = \varnothing$ *if* $T = \varnothing$,
(7) $f_S(\bar{S}_{p,m}) = \bigcup\{A_{p,n} \mid n \leqslant \max(T)+1\} \cup D_{p-1}$ *when* $T \neq \varnothing$ *and* T *is finite,*
(8) $f_S(\bar{S}_{p,m}) = D_p$ *if* T *is infinite,*
(9) $g_S(A_{p,1}) = U_p$,
(10) $g_S(A_{p,2}) = A_{p-1,1} \cup U_p$,
(11) $g_S(A_{p,m}) = U_{p+1} \cup S_{p,m-1} \cup \bar{S}_{p,m-1}$ *when* $m > 1$ *and* $m \notin S_{\mathbb{E}}$,
(12) $g_S(A_{p,m}) = A_{p-1,1} \cup U_{p+1} \cup S_{p,m-1} \cup \bar{S}_{p,m-1}$ *when* $m > 2$ *and* $m \in S_{\mathbb{E}}$,
(13) $g_S(D_p) = V$,
(14) $g_S(U_p) = A_{p-1,1} \cup U_p$,
(15) $g_S(S_{p,m}) = A_{p-1,1} \cup U_p$,
(16) $g_S(\bar{S}_{p,m}) = \varnothing$ *when* $T = \varnothing$,
(17) $g_S(\bar{S}_{p,m}) = S_{p,\min(T)-1} \cup \bar{S}_{p,\min(T)-1} \cup U_{p+1}$ *when* $T \neq \varnothing$.

Proof. Let $X \subseteq V$. By definition, $f_S(X)$ is the set of vertices that are pointed at by X, while $g_S(X)$ is the set of all vertices that point at X. From this observation and Figure 3.1, the required results follow immediately. □

Next we will check that the subsets we defined are in fact subuniverses.

Lemma 3.10. *Let* $S \subseteq \mathbb{O}$. *Then* $\mathscr{B}_S$ *is the subuniverse of* $\mathbf{F}_S$ *generated by* $\mathscr{S}_S$.

Proof. It is clear that $\mathscr{S}_S \subseteq \mathscr{B}_S$ and that all subuniverses of $\mathbf{F}_S$ extending $\mathscr{S}_S$ extend $\mathscr{B}_S$, so it remains to show that $\mathscr{B}_S$ is a subuniverse of $\mathbf{F}_S$.

It is clear that $\mathscr{B}_S$ is closed under (binary) union and $\varnothing \in \mathscr{B}_S$.

By distributivity, to show that $\mathscr{B}_S$ is closed under (binary) intersection, we only need to show that $X \cap Y \in \mathscr{B}_S$, for every $X, Y \in \mathscr{S}_S$. Let $p \in \mathbb{Z}$ and let $m \in \mathbb{N}$. If $X \in \mathscr{S}_S$, then $A_{p,m} \cap X = \varnothing$ or $A_{p,m} \cap X = A_{p,m}$, so $A_{p,m} \cap X \in \mathscr{B}_S$. Clearly, $U_p \cap U_q = U_{\max(p,m)}$ and $U_p \cap D_q = \bigcup\{V_r \mid p \leqslant r \leqslant q\}$, for all $q \in \mathbb{Z}$. If $q \in \mathbb{Z}$, $n \in \mathbb{N}$ and $X \in \{S_{q,n}, \bar{S}_{q,n}\}$, then $U_p \cap X = X$ when $q \geqslant p$ and $U_p \cap X = \varnothing$ if $q < p$. Thus, $U_p \cap X \in \mathscr{B}_S$ when $X \in \mathscr{S}_S$. If $q \in \mathbb{Z}$, then it is clear that $D_p \cap D_q = D_{\min(p,q)}$. If we have $X \in \{S_{q,n}, \bar{S}_{q,n}\}$, for some $q \in \mathbb{Z}$ and $n \in \mathbb{N}$, then $D_p \cap X = X$ if $q \leqslant p$ and $D_p \cap X = \varnothing$ when $q > p$. From this, it follows that $D_p \cap X \in \mathscr{B}_S$, for all $X \in \mathscr{S}_S$. If $q \in \mathbb{Z}$ and $n \in \mathbb{N}$, then $S_{p,m} \cap S_{q,n} = S_{p,\max(m,n)}$ when $q = p$ and $S_{p,m} \cap S_{q,n} = \varnothing$ when $q \neq p$. Clearly, $S_{p,m} \cap \bar{S}_{q,n} = \varnothing$, for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, hence $S_{p,m} \cap X \in \mathscr{B}_S$ if $X \in \mathscr{S}_S$. If $q \in \mathbb{Z}$ and $n \in \mathbb{N}$, then we have $\bar{S}_{p,m} \cap \bar{S}_{q,n} = S_{p,\max(m,n)}$ whenever $q = p$ and $\bar{S}_{p,m} \cap \bar{S}_{q,n} = \varnothing$ otherwise, so $\bar{S}_{p,m} \cap X \in \mathscr{B}_S$, for all $X \in \mathscr{S}_S$. From these results, it follows that $\mathscr{B}_S$ is closed under intersection.

Based on De Morgan's laws and the observations above, to show that $\mathscr{B}_S$ is closed under forming complements, it will be enough to show that $X^c \in \mathscr{B}_S$ when $X \in \mathscr{S}_S$. Using Figure 3.1, we find that

$$\begin{aligned}
A^c_{p,m} &= \bigcup\{A_{p,n} \mid n < m\} \cup U_{p+1} \cup D_{p-1} \cup S_{p,m+1} \cup \bar{S}_{p,m+1},\\
D^c_p &= U_{p+1},\\
U^c_p &= D_{p-1},\\
S^c_{p,m} &= \bigcup\{A_{p,n} \mid n < m\} \cup \bar{S}_{p,m} \cup U_{p+1} \cup D_{p-1},\\
\bar{S}^c_{p,m}T &= \bigcup\{A_{p,n} \mid n < m\} \cup S_{p,m} \cup U_{p+1} \cup D_{p-1},
\end{aligned}$$

so $\mathscr{B}_S$ is closed under forming complements.

As $\varnothing \in \mathscr{B}_S$ and $\mathscr{B}_S$ is closed under forming complements, we have $V \in \mathscr{B}_S$.

Since f_S and g_S preserve unions, to show that $\mathscr{B}_S$ is closed under both f_S and g_S, it will be enough to show that we have $f_S(X), g_S(X) \in \mathscr{B}_S$, for all $X \in \mathscr{S}_S$. Clearly, this follows from Lemma 3.9, so $\mathscr{B}_S$ is indeed closed under f_S and g_S.

Based on these results, $\mathscr{B}_S$ is a subuniverse of $\mathbf{F}_S$, which is what we wanted. $\square$

Thus, we can make the following definition.

Notation 3.11. For each $S \subseteq \mathbb{O}$, the subalgebra of $\mathbf{F}_S$ with universe $\mathscr{B}_S$ will be denoted by $\mathbf{B}_S$ and $\mathrm{Var}(\mathbf{B}_S)$ will be denoted by B_S.

Before we look at the varieties that these algebras generate, it will be convenient to show that they are generated by any element of the form V_p, for some $p \in \mathbb{Z}$.

Lemma 3.12. *Let $S \subseteq \mathbb{O}$, let $X \subseteq V$ and assume that there is a maximal $p \in \mathbb{Z}$ with $V_p \cap X \neq \varnothing$, say q. Then:*

(1) $f^4_S(X) \cap f^2_S(X)^c = V_{q+2}$ *when* $a_{q,1} \in X$*;*
(2) $f^5_S(X) \cap f^3_S(X)^c = V_{q+2}$ *when* $a_{q,1} \notin X$*.*

Proof. Assume that $a_{q,1} \in X$. From Figure 3.1, $f^2_S(X) = D_{q+1}$ and $f^4_S(X) = D_{q+2}$, which implies that $f^4_S(X) \cap f^2_S(X)^c = V_{q+2}$. Thus, (1) holds.

Assume that $a_{q,1} \notin X$. Similarly to the above, $f^3_S(X) = D_{q+1}$ and $f^5_S(X) = D_{q+2}$, so we have $f^5_S(X) \cap f^3_S(X)^c = V_{q+2}$. Thus, (2) also holds. $\square$

The following portion of our argument will be useful later, so we will isolate it here. Below we define a set of terms that describe how each element of $\{A_{p,1} \mid p \in \mathbb{Z}\}$ generates each element of V_p, for each $p \in \mathbb{Z}$ and $S \subseteq \mathbb{O}$. In an argument below, these terms are used to show that distinct choices of $\mathbb{O}$ yield nonisomorphic algebras.

Definition 3.13. (1) Let $\beta(x) := f^4(x) \wedge f^2(x)'$.
(2) Let $\sigma(x) := f(x) \wedge (x \vee g^2(\beta(x)) \vee f^4(g^{10}(\beta(x)) \wedge g^8(\beta(x))'))'$.
(3) Let $\nu_3(x) := f(\sigma(x)) \wedge f(x)'$.
(4) Let $\nu_4(x) := f(\nu_3(x)) \wedge f(\sigma(x))'$.
(5) For each $n \geqslant 5$, let $\nu_n := f(\nu_{n-1}(x)) \wedge f(\nu_{n-2}(x))'$.

Lemma 3.14. *Let $S \subseteq \mathbb{O}$ and let $p \in \mathbb{Z}$. Then $\sigma(A_{p,1}) = A_{p,2}$ and $\nu_n(\sigma(x)) = A_{p,n}$, for all $n \geqslant 3$.*

PROOF. By Lemma 3.12(1), we have $\beta(A_{p,1}) = V_{p+2}$. Hence, based on Figure 3.1, we must have $g_S^2(\beta(A_{p,1})) = U_{p+1}$. Similarly, $g_S^8(V_{p+2}) = U_{p-2}$ and $g_S^{10}(V_{p+2}) = U_{p-3}$, hence

$$\begin{aligned} f_S^4(g_S^{10}(\beta(A_{p,1}) \cap g_S^8(\beta(A_{p,1}))^c) &= f_S^4(V_{p-3}) \\ &= D_{p-1}. \end{aligned}$$

By Lemma 3.9(1), $\sigma(A_{p,1}) = A_{p,2}$, as claimed.

For the second claim we use strong induction. By Lemma 3.9(1) and Lemma 3.9(2), we have

$$\begin{aligned} \nu_3(A_{p,1}) &= f_S(A_{p,2}) \cap f_S(A_{p,1})^c \\ &= A_{p,3}, \\ \nu_4(A_{p,1}) &= f_S(A_{p,3}) \cap f_S(A_{p,2})^c \\ &= A_{p,3}, \end{aligned}$$

as $\sigma(A_{p,1}) = A_{p,2}$. Now, let $n \geqslant 5$ and assume that $\nu_m(A_{p,1}) = A_{p,m}$, for all $4 \leqslant m \leqslant n$. Then by Lemma 3.9(2), we must have

$$\begin{aligned} \nu_{n+1}(A_{p,1}) &= f_S(A_{p,n}) \cap f_S(A_{p,n-1})^c \\ &= A_{p,n+1}. \end{aligned}$$

It follows that $\nu_m(A_{p,1}) = A_{p,m}$, for all $m \geqslant 3$, as claimed. □

Lemma 3.15. *Let $S \subseteq \mathbb{O}$ and let $p \in \mathbb{Z}$. Then $\mathscr{B}_S$ is the subuniverse of $\mathbf{B}_S$ generated by V_p.*

PROOF. Let $\mathscr{V}_p$ denote the subuniverse of $\mathbf{B}_S$ generated by V_p. Based on Lemma 3.10, it will be enough to show that $\mathscr{S}_S \subseteq \mathscr{V}_p$.

Firstly, we claim that $V_q \in \mathscr{V}_p$, for every $q \in \mathbb{Z}$. From Figure 3.1, Lemma 3.9(3), and Lemma 3.9(5), it follows that $f_S^{2m}(V_p) = D_{p+m}$, for each $m \in \mathbb{N}$. Therefore

$$\begin{aligned} V_{p+m+1} &= f_S^{2m+2}(V_p) \cap f_S^{2m}(V_p)^c \\ &\in \mathscr{V}_p, \end{aligned}$$

for all $m \in \mathbb{N}$, so we have $V_q \in \mathscr{V}_p$, for every $q \geqslant m+2$. Similarly, if $q \in \mathbb{Z}$ and $m \in \mathbb{N}$, then $g_S^{2m}(V_q) = U_{q-m}$, hence

$$\begin{aligned} V_{q-m-1} &= g_S^{2m+2}(V_q)^c \cap g_S^{2m}(V_q) \\ &\in \mathscr{V}_p. \end{aligned}$$

It follows that $V_q \in \mathscr{V}_p$, for all $q \in \mathbb{Z}$, as claimed.

Based on Figure 3.1, Lemma 3.9(3), and Lemma 3.9(5), we have $f_S^2(V_{q-1}) = D_q$, for all $q \in \mathbb{Z}$. Similarly, $g_S^2(V_{q+1}) = U_q$, for all $q \in \mathbb{Z}$, so by the previous result, $D_q, U_q \in \mathscr{V}_p$, for all $q \in \mathbb{Z}$.

By Lemma 3.9(13) and the previous result, we have $A_{p,1} = g(U_{q+1}) \cap U_{q+1}^c \in \mathscr{V}_p$, for all $q \in \mathbb{Z}$. So, by Lemma 3.14, we have $A_{p,n} \in \mathscr{V}_p$, for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Based on Lemma 3.9(3), we have

$$S_{q,m} = f_S(D_{q-1}) \cap \left(\bigcup\{A_{q,n} \mid n < m\} \cup D_{q-1}\right)^c.$$

Hence, by the above results, we must have $S_{q,m} \in \mathscr{V}_p$, for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$.

Clearly, we have

$$\bar{S}_{q,m} = \left(\bigcup\{A_{q,n} \mid n < m\} \cup D_{q-1} \cup U_{q+1} \cup S_{q,m}\right)^c,$$

for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$. Hence, based on the above results, we must have $\bar{S}_{q,m} \in \mathscr{V}_p$, for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$.

Combining these results, we find that $\mathscr{S}_S \subseteq \mathscr{V}_p$, which is what we wanted. □

Using Lemma 3.12 and Lemma 3.15, it is not too hard to show that $\mathbf{B}_S$ is indeed the subalgebra of $\mathbf{F}_S$ generated by the set of atoms of $\mathbf{F}_S$ (or any element of $\mathscr{B}_S$), for each $S \subseteq \mathbb{O}$.

Now we can shift our attention to varieties. First, we will need four intermediate results.

Lemma 3.16. *Let $S \subseteq \mathbb{O}$ and let $X \in \mathscr{B}_S$. Then $f_S(X) \neq V$ or $f_S(X^c) \neq V$.*

Proof. By Lemma 3.10, X and X^c can be written as unions of finite subsets of $\mathscr{S}_S$. Clearly, only one of the unions will involve an element of $\{U_p \mid p \in \mathbb{Z}\}$. By Figure 3.1, we will either have $f_S(X) \neq V$ or $f_S(X^c) \neq V$, as required. □

Lemma 3.17. *Let $S \subseteq \mathbb{O}$ and let $X \in \mathscr{B}_S$ with $X \neq \varnothing$ and $f_S(X) \neq V$. Then there is a maximal $p \in \mathbb{Z}$ with $V_p \cap X \neq \varnothing$.*

Proof. From Lemma 3.10, X can be written as the union of a finite subset of $\mathscr{S}_S$. By assumption, $f_S(X) \neq V$, so Lemma 3.9(iii) tells us that no element of $\{U_p \mid p \in \mathbb{Z}\}$ appears in such a union. So, as $X \neq \varnothing$, there is maximal $p \in \mathbb{Z}$ with $V_p \cap X \neq \varnothing$, which is what we wanted to show. □

Lemma 3.18. *Let $S \subseteq \mathbb{O}$ and let $p, q \in \mathbb{Z}$. Then there is an automorphism of $\mathbf{B}_S$ that maps V_p to V_q.*

Proof. It is clear that map given by $a_{r,m} \mapsto a_{r+q-p,m}$ is an automorphism of $\langle V; R_S \rangle$. The image function of this map is clearly an automorphism of $\mathbf{B}_S$ that maps V_p to V_q, which is what we wanted. □

Lemma 3.19. *Let $S \subseteq \mathbb{O}$, let I be a non-empty set, let $\mathscr{U}$ be an ultrafilter over I, and let $X \in \mathscr{B}_S^I$ with $X/\mathscr{U} \neq 0$ and $X/\mathscr{U} \neq 1$. Then $\mathbf{B}_S$ embeds into the subalgebra of $\mathbf{B}_S^I/\mathscr{U}$ generated by $X/\mathscr{U}$.*

Proof. By Lemma 3.16 and Łoś's Theorem, we have $f(X/\mathscr{U}) \neq 1$ or $f(X/\mathscr{U}') \neq 1$. Hence, we can assume without any loss of generality that we have $f(X/\mathscr{U}) \neq 1$. So, by Lemma 3.12 and Lemma 3.17, we either have

$$\begin{aligned} &\{i \in I \mid f_S^4(X(i)) \cap f_S^2(X(i))^c = V_p, \text{ for some } p \in \mathbb{Z}\} \in \mathscr{U} \\ \text{or } &\{i \in I \mid f_S^5(X(i)) \cap f_S^3(X(i))^c = V_p, \text{ for some } p \in \mathbb{Z}\} \in \mathscr{U}. \end{aligned}$$

Combining this observation with Proposition 1.36, Lemma 3.15, and Lemma 3.18, we can conclude that $\mathbf{B}_S$ embeds into the subalgebra of $\mathbf{B}_S^I/\mathscr{U}$ generated by $X/\mathscr{U}$, as claimed. ☐

Now we have all of the tools to show that B_S is in fact a join-irreducible cover of T_0, for each $S \subseteq \mathbb{O}$.

Lemma 3.20. *Let $S \subseteq \mathbb{O}$. Then B_S is a join-irreducible cover of T_0 in $\mathbf{\Lambda}_{\mathsf{TTA}}$.*

Proof. It is clear that $\mathbf{T}_0$ embeds into $\mathbf{B}_S$, hence by Corollary 3.7, both of $\mathbf{T}_0$ and $\mathbf{B}_S$ are simple. Based on Proposition 1.34, we have $\mathbf{A} \notin \mathsf{T}_0$ and $\mathsf{T}_0 \subseteq \mathsf{A}_S \subseteq \mathsf{TTA}$. Now, let $\mathbf{B} \in \mathbb{ISU}(\mathbf{B}_S) \setminus \mathsf{T}_0$. By Lemma 3.19, $\mathbf{B}_S$ embeds into $\mathbf{B}$, so we have $\mathbf{B}_S \in \mathbb{ISU}(\mathbf{B})$. Based on Proposition 1.35(1), A_S is in fact a join-irreducible cover of T_0 in $\mathbf{\Lambda}_{\mathsf{TTA}}$, which is what we wanted to show. ☐

To get the main result of this section, it remains to show that these covers are distinct for distinct choices of $\mathbb{O}$. Using Lemma 3.14 and the terms from Definition 3.13, we will construct first-order formulæ that will be used to distinguish between the generators of these covers, and hence the varieties that they generate.

Definition 3.21. (1) Let $\alpha(x) := x \not\approx 0 \curlywedge (\forall y\colon x \wedge y \approx 0 \curlyvee x \wedge y \approx x)$.

(2) Let $\varphi(x) := \alpha(x) \curlywedge \neg(\exists w, y, z\colon \alpha(w) \curlywedge \alpha(y) \curlywedge \alpha(z) \curlywedge f(x) \wedge g(x) \approx w \vee y \vee z)$.

(3) For each $n \geqslant 3$, let $\tau_n(x) := \varphi(x) \curlywedge \nu_n(x) \wedge f(g^2(x) \wedge g(x)') \not\approx 0$.

Lemma 3.22. *Let $S \subseteq \mathbb{O}$, let $n \geqslant 3$, and let $X \in \mathscr{B}_S$. Then:*

(1) $\mathbf{B}_S \models \varphi[X]$ *if and only if* $X = A_{p,1}$, *for some* $p \in \mathbb{Z}$;

(2) $\mathbf{B}_S \models \tau_n[X]$ *if and only if* $n \in S_{\mathbb{E}}$ *and* $X = A_{p,1}$, *for some* $p \in \mathbb{Z}$.

Proof. If $\mathbf{T}$ is a tense algebra and $x \in T$, then $\mathbf{T} \models \alpha[x]$ if and only if x is an atom, so this implies that $\mathbf{B} \models \alpha[X]$ if and only if $X = A_{p,n}$, for some $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.

By Lemma 3.9(1) and Lemma 3.9(9),

$$f_S(A_{p,n}) \cap g_S(A_{p,n}) = A_{p,1} \cup A_{p,2} \cup S_{p,1}$$

if $n = 1$ and $p \in \mathbb{Z}$. Similarly,

$$f_S(A_{p,n}) \cap g_S(A_{p,n}) = A_{p,n-1} \cup A_{p,n} \cup A_{p,n+1}$$

if $n > 1$ and $p \in \mathbb{Z}$. From the above, (1) holds. By Lemma 3.9(9) and Lemma 3.9(13),

$$\begin{aligned} g_S^2(A_{p,1}) &= g_S(U_p) \\ &= A_{p-1,1} \cup U_p, \end{aligned}$$

for all $p \in \mathbb{Z}$. From Lemma 3.9(1),

$$\begin{aligned} f_S(g_S^2(A_{p,1}) \cap g_S(A_{p,1})^c) &= f_S(A_{p-1,1}) \\ &= A_{p-1,1} \cup A_{p-1,2} \cup D_{p-2} \cup S_{p,1}, \end{aligned}$$

for all $p \in \mathbb{Z}$. By Lemma 3.14,

$$\nu_n(A_{p,1}) \cap f_S(g_S^2(A_{p,1}) \cap g_S(A_{p,1})^c) \neq \varnothing$$

if and only if $n \in S_{\mathbb{E}}$, hence (2) follows from (1). Therefore (1) and (2) both hold, which is what we wanted to show. $\square$

Lemma 3.23. *Let $S, T \subseteq \mathbb{O}$. If $\mu\colon \mathbf{B}_S \to \mathbf{B}_T$ is a homomorphism, then μ is an isomorphism.*

Proof. Since $\mathbf{B}_S$ and $\mathbf{B}_T$ are both Non-trivial, the kernel of μ must be non-zero. Based on Corollary 3.7, $\mathbf{B}_S$ is simple, hence the kernel of μ is the identity relation. This implies that μ is an embedding, so $\varnothing \subsetneq \mu(V_0) \subsetneq V$. Combining Lemma 3.12, Lemma 3.15, Lemma 3.16, and Lemma 3.17, we find that $\mu[\mathscr{B}_S] = \mathscr{B}_T$. From this, it follows that μ is surjective, hence μ is an isomorphism, as required. $\square$

Lemma 3.24. *Let $S, T \subseteq \mathbb{O}$ with $S \neq T$. Then $\mathsf{B}_S \neq \mathsf{B}_T$.*

Proof. Without loss of generality, we can assume that we have $S \nsubseteq T$, since $S \neq T$. Thus, there exists some $n \in S \setminus N$. Based on Lemma 3.22, we have $\mathbf{B}_S \models \exists x\colon \tau_n(x)$ and $\mathbf{B}_T \not\models \exists x\colon \tau_n(x)$, which implies that $\mathbf{B}_S$ and $\mathbf{B}_T$ are not elementarily equivalent, and are therefore not isomorphic. So, by Lemma 3.23, $\mathbf{B}_S$ does not embed into $\mathbf{B}_T$. Based on Proposition 1.34(3), Corollary 3.7, and Lemma 3.19, we must have $\mathbf{B}_T \notin \mathsf{B}_S$, so $\mathsf{B}_S \neq \mathsf{B}_T$, as claimed. $\square$

Now we just need to put on the finishing touches.

Theorem 3.25. $\mathrm{Var}(\mathbf{T}_0)$ *has exactly* $2^{\aleph_0}$ *join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{TTA}}$*. Thus,* $\mathrm{Var}(\mathbf{T}_0)$ *has* $2^{\aleph_0}$ *join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{TA}}$*.*

Proof. Let C denote the set of all join-irreducible covers of $\mathrm{Var}(\mathbf{T}_0)$ in $\mathbf{\Lambda}_{\mathrm{TTA}}$. Then, based on Lemma 3.20 and Lemma 3.24, we must have $|C| \geqslant 2^{\aleph_0}$. Clearly, there are at most $2^{\aleph_0}$ sets of equations in a countable signature, so $|C| \leqslant |\Lambda_{\mathrm{TTA}}| \leqslant 2^{\aleph_0}$. Combining these results, we find that $|C| = 2^{\aleph_0}$, which is what we wanted to show. $\square$

3.3. Varieties of relation-type algebras

In this section, we will study varieties of low height in the subvariety lattices of five varieties of relation-type algebras. Theorem 3.32 appears in Koussas and Kowalski [**58**], and the majority of the results in this section are set to appear in Hirsch, Jackson, Koussas, and Kowalski [**39**]. Lemma 3.42 is partially due to Tomasz Kowalski. Unless stated otherwise, the remaining results are the sole work of the author.

Using the main result from Section 3.2, we will solve the classification problem from Section 3.1 for several varieties of relation-type algebras (up to cardinality). We will first look at the subvariety lattices of NA, SA, and a certain subvariety of SA. We begin by recalling some definitions from Jipsen, Kramer, and Maddux [**45**].

Definition 3.26 (Reflexive and subadditive)**.** We call a nonassociative relation algebra $\mathbf{A}$ *reflexive* or *subadditive* if we have $x \leqslant x^2$, for all $x \in A$ or $x(x' \wedge y) \leqslant x \vee y$, for all $x, y \in A$, respectively. The class (variety) of all reflexive subadditive symmetric semiassociative nonassociative relation algebras will be denoted by RSA.

Definition 3.27 (Reflexive subadditive semiassociative symmetric r-algebra)**.** We call an algebra $\mathbf{A} = \langle A; \vee, \wedge, \cdot, ', \breve{}, 0, 1\rangle$ of signature $(2,2,2,1,1,0,0)$ a *reflexive subadditive semiassociative symmetric r-algebra* if $\mathbf{A}$ satisfies all defining properties of a reflexive subadditive semiassociative symmetric relation algebra not involving e, i.e., $\mathbf{A}^\flat$ is a Boolean algebra, $x \mapsto xy$ and $x \mapsto yx$ are conjugates of themselves, for all $y \in A$, $\mathbf{A} \models (x1)1 \approx x1$, $x \leqslant x^2$, for every $x \in A$, $x(x' \wedge y) \leqslant x \vee y$, for all $x, y \in A$, and $\mathbf{A} \models x^{\smile} \approx x$. The class (variety) of all reflexive subadditive semiassociative symmetric r-algebras will be denoted by RSR.

Notation 3.28. The subvariety lattices of NA, SA, RSA, RA, and RRA will be denoted by $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RSA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, and $\mathbf{\Lambda}_{\mathsf{RRA}}$, respectively. Let $\mathbf{A}_i$ be the minimal (constant) subalgebra of $\mathbf{Re}(n)$ and let $\mathsf{A}_n := \mathrm{Var}(\mathbf{A}_n)$, for $1 \leqslant n \leqslant 3$. Let $\mathbf{N}_1 := \mathbf{Re}(\{0,1\})$ and let $\mathsf{N}_1 := \mathrm{Var}(\mathbf{N}_1)$; we follow Table 6 of Jipsen [**43**].

It is easy to see that $\mathbf{A}_1, \mathbf{A}_3 \in \mathsf{RSA}$, $\mathbf{A}_2, \mathbf{N}_1 \notin \mathsf{RSA}$, and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{N}_1 \in \mathsf{RRA}$. Thus, the results from Section 3.1 give the following result.

Proposition 3.29. (1) A_1, A_2, *and* A_3 *are atoms of* $\mathbf{\Lambda}_{\mathsf{NA}}$.
(2) A_1, A_2, *and* A_3 *are the only atoms of* $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RRA}}$.
(3) A_1 *and* A_3 *are the only atoms of* $\mathbf{\Lambda}_{\mathsf{RSA}}$.
(4) A_1 *has no join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RSA}}$.
(5) N_1 *is a join-irreducible cover of* A_2 *in* $\mathbf{\Lambda}_{\mathsf{NA}}$.
(6) N_1 *is the unique join-irreducible cover of* A_2 *in* $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RRA}}$.
(7) A_3 *has at least* $\aleph_0$ *join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RSA}}$.
(8) N_1 *has no join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RSA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RRA}}$.

In $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RSA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, and $\mathbf{\Lambda}_{\mathsf{RRA}}$, the join-irreducible varieties not including $\mathbf{A}_3$ are known, so looking at the corresponding varieties in $\mathbf{\Lambda}_{\mathsf{NA}}$ and classifying the join-irreducible covers of A_3 are two natural ways of continuing the study of the bottoms of these subvariety lattices; we will focus on the latter problem in $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, and $\mathbf{\Lambda}_{\mathsf{RSA}}$. The following result is essentially from Section 4 of [**45**].

Proposition 3.30. (1) *Let* $\langle A; \vee, \wedge, \cdot, ', \breve{}, 0, 1\rangle \in \mathsf{RSR}$. *Define two unary operations* f *and* g *on* A *by* $x \mapsto x^2$ *and* $x \mapsto x \vee xx'$, *respectively. Then we have* $\langle A; \vee, \wedge, ', f, g, 0, 1\rangle \in \mathsf{TTA}$.

(2) *Let* $\langle A; \vee, \wedge, ', f, g, 0, 1\rangle \in \mathsf{TTA}$. *Define a binary operation* $\cdot$ *and a unary operation* $\breve{}$ *on* A *by* $(x, y) \mapsto f(x \wedge y) \vee (x \wedge g(y)) \vee (g(x) \wedge y)$ *and* $x \mapsto x$, *respectively. Then* $\langle A; \vee, \wedge, \cdot, ', \breve{}, 0, 1\rangle \in \mathsf{RSR}$.

(3) RSR *and* TTA *are term-equivalent varieties.*

Proposition 3.30 can be used to establish the following result; we refer the reader to [**45**] for more details.

Proposition 3.31. A_3 *has at least as many join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{RSA}}$ *as has* T_0 *in* $\mathbf{\Lambda}_{\mathsf{TTA}}$.

Combining Theorem 3.25 and Proposition 3.31, we obtain the following result, which improves Proposition 3.29(7).

Theorem 3.32. A_3 *has exactly* $2^{\aleph_0}$ *join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, *and* $\mathbf{\Lambda}_{\mathsf{RSA}}$.

The join irreducibles of low height in $\mathbf{\Lambda}_{\mathsf{SA}}$ and $\mathbf{\Lambda}_{\mathsf{RSA}}$ are illustrated in Figure 3.3 below.

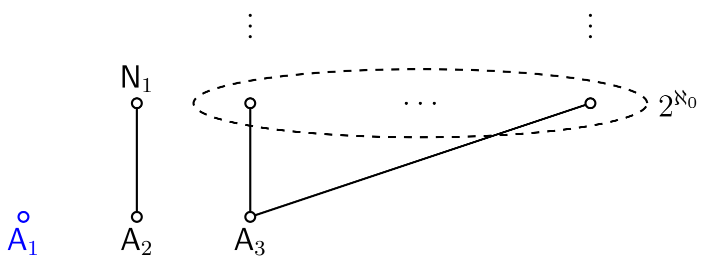

FIGURE 3.2. $\mathbf{J}(\mathbf{\Lambda}_{\mathsf{SA}})$ and $\mathbf{J}(\mathbf{\Lambda}_{\mathsf{RSA}})$, with $J(\mathbf{\Lambda}_{\mathsf{SA}}) \setminus J(\mathbf{\Lambda}_{\mathsf{RSA}})$ shown in blue.

Now we will shift our attention to the subvariety lattices of the varieties from Chapter 2. Firstly, we must introduce some notation.

Notation 3.33 ($\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$)**.** The subvariety lattices of GFRA and GQRA will be denoted by $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$, respectively.

Next, we extend the notion of a class from Jónsson and Tarski [**55**] to NA

Definition 3.34 (Classes)**.** We say that an algebra $\mathbf{A} \in \mathsf{NA}$ is of *class* 1, *class* 2, or *class* 3 if $\mathbf{A}$ satisfies $d^2 \approx 0$, $d^2 \approx e$, or $d^2 \approx 1$, respectively.

The following result is rather easy to prove; we refer to Lemma 13.1 and Lemma 13.3 of Givant [**32**], Theorem 4.35 of [**55**], Theorem 236 of Maddux [**70**].

Proposition 3.35. (1) *An algebra* $\mathbf{A} \in \mathsf{NA}$ *is of class 1 if and only if* $\mathbf{A} \models d \approx 0$.

(2) $\mathbf{A}_n$ *is of class* n*, for each* $1 \leqslant n \leqslant 3$.

The following result shows how the size of a base determines the class of a feebly representable algebra. This turns out to be different to the strong representation case, where $|D| = 2$ implies that $\mathbf{A}$ is of class 2.

Lemma 3.36. *Let* $\mathbf{A} \in \mathsf{NA}$ *and let* ϕ *be a feeble representation of* $\mathbf{A}$ *over a base* D.

(1) *If* $|D| = 1$*, then* $\mathbf{A}$ *is of class* 1.
(2) *If* $|D| = 2$*, then* $\mathbf{A}$ *is of class* 2 *or class* 3.
(3) *If* $|D| > 2$*, then* $\mathbf{A}$ *is of class* 3.

Proof. Firstly, we will assume that $|D| = 1$. By Proposition 3.35, we have $\mathrm{di}_D = \varnothing$, hence $\mathbf{A} \models d \approx 0$. This implies that $\mathbf{A} \models d^2 \approx 0$, so $\mathbf{A}$ is of class 1 and (1) holds.

Next, we will assume that $|D| = 2$. Then we must have

$$\begin{aligned} \phi(d^2) &\supseteq \phi(d) \circ \phi(d) \\ &= \mathrm{di}_D \circ \mathrm{di}_D \\ &= \mathrm{id}_D. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \phi(d^2)^{-1} &= \phi((d^2)^{\smile}) \\ &= \phi((d^{\smile})^2) \\ &= \phi(d^2). \end{aligned}$$

These results tell us that $\phi(d^2)$ is a symmetric element extending id_D. Since $|D| = 2$, we have $\phi(d^2) = \mathrm{id}_D$ or $\phi(d^2) = D^2$. This implies that $\mathbf{A} \models d^2 \approx e$ or $\mathbf{A} \models d^2 \approx 1$, hence $\mathbf{A}$ is of class 2 or class 3 and (2) holds.

Lastly, we will assume that $|D| > 2$. Then

$$\begin{aligned} \phi(d^2) &\supseteq (d)^{\phi} \circ (d)^{\phi} \\ &= \mathrm{di}_D \circ \mathrm{di}_D \\ &= D^2, \end{aligned}$$

so we have $\phi(d^2) = D^2$, which implies that $\mathbf{A} \models d^2 \approx 1$. Thus, $\mathbf{A}$ is of class 3, so (1), (2), and (3) all hold, as required. $\square$

It is not too hard to show that $\mathbf{A}_3$ has a feeble representation on a two-element base, so the conclusion of Lemma 3.36(2) cannot be strengthened.

It is shown in Section 13.1 of [**32**] that a non-trivial relation algebra has at most one class. This proof does not require associativity, so we have the following result.

Corollary 3.37. *Every simple member of* GFRA *and* GQRA *has a unique class.*

It follows that every simple member of GFRA has a subalgebra isomorphic to $\mathbf{A}_1$, $\mathbf{A}_2$, or $\mathbf{A}_3$. Since $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathsf{RRA} \subseteq \mathsf{GQRA}$, we get the following result.

Corollary 3.38. A_1, A_2, *and* A_3 *are the only atoms of* $\mathbf{\Lambda}_{\mathsf{F}}$ *and* $\mathbf{\Lambda}_{\mathsf{Q}}$.

From Proposition 3.29(4), we obtain the following result, which characterises the covers of A_1 in $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$.

Theorem 3.39. A_1 *has no join-irreducible covers in both* $\mathbf{\Lambda}_{\mathsf{F}}$ *and* $\mathbf{\Lambda}_{\mathsf{Q}}$.

Next, we aim to determine all of the join-irreducible varieties that extend A_2, but not A_1 or A_3. Firstly, we define four algebras.

Definition 3.40 ($\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{F}_1$, and $\mathbf{F}_2$)**.** Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the algebras #19 and #21 from Section 4.4 of Neuzerling [**79**], respectively. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the algebras #49 and #50 from Section A.2 of [**79**], respectively. For each $1 \leqslant n \leqslant 2$, let F_n and Q_n denote $\mathrm{Var}(\mathbf{F}_n)$ and $\mathrm{Var}(\mathbf{Q}_n)$, respectively.

The atom tables and representations of these algebras are illustrated below.

#19	e_1	e_2	a
e_1	e_1	0	a
e_2	0	e_2	a
a	a	a	e

#21	e	r	$r^{\smile}$
e	e	r	$r^{\smile}$
r	r	0	e
$r^{\smile}$	$r^{\smile}$	e	0

#49	e_1	e_2	r	$r^{\smile}$
e_1	e_1	0	r	$r^{\smile}$
e_2	0	e_2	r	0
r	r	0	0	e
$r^{\smile}$	$r^{\smile}$	$r^{\smile}$	e_1	0

#50	e_1	e_2	r	$r^{\smile}$
e_1	e_1	0	r	$r^{\smile}$
e_2	0	e_2	r	$r^{\smile}$
r	r	r	0	e
$r^{\smile}$	$r^{\smile}$	$r^{\smile}$	e	0

FIGURE 3.3. Atom tables for $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{F}_1$, and $\mathbf{F}_2$.

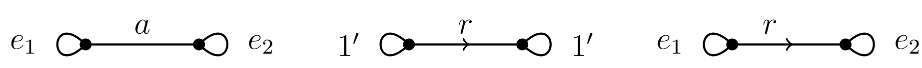

FIGURE 3.4. Networks that yield qualitative representations $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{N}_1$. The third network gives feeble representations of $\mathbf{F}_1$ and $\mathbf{F}_2$.

Now we are in the position to determine all of the join-irreducible varieties that extend A_2, but not A_1 or A_3.

Theorem 3.41. (1) N_1, Q_1, Q_2, *and* F_1 *are the only join-irreducible covers of* A_2 *in* $\mathbf{\Lambda}_{\mathsf{F}}$.

(2) F_2 *is the only join-irreducible cover of* Q_1 *in* $\mathbf{\Lambda}_{\mathsf{F}}$.

(3) N_1, Q_1, F_1, *and* F_2 *have no join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{F}}$.

(4) N_1, Q_1, *and* Q_2 *are the only join-irreducible covers of* A_2 *in* $\mathbf{\Lambda}_{\mathsf{Q}}$.
(5) N_1 *and* Q_1 *have no join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{Q}}$.
(6) N_1, Q_1, *and* Q_2 *have no join-irreducible covers in* $\mathbf{\Lambda}_{\mathsf{Q}}$.

Proof. Firstly, we note that $\mathbf{N}_1, \mathbf{Q}_1, \mathbf{Q}_2 \in \mathsf{QRA}$ and $\mathbf{F}_1, \mathbf{F}_2 \in \mathsf{GFRA} \setminus \mathsf{QRA}$.

That N_1 is a join-irreducible cover of A_2 in $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$ follows from Proposition 3.29(6).

Clearly, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ have no non-constant proper subalgebras, so by Proposition 1.35(2), Q_1 and Q_2 are join-irreducible covers of A_2 in $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$.

By definition, we have $r^{\smile}r = e_1$ and

$$\begin{aligned} e_2(r \vee r^{\smile}) &= e_2 r \vee e_2 r^{\smile} \\ &= r \end{aligned}$$

in $\mathbf{F}_1$. Based on this, it is easy to see that $\mathbf{F}_1$ has no non-constant proper subalgebras, so by Proposition 1.35(2), F_1 is a join-irreducible cover of A_2 in $\mathbf{\Lambda}_{\mathsf{F}}$.

Clearly, $\mathbf{F}_2$ has no non-constant subalgebras except for the one generated by $r \vee r^{\smile}$. It is easy to see that this subalgebra is isomorphic to $\mathbf{Q}_1$. So, by Proposition 1.35(2), F_2 is indeed a join-irreducible cover of Q_1 in $\mathbf{\Lambda}_{\mathsf{F}}$.

Assume that an algebra $\mathbf{A} \in \mathsf{FRA}$ generates a join-irreducible subvariety containing A_2. This implies that $\mathbf{A}_2$ is the only member of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ that is in $\mathrm{Var}(\mathbf{A})$, hence $\mathbf{A}$ is of class 2. Based on this fact and Lemma 3.36, $\mathbf{A}$ has a feeble representation on a two-element base. This implies that $\mathbf{A}$ is isomorphic to an algebra from Section 4.4 or Section A.2 of [**79**]. Thus, $\mathbf{A}$ must be isomorphic to $\mathbf{N}_1$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{F}_1$, or $\mathbf{F}_2$, hence there are no further join-irreducible varieties above A_2, but not A_1 or A_3.

Combining all of these results, we find that (1), (2), (3), (4), (5), and (6) all hold, which is what we wanted to show. □

We can also prove the above claim that an algebra $\mathbf{A} \in \mathsf{FRA}$ that generates a join-irreducible subvariety containing A_2 is isomorphic to $\mathbf{N}_1$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{F}_1$, or $\mathbf{F}_2$.

Proof. As in Theorem 3.41, if $\mathbf{A} \in \mathsf{FRA}$ generates a join-irreducible variety extending A_2, then $\mathbf{A}$ must have a feeble representation on a two-element base set. Firstly, assume that this feeble representation is a qualitative representation. By Lemma 2.15, $\mathbf{A}$ has a network with two nodes that gives a qualitative representation of $\mathbf{A}$. Clearly, the networks shown in Figure 3.3 are the only networks of this form (up to relabeling), since only one diversity cycle can be represented in a network with only two nodes. Now, based on Lemma 2.14, all choices of $\mathbf{A}$ are represented by the same networks, and are therefore obtained from $\mathbf{N}_1$, $\mathbf{Q}_1$ or $\mathbf{Q}_2$ by adding consistent cycles. Clearly, $\mathbf{F}_1$ and $\mathbf{F}_2$ are the only results with $d^2 \approx e$, so this list is exhaustive, as required. □

Lastly, using Theorem 3.32, we aim to show that A_3 has $2^{\aleph_0}$ join-irreducible covers in both $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$, which solves the classification problem from Section 3.1 for $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$, up to cardinality. Clearly, it will be enough to show that the semiassociative relation algebra corresponding to $\mathbf{B}_S$ has a qualitative representation, for each $S \subseteq \mathbb{O}$; see Proposition 3.30. To this end, we state and prove the following intermediate result.

Lemma 3.42. *Let $\mathbf{A} \in \mathsf{INA}$ be symmetric and let U be the set of all atoms of $\mathbf{A}$. Assume that there is a chain $\langle I; \leqslant \rangle$ and sets $\{a_i \mid i \in I\} \subseteq U$ and $\{X_i \mid i \in I\} \subseteq \wp(U)$ with the following properties:*

(1) $\bigcup_{i \in I} U_i = U$;
(2) *if $i, j \in I$, then $U_i \subseteq U_i$ if and only if $i \leqslant j$;*
(3) *if $i \in I$, then $[a_i, a_i, a]$ is consistent, for all $a \in U_i \cup \{a_i\}$.*

Then we have $\mathbf{A} \in \mathsf{QRA}$.

Proof. For each $i \in I$, we define $T_i := T \cap U_i^3$ and $N_i := \{t_{i,m} \mid t \in T_i, m \in \{1,2,3\}\}$. For all $t \in T$ and distinct $1 \leqslant m, n \leqslant 3$, let $a_{t,m,n}$ be the atom in the o^{th} entry of t, where $o \in \{1,2,3\} \setminus \{m,n\}$. Now, let $N := \bigcup_{i \in I} N_i$ and define $\lambda\colon N^2 \to U$ by

$$\lambda(t_{i,m}, s_{j,n}) = \begin{cases} e & \text{if } t_{i,m} = s_{j,n} \\ a_{t,m,n} & \text{if } s = t, i = j, m \neq n \\ a_{\max\{i,j\}} & \text{if } i \neq j. \end{cases}$$

Assume that $u, v, w \in N$ are mutually distinct. If we have $\{u, v, w\} = \{t_{i,1}, t_{i,2}, t_{i,3}\}$, for some $i \in I$ and $t \in T_i$, then $(\lambda(u,v), \lambda(v,w), \lambda(u,w))$ is a Peircean transform of t, and is therefore consistent. Next, assume that we have $\{u, v, w\} \neq \{t_{i,1}, t_{i,2}, t_{i,3}\}$, for all $i \in I$ and $t \in T_m$. By (1) and (2), there is a minimal $i \in I$ with $u, v, w \in U_i$. Clearly, we have $(\lambda(u,v), \lambda(v,w), \lambda(u,w)) \in [a_i, a_i, a]$, for some $a \in U_i \cup \{a_i\}$. Hence, by (3), $(\lambda(u,v), \lambda(v,w), \lambda(u,w))$ is a consistent triple. Now, $(\lambda(u,v), \lambda(v,w), \lambda(u,w))$ is a consistent identity triple whenever u, v, and w are not mutually distinct, so $\langle N; \lambda \rangle$ is a consistent atomic network.

Now, let $t \in T$. Then we have $t = (a, b, c)$, for some $a, b, c \in U$. From (1) and (2), it follows that there is some $i \in I$ with $a, b, c \in U_i$. By construction,

$$\begin{aligned} (\lambda(t_{i,2}, t_{i,3}), \lambda(t_{i,3}, t_{i,1}), \lambda(t_{i,1}, t_{i,2})) &= (a_{t,2,3}, a_{t,3,1}, a_{t,1,2}) \\ &= (a, b, c) \\ &= t, \end{aligned}$$

so every consistent triple is witnessed in $\langle N; \lambda \rangle$.

By Proposition 2.15, $\mathbf{A}$ has a qualitative representation, as claimed. □

Notation 3.43 (A_S)**.** For each $S \subseteq \mathbb{O}$, let A_S be the variety generated by the semiassociative relation algebra corresponding to $\mathbf{B}_S$.

Lemma 3.44. A_S *is a join-irreducible cover of* A_3 *in both* $\mathbf{\Lambda}_{\mathsf{F}}$ *and* $\mathbf{\Lambda}_{\mathsf{Q}}$*, for all* $S \subseteq \mathbb{O}$*. Further,* $S \neq T$ *implies that* $\mathsf{A}_S \neq \mathsf{A}_T$*, for all* $S, T \subseteq \mathbb{O}$*.*

Proof. Using Proposition 3.30, we can see that

$$AB = (A \cap g_S(B)) \cup (f_S(A) \cap B),$$

for distinct atoms A and B in the semiassociative relation algebra given by $\mathbf{A}_S$. Therefore $[A_{p,m}, A_{p,m}, A_{q,n}]$ is consistent when $a_{q,n}$ points at $a_{p,m}$ in $\mathbf{F}_S$; in particular, when $m < n$. Similarly, $[A_{p,m}, A_{p,m}, A_{p,m}]$ is always consistent, since we have

$$\begin{aligned} A_{p,m} &\subseteq f_S(A_{p,m}) \\ &\subseteq A_{p,m}^2. \end{aligned}$$

So, if we put $\langle I; \leqslant\rangle := \langle \mathbb{Z}; \leqslant\rangle$, $a_i := A_{i,1}$, and $X_i := \{A_{p,m} \mid p < i\}$, the conditions (1), (2), and (3) of Lemma 3.42 are satisfied, so the corresponding algebra belongs to QRA. So, based on Theorem 3.32, we have the required result. $\square$

Using this result, we get the following result on the join-irreducible covers of A_3 in $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$.

Theorem 3.45. A_3 *has* $2^{\aleph_0}$ *join-irreducible covers in both* $\mathbf{\Lambda}_{\mathsf{F}}$ *and* $\mathbf{\Lambda}_{\mathsf{Q}}$*.*

The results from Theorem 3.39, Theorem 3.41, and Theorem 3.45 are illustrated in Figure 3.3 below, which depicts the join irreducibles of low height in $\mathbf{\Lambda}_{\mathsf{F}}$ and $\mathbf{\Lambda}_{\mathsf{Q}}$.

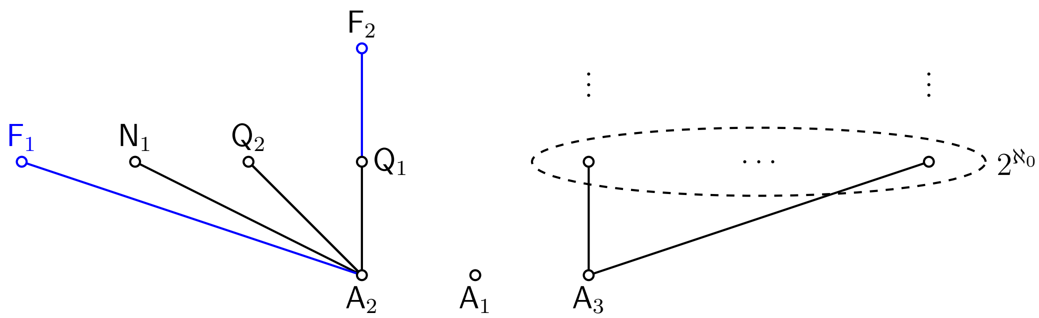

Figure 3.5. $\mathbf{J}(\mathbf{\Lambda}_{\mathsf{F}})$ and $\mathbf{J}(\mathbf{\Lambda}_{\mathsf{Q}})$, with $J(\mathbf{\Lambda}_{\mathsf{F}}) \setminus J(\mathbf{\Lambda}_{\mathsf{Q}})$ shown in blue.

3.4. Open problems

In this chapter, we studied the join-irreducible covers of some known varieties in a number of subvariety lattices. There are many open problems related to these results. In Section 3.3, we saw that A_3 has $2^{\aleph_0}$ covers in $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{F}}$, and $\mathbf{\Lambda}_{\mathsf{Q}}$. However, based on Lemma 3.9 and Proposition 3.30, we have

$$\begin{aligned} A_{0,1}(A_{0,1}A_{1,1}) &= A_{0,1}A_{1,1} \\ &= A_{1,1} \end{aligned}$$

and

$$\begin{aligned}(A_{0,1}A_{0,1})A_{1,1} &= (A_{0,1} \cup A_{0,2} \cup D_{-1} \cup S_{1,1})A_{1,1} \\ &= A_{1,1} \cup S_{1,1}\end{aligned}$$

in the relation-type algebras that correspond to $\mathbf{B}_S$, so these algebras are never associative, and hence not members of RA or RRA. Thus, the number of covers of A_3 in $\mathbf{\Lambda}_{\mathsf{RA}}$ and $\mathbf{\Lambda}_{\mathsf{RRA}}$ cannot be determined from this construction, and are hence unknown. Thus, the following problem is still one of the most important open problems in the study of subvariety lattices of varieties of relation-type algebras; see Maddux [**68**], Chapter 18 of Givant [**30**], and Chapter 21 of Hirsch and Hodkinson [**36**].

Problem 4. Determine the number of covers of A_3 in $\mathbf{\Lambda}_{\mathsf{RA}}$ and $\mathbf{\Lambda}_{\mathsf{RRA}}$.

A list of finite algebras in SA that generate covers of A_3 is given in Appendix 4.5. Another interesting problem would be determining whether or not such algebras exist with n atoms, for each $n > 9$.

Problem 5. Determine whether or not A_3 has infinitely many covers in $\mathbf{\Lambda}_{\mathsf{SA}}$ that are generated by a finite algebra.

Taking these problems and the results we saw in this chapter another step further, one may wish to find the complete list of covers rather than just the number of them. Extending Theorem 14 of Jipsen and Lukács [**46**] to finite joins of diversity atoms is a possible starting point for the relation algebra case; see Section 4 of the same article. Determining the feeble and qualitative representability of $\mathbf{B}_{10}$, $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{C}_4$ and $\mathbf{C}_5$ from Tables 4 and 5 of Jipsen [**43**] could be a starting point for $\mathbf{\Lambda}_{\mathsf{Q}}$ and $\mathbf{\Lambda}_{\mathsf{F}}$.

Problem 6. Classify the covers of A_3 in $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RSA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, $\mathbf{\Lambda}_{\mathsf{F}}$, $\mathbf{\Lambda}_{\mathsf{Q}}$, or $\mathbf{\Lambda}_{\mathsf{RRA}}$.

Problem 7. Classify the covers of T_0 in $\mathbf{\Lambda}_{\mathsf{TA}}$ or $\mathbf{\Lambda}_{\mathsf{TTA}}$.

Equational bases for the varieties we saw in Section 3.2 may also be of interest. The following problem is an obvious one; recursive subsets seem to be a likely answer.

Problem 8. Determine the subsets S of $\mathbb{O}$ for which the variety B_S is finitely based.

We also mention the following problems that are inspired by results in Section 3.1. If they are solved, studying the lower covers of these coatoms would be the next step. We refer to Jónsson [**51**] and [**68**].

Problem 9. Determine the number of coatoms in $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, $\mathbf{\Lambda}_{\mathsf{F}}$, $\mathbf{\Lambda}_{\mathsf{Q}}$, or $\mathbf{\Lambda}_{\mathsf{RRA}}$.

Problem 10. Classify the coatoms of $\mathbf{\Lambda}_{\mathsf{NA}}$, $\mathbf{\Lambda}_{\mathsf{SA}}$, $\mathbf{\Lambda}_{\mathsf{RA}}$, $\mathbf{\Lambda}_{\mathsf{F}}$, $\mathbf{\Lambda}_{\mathsf{Q}}$, or $\mathbf{\Lambda}_{\mathsf{RRA}}$.

CHAPTER 4

Probability

In this chapter, we will revise some probabilistic concepts from finite model theory, and then we will study relation-type algebras and atom-type structures in this context. More precisely, we will look at 'almost all' results and 0–1 laws for these structures. This approach has been taken in graph theory, semigroup theory, universal algebra, and many other fields; we refer to Section 11.3 of Diestel [**24**], Jackson and Volkov [**42**], and Section 6.2 of Bergman [**13**], for example. Unless otherwise stated, all results in this chapter are due to the author. The results original results from Section 4.2 and Section 4.3 are set to appear in Koussas [**57**].

4.1. Background

The idea of defining the probability of a property holding in a class of finite structures as the limit of the fraction of isomorphism classes of n-element structures that satisfy it first appeared explicit;y in writing in the nineteen fifties in Carnap [**17**], but did not attract much attention until the late nineteen sixties and early seventies. In [**25**] and [**26**], Fagin shows that the class of all finite structures of a finite purely relational signature that has at least one symbol that is not unary has a 0–1 law, i.e., first-order sentences either hold in almost all structures or fail in almost all structures; that almost all of these structures are rigid, i.e., have a trivial automorphism group; and finds an asymptotic formula for the number of isomorphism types of structures. The former results were found independently in Glebskiĭ, Kogan, and Liogon'kiĭ [**34**], and Liogon'kiĭ [**63**] without the assumption of the existence of a nonunary symbol, but with a more difficult proof. This area became active again in the nineteen eighties, when Compton published two articles on 0–1 laws. In [**20**] and [**21**], Compton found conditions that many classes of purely relational structures satisfy that guarantee the existence of 0–1 laws for first-order sentences and monadic second order sentences, respectively. The most well known results for algebraic signatures are those of Murskiĭ. In [**75**], [**76**], and [**77**], Murskiĭ showed that almost all finite algebras of a finite signature with at least two operation symbols and at least one nonunary symbol have discriminator terms and are finitely based, that almost all finite binars are simple, and that finite binars have no idempotent element with probability 1/e, for example; an English presentation of this work can be found in Section 6.2 of [**13**]. In [**28**], Freese examines the relationship between two definitions of probabilities as limits, and shows that they are equivalent for classes where almost all structures are rigid; the class of all finite structures of finite signatures with at least three unary operation symbols or an operation of arity two or greater are shown to have this property. It is also shown that the definitions differ by at most 0.001 in the class of all finite structures with just two unary operation symbols.

Surprisingly little results of this nature have been published on relation-type algebras. The seminal (and essentially the only) article on this topic is Maddux [**67**]. In this article, Maddux shows that almost all finite nonassociative relation algebras are rigid, and that almost all nonassociative algebras satisfy each finite subset of the equational theory of RRA, which means that almost all of these algebras are in SA, RA, and RA_n, for each fixed $n \geqslant 5$; recall Proposition 2.3(5). Another (unpublished) article is Alm [**2**]. The main open problem in this area is to extend the latter result by showing that almost all nonassociative relation algebras are strongly representable; this problem is stated by Maddux in [**67**] and [**68**], by Hirsch and Hodkinson in [**36**], and by Alm in [**2**].

4.2. Symmetric and integral algebras

In this section, we will show that almost all nonassociative relation algebras are symmetric and have e as an atom. Combining this with results from Maddux [**67**], we obtain a simple asymptotic formula for the number of relation algebras. Firstly, we will need to define the concepts from the previous section with more formality. Our presentation will mostly follow Freese [**28**].

Definition 4.1 (Labelled and unlabelled probabilities)**.** Let $\mathcal{K}$ be a class of finite structures of a finite signature F that is closed under isomorphism and has no upper bound on the size of its members. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a set with precisely one representative from each isomorphism class of n-element structures from $\mathcal{K}$ and let $\mathcal{L}_n$ be the set of all structures in $\mathcal{K}$ with universe $\{1, \dots, n\}$. Let P be some property of F-structures that is invariant under isomorphisms (for example, a first-order property). Let $s\colon \mathbb{N} \to \mathbb{N}$ be the increasing sequence of values of n with $\mathcal{U}_n \neq \varnothing$. If the limit

$$\lim_{n\to\infty} \frac{|\{\mathbf{A} \in \mathcal{U}_{s(n)} \mid \mathbf{A} \models P\}|}{|\mathcal{U}_{s(n)}|}$$

exists, then we will call it the *unlabelled probability of* P and denote it by $\mathrm{Pr}_{\mathrm{U}}(P, \mathcal{K})$. If the limit

$$\lim_{n\to\infty} \frac{|\{\mathbf{A} \in \mathcal{L}_{s(n)} \mid \mathbf{A} \models P\}|}{|\mathcal{L}_{s(n)}|}$$

exists, then we will call it the *labelled probability of* P and denote it by $\mathrm{Pr}_{\mathrm{L}}(P, \mathcal{K})$. When $\mathrm{Pr}_{\mathrm{U}}(P, \mathcal{K}) = 1$, we say that *almost all structures in* $\mathcal{K}$ *satisfy* P.

We will usually work with classes where there are elements of every possible cardinality, so the sequence s will be the identity sequence.

The main result from [**28**] that we will need is stated below. While this result is only proved for algebraic signatures, the arguments apply to more general signatures.

Proposition 4.2. *Let $\mathcal{K}$ be a class of similar finite structures of a finite signature F that is closed under isomorphism and has no upper bound on the size of its members, let R denote the property of being rigid, let P be some arbitrary property of F-structures that is isomorphism invariant, and assume that we have* $\mathrm{Pr}_{\mathrm{L}}(R, \mathcal{K}) = 1$. *If one of* $\mathrm{Pr}_{\mathrm{U}}(P, \mathcal{K})$ *and* $\mathrm{Pr}_{\mathrm{L}}(P, \mathcal{K})$ *exists, then both quantities exist and are equal.*

Next, we recall the following definition, which we will need later in Section 4.3.

Definition 4.3 (Almost sure theory)**.** Let $\mathcal{K}$ be a class of finite structures of a finite signature F that is closed under isomorphism and has no upper bound on the size of its members. We call the set of all first-order sentences σ that satisfy $\mathrm{Pr}_{\mathrm{L}}(\sigma, \mathcal{K}) = 1$ the *almost sure theory of* $\mathcal{K}$.

Based on Proposition 1.30 and Proposition 1.28, once some finite set U, some $e \in U$, and an involution $f \colon U \to U$ with $f(e) = e$, any $\mathbf{U} \in \mathsf{FAS}$ that is an expansion of $\langle U; f, \{e\}\rangle$ is completely determined by which cycles are consistent or forbidden. Using this observation, it is possible count the number of atom-structures of a given (finite) size. Indeed, in [**67**], Maddux obtains asymptotic formulæ using this method. The results we will need from [**67**] are summarised below.

Proposition 4.4. *Let U be an n-element set, for some $n \in \mathbb{N}$, let $e \in U$, let f be an involution of U with $f(e) = e$, and let $s := |\{a \in U \mid f(a) = a\}|$.*

(1) *There are $s - 1$ diversity cycles with 1 triple: those of the form $[a, a, a]$.*
(2) *There are $(n{-}s)/2$ diversity cycles with 2 triples: those of the form $[a, a, f(a)]$, where $f(a) \neq a$.*
(3) *There are $(s - 1)(n - 2)$ diversity cycles with 3 triples: those of the form $[a, b, b]$, where $f(a) = a$ and $a \neq b$.*
(4) *There are $(n - 1)((n - 1)^2 - 3s + 2)/6 + (s - 1)/2$ diversity cycles with 6 triples.*
(5) *There are $Q(n, s) := (n - 1)((n - 1)^2 + 3s - 1)/6$ diversity cycles in total.*
(6) *There are $P(n, s) := (s{-}1)!\,((n{-}s)/2)!\,2^{(n-s)/2}$ automorphisms of $\langle U; f, \{e\}\rangle$.*

Definition 4.5 (d_t)**.** For each $t \in \mathbb{N}$, let d_t be the following (first-order) property: For all $a_1, \dots, a_t, b_1, \dots, b_t \neq e$, there is some $c \neq e$ such that $[a_1, b_1, c], \dots, [a_t, b_t, c]$ are all consistent.

Proposition 4.6. (1) *Almost all labelled integral structures in* FAS *are rigid.*
(2) *Almost all labelled integral structures in* FAS *satisfy d_t, for each fixed $t \in \mathbb{N}$.*
(3) *If $t \geqslant 2$ and $\mathbf{U} \in \mathsf{FAS}$ is integral and satisfies d_t, then $\mathbf{Cm}(\mathbf{U}) \in \mathsf{RA}_n$.*
(4) *If ϵ is the conjunction of some finite set of equations that hold in all members of* RRA*, then ϵ will hold in almost all finite members of* NA*. In particular, almost all members of* NA *belong to* SA*,* RA*, and* RA_n*, for each fixed $n \geqslant 5$.*

Proposition 4.7. *Let $F(n, s)$ be the number of n-atom integral relation algebras with s atoms satisfying $x^{\smile} = x$. Then $2^{Q(n,s)}/P(n, s)$ is an asymptotic formula for $F(n, s)$,* *i.e., for all $\varepsilon > 0$, there is some integer N such that if $n > N$ and $1 \leqslant s \leqslant n$ is even, then we must have*

$$\left| 1 - \frac{F(n,s)P(n,s)}{2^{Q(n,s)}} \right| < \varepsilon.$$

The same statement holds for nonassociative relation algebras in which e is an atom, integral semiassociative relation algebras, and integral relation algebras from RA_m, for each fixed $m > 3$.

Now we have all of the tools we need to prove the main result of this section. Firstly, we will prove the equivalent result for atom-type structures.

Theorem 4.8. *Almost all members of* FAS *are in* FSIAS.

Proof. Let $n \geqslant 5$, let U be an n-element set, and assume that $1 \leqslant i < n$. Clearly, there are $\binom{n}{i}$ ways to select i identity atoms from U. Assume that $0 \leqslant p \leqslant \lfloor (n-i)/2 \rfloor$. Clearly, there are at most $\binom{n-i}{2}^p$ involutions of U with p non-fixed pairs of elements, i.e., sets of the form $\{u, f(u)\}$ with $u \neq f(u)$, since $\binom{n-i}{2}^p$ is the number of p *independent* selections of 2-element sets of diversity atoms. Based on Proposition 1.30(2), there are $(2^i - 1)^n$ possible ways of selecting identity cycles to define an member of FAS, since each element of U must appear in at least one of the i cycles of the given form. Lastly, by Proposition 4.4(5), there are $2^{Q(n-i+1,n-i+1-2p)}$ ways to pick diversity cycles; the number of choices of diversity cycles in a $(n-i+1)$-element structure with $|I| = 1$ and a n-element structure such that $|I| = i$ and $|U \setminus I| = n - i$ is clearly the same. Hence, by Proposition 1.22(5), the fraction of members of FAS with universe $\{1, \ldots, n\}$ that belong to FSIAS is bounded below by

$$\frac{n 2^{Q(n,n)}}{\sum_{i=1}^{n} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} \binom{n}{i} \binom{n-i}{2}^p (2^i - 1)^n 2^{Q(n-i+1,n-i+1-2p)}}$$
$$= \frac{1}{\left(\sum_{i=1}^{n} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} \binom{n}{i} \binom{n-i}{2}^p (2^i - 1)^n 2^{Q(n-i+1,n-i+1-2p) - Q(n,n)} \right) / n}$$

Now, we have

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^{n} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} \binom{n}{i} \binom{n-i}{2}^p (2^i - 1)^n 2^{Q(n-i+1,n-i+1-2p) - Q(n,n)} \\ = &\frac{1}{n} \sum_{p=0}^{\lfloor (n-1)/2 \rfloor} n \binom{n-1}{2}^p 2^{Q(n,n-2p) - Q(n,n)} \\ &\quad + \frac{1}{n} \sum_{i=2}^{n-1} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} \binom{n}{i} \binom{n-i}{2}^p (2^i - 1)^n 2^{Q(n-i+1,n-i+1-2p) - Q(n,n)}. \end{aligned}$$

Firstly, we show that the first term has a limit less than 1. Clearly, we have

$$\binom{n-1}{2}^p \leqslant \left(\frac{n^2}{2} \right)^p.$$

Now,

$$\begin{aligned}
&Q(n, n-2p) - Q(n,n) \\
&= \frac{1}{6}(n-1)((n-1)^2 + 3(n-2p) - 1) - \frac{1}{6}(n-1)((n-1)^2 + 3n - 1) \\
&= \frac{1}{6}(n-1)((n-1)^2 + 3n - 6p - 1 - ((n-1)^2 + 3n - 1)) \\
&= \frac{1}{6}(n-1)(3n - 6p - 3n) \\
&= (1-n)p,
\end{aligned}$$

hence

$$\begin{aligned}
\left(\frac{n^2}{2}\right)^p 2^{Q(n,n-2p)-Q(n,n)} &= \left(\frac{n^2}{2}\right)^p 2^{(1-n)p} \\
&= \left(\frac{n^2}{2^n}\right)^p.
\end{aligned}$$

Since $n \geqslant 5$, we have $0 < n^2/2^n < 1$, so the formula for a geometric sum gives

$$\begin{aligned}
\frac{1}{n}\sum_{p=0}^{\lfloor (n-1)/2 \rfloor} n\binom{n-1}{2}^p 2^{Q(n,n-2p)-Q(n,n)} &\leqslant \sum_{p=0}^{\lfloor (n-1)/2 \rfloor} \left(\frac{n^2}{2^n}\right)^p \\
&= \frac{1-(n^2/2^n)^{\lfloor (n-1)/2 \rfloor + 1}}{1 - n^2/2^n}.
\end{aligned}$$

Clearly, both $n^2/2^n$ and $(n^2/2^n)^{\lfloor (n-1)/2 \rfloor + 1}$ tend to 0, so we have

$$\lim_{n\to\infty}\left(\frac{1-(n^2/2^n)^{\lfloor (n-1)/2 \rfloor + 1}}{1 - n^2/2^n}\right) = 1.$$

Now we will consider the second term. Define $S(m) := Q(m,m)$, for each $m \in \mathbb{N}$. We have

$$\begin{aligned}
S(m) &= \frac{1}{6}(m-1)((m-1)^2 + 3m - 1) \\
&= \frac{1}{6}(m-1)(m^2 - 2m + 1 + 3m - 1) \\
&= \frac{1}{6}(m-1)(m^2 + m) \\
&= \frac{1}{6}(m^3 - m), \qquad (1)
\end{aligned}$$

for all $m \in \mathbb{N}$. By the difference of cubes formula, i.e., $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$,

$$\begin{aligned}
S(n-i+1) - S(n) &= \frac{1}{6}((n-i+1)^3 - (n-i+1) - n^3 + n) \\
&= \frac{1}{6}((n-i+1-n)((n-i+1)^2 + n(n-i+1) + n^2) + i - 1) \\
&= \frac{1}{6}(-(i-1)(n^2 - 2(i-1)n + (i-1)^2 + n^2 - (i-1)n + n^2) + i - 1) \\
&= \frac{1}{6}(-3(i-1)n^2 + 3(i-1)^2 n - (i-1)^3 + i - 1).
\end{aligned}$$

In particular,

$$i = 2 \implies S(n-i+1) - S(n) = -\frac{1}{2}n^2 + \frac{1}{2}n, \tag{2}$$

$$i = 3 \implies S(n-i+1) - S(n) = -n^2 + 2n - 1, \tag{3}$$

$$i = 4 \implies S(n-i+1) - S(n) = -\frac{3}{2}n^2 + \frac{9}{2}n - 4. \tag{4}$$

If $1 \leqslant i < n$ and $1 \leqslant p \leqslant \lfloor (n-i)/2 \rfloor$, then we have

$$\lfloor (n-i)/2 \rfloor \leqslant n, \tag{5}$$

$$\binom{n}{i} \leqslant n^n, \tag{6}$$

$$(2^i - 1)^n \leqslant 2^{in}, \tag{7}$$

$$\begin{aligned} \binom{n-i}{2}^p &\leqslant (n^2)^n \\ &= n^{2n}, \end{aligned} \tag{8}$$

and

$$Q(n-i+1, n-i+1-2p) \leqslant S(n-i+1). \tag{9}$$

On the interval $[1, \infty)$, the function $x \mapsto (x^3 - x)/6$ is increasing, so by (1), $m \mapsto S(m)$ is an increasing sequence. Thus, $S(n-i+1)$ is maximised when $n-i+1$ is maximised, i.e., when i is mimimised. So, based on (4), we have that

$$4 \leqslant i \leqslant n \implies S(n-i+1) - S(n) \leqslant -\frac{3}{2}n^2 + \frac{9}{2}n - 4. \tag{10}$$

Combining these results, we find that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=2}^{n} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} \binom{n}{i}\binom{n-i}{2}^p (2^i-1)^n 2^{Q(n-i+1,n-i+1-2p)-Q(n,n)} \\
\leqslant{} &\frac{1}{n}\sum_{i=2}^{n} \sum_{p=0}^{\lfloor (n-i)/2 \rfloor} n^n n^{2n} (2^i-1)^n 2^{S(n-i+1)-S(n)} && (6,\ 8,\ 9) \\
\leqslant{} &\frac{1}{n}\sum_{i=2}^{n} n\, n^{3n} (2^i-1)^n 2^{S(n-i+1)-S(n)} && (5) \\
={} &\sum_{i=2}^{n} n^{3n} (2^i-1)^n 2^{S(n-i+1)-S(n)} \\
={} &n^{3n} 3^n 2^{-n^2/2+n/2} + n^{3n} 7^n 2^{-n^2+2n-1} + \sum_{i=4}^{n} n^{3n} (2^i-1)^n 2^{S(n-i+1)-S(n)} && (2,\ 3) \\
\leqslant{} &n^{3n} 3^n 2^{-n^2/2+n/2} + n^{3n} 7^n 2^{-n^2+2n-1} + \sum_{i=4}^{n} n^{3n} 2^{in} 2^{-3n^2/2+9n/2-4} && (7,\ 10) \\
\leqslant{} &n^{3n} 3^n 2^{-n^2/2+n/2} + n^{3n} 7^n 2^{-n^2+2n-1} + n^{3n} 2^{-3n^2/2+9n/2-4} \sum_{i=0}^{n} (2^n)^i.
\end{aligned}$$

Firstly, we have

$$n^{3n} 3^n 2^{-n^2/2+n/2} = 2^{3n\log_2(n) + \log_2(3)n - n^2/2 + n/2},$$

which clearly tends to 0. Similarly,

$$n^{3n}7^n2^{-n^2+2n+1} = 2^{3n\log_2(n)+\log_2(7)n-n^2+2n-1},$$

which tends to 0. As $n \geqslant 5$, we have $2^n > 1$. Using the formula for a geometric sum, we get

$$\begin{aligned} n^{3n}2^{-3n^2/2+9n/2-4}\sum_{i=0}^{n}(2^n)^i &= n^{3n}2^{-3n^2/2+9n/2-4}\frac{2^{n(n+1)}-1}{2^n-1} \\ &= 2^{3n\log_2(n)-n^2/2+11n/2-4}\frac{2^{-n^2-n}(2^{n^2+n}-1)}{2^n-1} \\ &= 2^{3n\log_2(n)-n^2/2+11n/2-4}\frac{1-2^{-n^2-n}}{2^n-1}. \end{aligned}$$

It is clear that $2^{3n\log_2(n)-n^2/2+9n/2-4}$ tends to 0 and $(1-2^{-n^2-n})/(2^n-1)$ tends to 0, hence the term above has limit 0. Combining these results with basic limits, we get

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{p=1}^{\lfloor (n-i)/2\rfloor}\binom{n}{i}\binom{n-i}{2}^p(2^i-1)^n2^{Q(n-i+1,n-i+1-2p)-Q(n,n)}\right) \leqslant 1,$$

so

$$\lim_{n\to\infty}\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{p=1}^{\lfloor (n-i)/2\rfloor}\binom{n}{i}\binom{n-i}{2}^p(2^i-1)^n2^{Q(n-i+1,n-i+1-2p)-Q(n,n)}\right)/n}\right) \geqslant 1.$$

The fraction of members of FAS with universe $\{1,\dots,n\}$ in FSIAS is always below 1, and this fraction is an upper bound for the corresponding term of this sequence, so

$$\lim_{n\to\infty}\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{p=1}^{\lfloor (n-i)/2\rfloor}\binom{n}{i}\binom{n-i}{2}^p(2^i-1)^n2^{Q(n-i+1,n-i+1-2p)-Q(n,n)}\right)/n}\right) \leqslant 1.$$

Combining these inequalities, we find that the limit of this sequence is equal to 1. The fraction of members of FAS with universe $\{1,\dots,n\}$ in FSIAS is always below 1, and above the corresponding term of the sequence above, so by the Squeeze Theorem, the fraction of members of FAS with universe $\{1,\dots,n\}$ in FSIAS tends to 1. Now, combining this observation, Proposition 1.28, Proposition 4.2, and Proposition 4.6(1), we find that almost all members of FAS belong to FSIAS, which is what we wanted. □

Combining this with Proposition 4.6(2), we obtain the following.

Theorem 4.9. *Almost all finite members of* NA *are symmetric and belong to* IRA *and* RA_n*, for each fixed* $n \geqslant 5$.

Further, using Proposition 1.30, Proposition 1.28, and Proposition 4.7, we obtain the following result, which gives the promised asymptotic formula for relation algebras.

Theorem 4.10. $2^{(n^3-n)/6}/(n-1)!$ *is an asymptotic formula for the number of* n*-atom nonassociative relation algebras. This formula also applies if we add the assumption of semiassociativity, associativity, being a member of* RA_m*, for some fixed* $m > 3$*, or* e *being an atom.*

4.3. A Fraïssé limit and its applications

In this section, we will show that FAS has a 0–1 law. We follow the approach outlined by Burris and Bell in [**12**]. For the completeness component of this method, we will make use of the Fraïssé limit construction that we covered in Section 1.3. The majority of this section deals with the class FSIAS_e, since it has a Fraïssé limit; the results we obtain for this class will subsequently be lifted to FSIAS and FAS. Firstly, we will show that the limit of FSIAS_e does indeed exist.

Lemma 4.11. FSIAS_e *has a Fraïssé limit.*

Proof. Based on Theorem 1.41, it is enough to show that FSIAS_e has the HP, JEP, and AP.

By definition, FSIAS_e is the class of all finite members of a universal class. Thus, FSIAS_e is closed under forming substructures, so FSIAS_e clearly has the HP.

For the AP, let $\mathbf{S}, \mathbf{V}, \mathbf{W} \in \mathsf{FSIAS}_e$ and let $\mu\colon \mathbf{S} \to \mathbf{V}$ and $\nu\colon \mathbf{S} \to \mathbf{W}$ be embeddings. Without loss of generality, we can assume that $V \cap W = S$, and that μ and ν are inclusion maps. Thus, we can define $\mathbf{U} := \langle U; f^{\mathbf{U}}, e, T^{\mathbf{U}}\rangle$, where $U := V \cup W$, $f^{\mathbf{U}}$ is given by

$$f^{\mathbf{U}}(x) = \begin{cases} f^{\mathbf{V}}(x) & \text{if } x \in V \\ f^{\mathbf{W}}(x) & \text{if } x \in W, \end{cases}$$

and $T^{\mathbf{U}} := T^{\mathbf{V}} \cup T^{\mathbf{W}}$. Let $a, b, c \in U$ and assume that $(a, b, c) \in T^{\mathbf{U}}$. By construction,

$$(a, b, c \in V \text{ and } (a, b, c) \in T^{\mathbf{V}}) \text{ or } (a, b, c \in W \text{ and } (a, b, c) \in T^{\mathbf{W}}).$$

In the first case, we have $(f^{\mathbf{U}}(a), c, b), (c, f^{\mathbf{U}}(b), a) \in T^{\mathbf{U}}$, since $T^{\mathbf{V}} \subseteq T^{\mathbf{U}}$, $f^{\mathbf{U}}{\restriction}_V = f^{\mathbf{V}}$, and $\mathbf{V}$ satisfies (IP). Similarly, $(f^{\mathbf{U}}(a), c, b), (c, f^{\mathbf{U}}(b), a) \in T^{\mathbf{U}}$ in the second case, hence $\mathbf{U}$ satisfies (IP). Now, let $a \in U$. If $a \in V$, then we have $(a, e, a) \in T^{\mathbf{U}}$, since $T^{\mathbf{V}} \subseteq T^{\mathbf{U}}$ and $\mathbf{V}$ satisfies (II). Similarly, we have $(a, e, a) \in T^{\mathbf{U}}$ when $a \in W$. By construction, $U = V \cup W$, it follows that we have $(a, e, a) \in T^{\mathbf{U}}$ in every case. Lastly, assume that $(a, e, b) \in T^{\mathbf{U}}$. By construction,

$$(a, b \in V \text{ and } (a, e, b) \in T^{\mathbf{V}}) \text{ or } (a, b \in W \text{ and } (a, e, b) \in T^{\mathbf{W}}).$$

As $\mathbf{V}$ and $\mathbf{W}$ satisfy (II), we have $x = y$, so (II) holds. As $\mathbf{V}$ and $\mathbf{W}$ satisfy $f(x) \approx x$, it follows that $f^{\mathbf{V}}$ and $f^{\mathbf{W}}$ are identity maps. By construction, $f^{\mathbf{U}}$ is an identity map, so $\mathbf{U} \models f(x) \approx x$. By definition, $U = V \cup W$, so $|U| \leqslant |V| + |W|$. Thus, U is finite. Based the above results, we have $\mathbf{U} \in \mathsf{FSIAS}_e$. Clearly, the inclusion maps $\imath_V\colon V \to U$ and $\imath_W\colon W \to U$ are embeddings such that $\imath_V \circ \mu = \imath_W \circ \nu$. Combining these results, we find that FSIAS_e has the AP, which is what we wanted to show.

It is clear that FSIAS_e contains a trivial structure that embeds into all $\mathbf{A} \in \mathsf{FSIAS}_e$, so the JEP follows from the AP for this class.

Based on the above, FSIAS_e has the HP, JEP and AP, as required. ☐

This result allows us to make the following definition.

Definition 4.12 ($\mathbf{L}_{\mathsf{SI}}$, T_{SI}, and S_{SI})**.** Let $\mathbf{L}_{\mathsf{SI}}$ be a Fraïssé limit of the class FSIAS_e, let T_{SI} be the first-order theory of $\mathbf{L}_{\mathsf{SI}}$, and let S_{SI} be the almost-sure theory of FSIAS_e.

The members of FSIAS_e are symmetric, so subsets generate at most one additional element, namely e. By Proposition 1.44, we have the following.

Corollary 4.13. *T_{SI} is $\aleph_0$-categorical and has quantifier elimination.*

Next we introduce the sentences that Bell and Burris call extension axioms in [**12**]. These sentences essentially assert that a substructure can be extended by a single point in all possible ways. We define $\neg^c$ in the note on notation.)

Definition 4.14. Let A_{SI} be the set of first-order sentences of the form

$$\forall x_1, \dots, x_n \colon \bigcurlywedge_{i=1}^{n} x_i \not\approx e \to \exists y \colon y \not\approx e \curlywedge \left(\bigcurlywedge_{i=1}^{n} y \not\approx x_i \right) \curlywedge \neg^c T(y,y,y) \curlywedge$$
$$\left(\bigcurlywedge_{i=1}^{n} \neg^{c_i} T(x_i, y, y) \right) \curlywedge \left(\bigcurlywedge_{1 \leqslant i \leqslant j \leqslant n} \neg^{c_{ij}} T(x_i, x_j, y) \right),$$

where $n \in \omega$ and $c, c_i, c_{ij} \in \{0,1\}$, for all $1 \leqslant i \leqslant j \leqslant n$.

Next, we show that these sentences (effectively) axiomatise $\mathbf{L}_{\mathsf{SI}}$.

Lemma 4.15. *Let $\mathbf{L}$ be countable integral atom-type structure that is a model of* (II)*,* (IP)*, and $f(x) \approx x$. Then we have $\mathbf{L} \cong \mathbf{L}_{\mathsf{SI}}$ if and only if $\mathbf{L} \models A_{\mathsf{SI}}$.*

Proof. For the forward direction, say $\mathbf{L} \cong \mathbf{L}_{\mathsf{SI}}$. Then $\mathbf{L}$ is a Fraïssé limit of FSIAS_e, so the age of $\mathbf{L}$ is FSIAS_e and $\mathbf{L}$ is ultrahomogeneous. Let $n \in \omega$, let $c, c_i, c_{ij} \in \{0,1\}$, for all $1 \leqslant i \leqslant j \leqslant n$, and let $u_1, \dots, u_n \in L \setminus \{e^{\mathbf{L}}\}$. Let $\mathbf{U}$ be the substructure of $\mathbf{L}$ generated by $U := \{u_1, \dots, u_n\}$. Now, fix some $v \notin U$ and let $\mathbf{V} := \langle V; f^{\mathbf{V}}, e^{\mathbf{V}}, T^{\mathbf{V}} \rangle$, where $V := U \cup \{e^{\mathbf{F}}, v\}$, $f^{\mathbf{V}} = \mathrm{id}_V$, $e^{\mathbf{V}} = e^{\mathbf{L}}$, and $T^{\mathbf{V}}$ is given by

$$T^{\mathbf{U}} \cup [v, e^{\mathbf{L}}, v] \cup [v,v,v] \cup \left(\bigcup \{ [u_i, v, v] \mid c_i = 0 \} \right) \cup \left(\bigcup \{ [u_i, u_j, v] \mid c_{ij} = 1 \} \right)$$

if $c = 0$ and

$$T^{\mathbf{U}} \cup [v, e^{\mathbf{L}}, v] \cup \left(\bigcup \{ [u_i, v, v] \mid c_i = 0 \} \right) \cup \left(\bigcup \{ [u_i, u_j, v] \mid c_{ij} = 1 \} \right)$$

if $c = 1$. As the age of $\mathbf{L}$ is FSIAS_e, we must have $\mathbf{U} \in \mathsf{FSIAS}_e$, so we have $\mathbf{V} \in \mathsf{FSIAS}_e$. Again, the age $\mathbf{L}$ is FSIAS_e, so there is a substructure $\mathbf{W}$ of $\mathbf{L}$ that is isomorphic to $\mathbf{V}$. Now, let $\mu \colon \mathbf{V} \to \mathbf{W}$ be such an isomorphism. Then $\mu \circ \imath_U$ is an isomorphism from $\mathbf{U}$ to the substructure of $\mathbf{L}$ generated by $\mu[U]$. We saw above that $\mathbf{L}$ is ultrahomogeneous, so $\mu \circ \imath_U$ extends to an automorphism of $\mathbf{L}$, say ν. Then, by construction, $\nu^{-1}(\mu(v))$ is a suitable witness to the sentence from A_{SI} defined by n, c, and each c_i, and c_{ij}, when choosing $x_i = u_i$, for each $1 \leqslant i \leqslant n$. The choice of parameters was arbitrary, so this tells us that $\mathbf{L} \models A_{\mathsf{SI}}$.

For the converse, assume that $\mathbf{L} \models A_{\mathsf{SI}}$. By Theorem 1.41, to show that $\mathbf{L} \cong \mathbf{L}_{\mathsf{SI}}$, it is enough to show that FSIAS_e is the age of $\mathbf{L}$ and that $\mathbf{L}$ is ultrahomogeneous. Since $\mathbf{L}$ is a symmetric model of (II) and (IP), the age of $\mathbf{L}$ is a subset of FSIAS_e. Assume, for a contradiction, that the age of $\mathbf{L}$ is a proper subclass of FSIAS_e. Let $\mathbf{U}$ be a minimal member of FSIAS_e that is not in the age of $\mathbf{L}$. It is clear that $|U| > 1$. Now, let $u \in U \setminus \{e^{\mathbf{U}}\}$ and let $\mathbf{V}$ be the substructure of $\mathbf{U}$ generated by $V := U \setminus \{u\}$. By our minimality assumption, $\mathbf{V}$ embeds into $\mathbf{L}$. Let $\mu\colon \mathbf{V} \to \mathbf{L}$ be an embedding, let $n := |V| - 1$, let $\{v_1, \ldots, v_n\}$ be an enumeration of $V \setminus \{e^{\mathbf{V}}\}$, let

$$c := \begin{cases} 0 & \text{if } [u,u,u] \subseteq T^{\mathbf{U}} \\ 1 & \text{if } [u,u,u] \nsubseteq T^{\mathbf{U}}, \end{cases}$$

let

$$c_i := \begin{cases} 0 & \text{if } [v_i,u,u] \subseteq T^{\mathbf{U}} \\ 1 & \text{if } [v_i,u,u] \nsubseteq T^{\mathbf{U}}, \end{cases}$$

for all $1 \leqslant i \leqslant n$, and let

$$c_{ij} := \begin{cases} 0 & \text{if } [v_i,v_j,u] \subseteq T^{\mathbf{U}} \\ 1 & \text{if } [v_i,v_j,u] \nsubseteq T^{\mathbf{U}}, \end{cases}$$

for all $1 \leqslant i \leqslant j \leqslant n$. As $\mathbf{L} \models A_{\mathsf{SI}}$, $\mathbf{L}$ satisfies the sentence defined by c and each c_i and c_{ij}, so there is a witness, say y, for the choice of $x_i = \mu(v_i)$, for all $1 \leqslant i \leqslant n$. By construction, the substructure of $\mathbf{L}$ generated by $\mu[V] \cup \{y\}$ is isomorphic to $\mathbf{U}$. It follows that $\mathbf{U}$ embeds into $\mathbf{L}$, so $\mathbf{U}$ is in the age of $\mathbf{L}$, contradicting our assumption. Thus, FSIAS_e is in fact the age of $\mathbf{L}$, as claimed.

Based on Lemma 1.40, it will be enough to show that $\mathbf{L}$ is weakly homogeneous. Let $\mathbf{U} \leqslant \mathbf{V}$ be finitely generated substructures of $\mathbf{L}$ and let $\mu\colon \mathbf{U} \to \mathbf{L}$ be an embedding. Observe that since $\mathbf{L}$ is symmetric, generating sets and subuniverses coincide, except for the possible addition of the constant e. If $\mathbf{U} = \mathbf{V}$, then we are clearly done. Now, assume that $\mathbf{U} \neq \mathbf{V}$ and fix some $v \in V \setminus U$. Let $n := |U| - 1$, let $\{u_1, \ldots, u_n\}$ be an enumeration of $U \setminus \{e^{\mathbf{U}}\}$, let

$$c := \begin{cases} 0 & \text{if } [v,v,v] \subseteq T^{\mathbf{V}} \\ 1 & \text{if } [v,v,v] \nsubseteq T^{\mathbf{V}}, \end{cases}$$

let

$$c_i := \begin{cases} 0 & \text{if } [u_i,v,v] \subseteq T^{\mathbf{V}} \\ 1 & \text{if } [u_i,v,v] \nsubseteq T^{\mathbf{V}}, \end{cases}$$

for all $1 \leqslant i \leqslant n$, and let

$$c_{ij} := \begin{cases} 0 & \text{if } [u_i,u_j,v] \subseteq T^{\mathbf{V}} \\ 1 & \text{if } [u_i,u_j,v] \nsubseteq T^{\mathbf{V}}, \end{cases}$$

for all $1 \leqslant i \leqslant j \leqslant n$. As $\mathbf{L} \models A_{\mathsf{SI}}$, $\mathbf{L}$ satisfies the sentence defined by c and each c_i and c_{ij}, so there is a witness, say y, for the choice of $x_i = \mu(v_i)$, for all $1 \leqslant i \leqslant n$. By construction, the map $\nu\colon U \cup \{v\} \to L$ given by

$$\nu(x) = \begin{cases} \mu(x) & \text{if } x \in U \\ y & \text{if } x = v \end{cases}$$

embeds the substructure of $\mathbf{V}$ generated by $U \cup \{v\}$ into $\mathbf{L}$. By assumption, $\mathbf{V}$ is finite, hence μ can be extending to an embedding $\nu\colon \mathbf{V} \to \mathbf{L}$ by repeating this construction. Thus, $\mathbf{L}$ is weakly homogeneous, as claimed.

Combining these results, we find that $\mathbf{L} \cong \mathbf{L}_{\mathsf{SI}}$, so the two statements are equivalent, which is what we wanted to show. □

Based on Theorem 1.41, we have the following.

Corollary 4.16. *Together,* (II), (IP), $f(x) \approx x$, *and* A_{SI} *form a* $\aleph_0$*-categorical, and therefore complete, theory.*

Next, we show that these sentences belong to S_{SI}.

Lemma 4.17. $A_{\mathsf{SI}} \subseteq S_{\mathsf{SI}}$.

Proof. Let $n \in \mathbb{N}$, let $m \in \omega$, and let $c, c_i, c_{ij} \in \{0, 1\}$, for all $1 \leqslant i \leqslant j \leqslant m$. Clearly, if we are given non-identity elements $x_1, \ldots, x_m, y \in \{1, \ldots, n\}$ with $y \neq x_i$, for all $1 \leqslant i \leqslant n$, then at most

$$1 + m + (m^2 + m)/2 = (m^2 + 3m + 2)/2$$

cycles must be included for the sentence from A_{SI} given by these parameters to hold. So, the fraction of structures failing this sentence must be below $1 - 2^{-(m^2+3m+2)/2}$. There are $(n-1)^m$ ways to select $x_1, \ldots, x_m$, and then $n - m - 1$ ways to select y once given $x_1, \ldots, x_m$, so the fraction of structures not satisfying the sentence from A_{SI} is bounded above by

$$(n-1)^m (1 - 2^{-(m^2+3m+2)/2})^{n-m-1}.$$

Clearly, we have $n - m - 1 < n$ and

$$\begin{aligned} (n-1)^m &\leqslant n^m \\ &= 2^{m \log_2(n)}, \end{aligned}$$

so this quantity is below $2^{m \log_2(n) + n \log_2(1 - 2^{-(m^2+3m+2)/2})}$. Since $1 - 2^{-(m^2+3m+2)/1} < 1$, and since $m \in \omega$ is fixed, it follows that $\log_2(1 - 2^{-(m^2+3m+2)/2})$ is fixed and negative, hence

$$\lim_{n \to \infty} 2^{m \log_2(n) + n \log_2(1 - 2^{-(m^2+3m+2)/2})} = 0.$$

By the Squeeze Principle, the fraction of structures that do not satisfy the sentence defined by the given parameters tends to 0. It follows that we must have $A_{\mathsf{SI}} \subseteq S_{\mathsf{SI}}$, as claimed. □

So, based on Proposition 1.41 and Lemma 4.15, we have the following.

Corollary 4.18. *The theory S_{SI} is $\aleph_0$-categorical, and is therefore complete. Thus,* FSIAS_e *has a* 0–1 *law.*

Next, we translate this result to one on FSIAS.

Corollary 4.19. FSIAS *has a* 0–1 *law.*

Proof. Let $\mathbf{U} \in \mathsf{FSIAS}$, let $e^{\mathbf{U}}$ be the unique element of I, let $\mathbf{U}_e := \langle U; f^{\mathbf{U}}, e^{\mathbf{U}}, T^{\mathbf{U}}\rangle$, let σ be a $\{f, T, I\}$-sentence, and let σ_e denote the $\{f, e, T\}$-sentence obtained from φ by replacing all occurences of $I(x)$, for some variable x, with $x \approx e$. By construction, $\mathbf{U} \models \varphi$ if and only if $\mathbf{U}_e \models \varphi_e$. Hence, based on Proposition 1.28 and Corollary 4.18, the class FSIAS has a 0–1 law, as required. □

Theorem 4.8 allows us to translate this result to FAS, giving us the main result of this section.

Theorem 4.20. FAS *has a* 0–1 *law.*

4.4. Dead ends

In this section, we outline some results that are related to the problem mentioned in Section 4.1, namely, the problem of determining whether or not almost all nonassociative relation algebras are strongly representable.

One approach to solving this problem would be to make use of the following result, which is Theorem 437 of Maddux [**70**].

Proposition 4.21. *Let $\mathbf{A} \in \mathsf{INA}$ be complete and atomic. If there is some $a \in \mathrm{At}(\mathbf{A})$ with $a \leqslant bc$, for all diversity atoms b and c, then $\mathbf{A} \in \mathsf{RRA}$. Further, if $|\mathrm{At}(\mathbf{A})| = \aleph_0$, then there is an embedding μ of $\mathbf{A}$ into $\mathbf{Re}(\omega^2)$ with $\bigcup_{b\in\mathrm{At}(\mathbf{A})} \mu(b) = \omega^2$.*

However, in [**2**], Alm shows that almost all nonassociative algebras do not have such an atom, which is usually called a *flexible atom*. Similarly, the following result, which is Theorem 422 of [**70**], would be another approach.

Proposition 4.22. *Let $\mathbf{A} \in \mathsf{ISA}$. If any finite $S \subseteq A\setminus\{0\}$ with $\bigvee S = 1$ and $s\wedge t = 0$, for all distinct $s, t \in S$, contains an element s with $s^2 = 1$, then $\mathbf{A} \in \mathsf{RRA}$.*

To show that making use of this result cannot lead to a solution of this problem, we will use the following result.

Lemma 4.23. *Let $\mathbf{A} \in \mathsf{ISA}$ be finite. Then the following are equivalent:*

(1) *every $S \subseteq A \setminus \{0\}$ with $\bigvee S = 1$ and $s \wedge t = 0$, for all distinct $s, t \in S$, contains an element s with $s^2 = 1$;*
(2) *there is a diversity atom a with $a^2 = 1$.*

Proof. Firstly, assume that (1) holds. Since A is finite, it follows that $\bigvee \mathrm{At}(\mathbf{A}) = 1$. Since $a \wedge b = 0$, for all distinct $a, b \in \mathrm{At}(\mathbf{A})$, it is clear that (2) follows from (1).

Conversely, assume that (2) holds. Let $S \subseteq A \setminus \{0\}$ with $\bigvee S = 1$ and $s \wedge t = 0$, for all distinct $s, t \in S$. Since (2) holds, there is an atom a with $a^2 = 1$. As $\bigvee S = 1$, we have $a \leqslant s$, for some $s \in S$. Based on Proposition 1.22(1), we must have $a^2 \leqslant s^2$, so $s^2 = 1$ and (2) implies (1).

Thus, (1) and (2) are equivalent, as claimed. $\square$

Lemma 4.24. *Almost all finite members of* NA *fail to have an atom* a *with* $a^2 = 1$.

Proof. Let $n \geqslant 2$ and let $1 \leqslant s < n$. Based on Proposition 4.4(5), there are

$$n\binom{n-1}{s}2^{Q(n,n)-sn}$$

members $\mathbf{U}$ of FSIAS with $A = \{1, \ldots, n\}$ and s atoms whose square is 1 in $\mathbf{Cm}(\mathbf{U})$; first choose what is the unique element of I, then pick a s-element subset of $\mathrm{At}(\mathbf{A}) \setminus I$, then choose freely from the $Q(n, n) - sn$ cycles that are not the ones of the form $[a, a, b]$ that are forced to be included for the atoms to square to 1. Thus, there are

$$\sum_{s=1}^{n-1} n\binom{n-1}{s}2^{Q(n,n)-sn}$$

members of FSIAS with universe $\{1, \ldots, n\}$ and an atom a that satisfies $a^2 = 1$. Thus, the fraction of these structures is

$$\frac{\sum_{s=1}^{n-1} n\binom{n-1}{s}2^{Q(n,n)-sn}}{n2^{Q(n,n)}} = \sum_{s=1}^{n-1}\binom{n-1}{s}2^{-sn}.$$

Since $n \geqslant 2$, we have

$$\begin{aligned}\binom{n-1}{s} &\leqslant (n-1)^s \\ &= 2^{s\log_2(n-1)},\end{aligned}$$

hence

$$\sum_{s=1}^{n-1}\binom{n-1}{s}2^{-sn} \leqslant \sum_{s=1}^{n-1} 2^{s(\log_2(n-1)-n)}.$$

Since $n \geqslant 2$, we have $n \geqslant \log_2(n)$ and $\log_2(n) \geqslant \log_2(n-1)$, so $\log_2(n-1) - n \leqslant 0$. Combining these results, we find that

$$\begin{aligned}\sum_{s=1}^{n-1}\binom{n-1}{s}2^{-sn} &\leqslant (n-1)2^{\log_2(n-1)-n} \\ &= 2^{2\log_2(n-1)-n},\end{aligned}$$

which tends to 0, hence almost all labelled members of FSIAS have no such atom. Hence, from Theorem 4.8 and Proposition 4.6(2), almost all finite members of NA have no such atom, which is what we wanted to show. $\square$

Based on this result, not enough algebras have such an atom.

The other results in [**70**] that guarantee the strong representability of an algebra, namely Theorem 423, Theorem 424, Theorem 425, Theorem 426, Theorem 427, Theorem 431, Theorem 433, Theorem 532, Theorem 533, and the results in Section 77, are easily shown (with Theorem 4.9) to not hold in enough algebras to give a solution.

The corresponding problems for feeble and qualitative representability are of interest in their own right, and their solutions could lead to a negative solution or useful ideas for the strong representability problem. So, we will also look at these problems.

Based on Proposition 2.14, if $\mathbf{U}, \mathbf{V} \in \mathsf{FAS}$ such that $\mathbf{U} \leqslant \mathbf{V}$ and $\mathbf{Cm}(\mathbf{U}) \in \mathsf{FRA}$, then we have $\mathbf{Cm}(\mathbf{V}) \in \mathsf{FRA}$; simply use the same network. So, the following result, which is a special case of Theorem 10 of Maddux [**71**], along with Proposition 4.6(2), would give a possible approach to the feeble representability problem.

Definition 4.25. Let $\mathbf{A}$ be a relation algebra, let $a, b, c \in \mathrm{At}(\mathbf{A})$, and let $1 \leqslant n \leqslant 3$. We call $[a, b, c]$ a n-cycle if $|\{a, b, c\}| = n$.

Proposition 4.26. *Let $\mathbf{A} \in \mathsf{IRA}$ be finite and symmetric with no consistent 3-cycles. Then the following are equivalent:*

(1) $\mathbf{A} \in \mathsf{RRA}$*;*
(2) $\mathbf{A} \in \mathsf{RA}_5$.

We will see that almost none of these (labelled) algebras are even relation algebras. The following results are mostly folklore; we include proofs for completeness.

Lemma 4.27. *Let $\mathbf{A} \in \mathsf{INA}$. Then the following are equivalent:*

(1) $\mathbf{A} \in \mathsf{ISA}$*;*
(2) $a1 = 1$*, for every atom* $a \in A$*;*
(3) $x1 = 1$*, for every non-zero* $x \in A$.

Proof. Assume that (1) holds and that $a \in A$ is an atom. Since a is an atom, we have $0 < a \leqslant a1$, so $a1 \neq 0$. As $\mathbf{A}$ is symmetric, it follows that $\mathbf{A}$ is commutative. So, by the triangle laws,

$$\begin{aligned}(a1)'a &= (a1)'a \wedge 1 \\ &= 0,\end{aligned}$$

as $a1 \wedge (a1)' = 0$. Since $a \neq 0$, this implies that

$$\begin{aligned}ea = a &\nleqslant 0 \\ &= (a1)'a,\end{aligned}$$

hence $e \nleqslant (a1)'$. Thus, $e \wedge (a1)' = 0$, as a is an atom. This implies that $e \leqslant a1$, so

$$\begin{aligned} 1 &= e1 \\ &\leqslant (a1)1, \end{aligned}$$

which implies that $(a1)1 = 1$. As (1) holds, this implies that $a1 = 1$, so (1) implies (2).

Next, we will assume that (2) holds and let $x \in A$ with $x \neq 0$. Since $\mathbf{A}$ is atomic, there is an atom a with $a \leqslant x$. Thus,

$$\begin{aligned} 1 &= a1 \\ &\leqslant x1, \end{aligned}$$

which implies that $x1 = 1$. From this, it follows that (2) implies (3).

Assume that (3) holds and $x \in A$. If $x = 0$, then we have

$$\begin{aligned} (x1)1 &= (01)1 \\ &= 01 \\ &= x1. \end{aligned}$$

When $x \neq 0$, we have

$$\begin{aligned} (x1)1 &= 11 \\ &= 1 \\ &= x1, \end{aligned}$$

hence (3) implies (1).

Thus, (1), (2), and (3) are equivalent, which is what we wanted to show. □

Lemma 4.28. *Let $\mathbf{A} \in \mathsf{INA}$ be finite and symmetric. Then the following are equivalent:*

(1) $\mathbf{A} \in \mathsf{ISA}$*;*
(2) *for all atoms $a, b \in A$, there is an atom $c \in A$ with $b \leqslant ac$;*
(3) *for all distinct diversity atoms a and b, there is an atom c with $[a, b, c]$ consistent.*

Proof. Since $\mathbf{A}$ is finite, $\mathbf{A}$ has n diversity atoms, for some $n \in \omega$. Let $\{a_1, \ldots, a_n\}$ be enumeration of $\mathrm{At}(\mathbf{A})$. Since $\mathbf{A}$ is finite, $\mathbf{A}$ must be atomic, so by Lemma 4.27, we have $\mathbf{A} \in \mathsf{ISA}$ if and only if $a1 = 1$, for every atom a, i.e.,

$$\begin{aligned} 1 &= a1 \\ &= a(e \vee a_1 \vee \cdots \vee a_n) \\ &= ae \vee aa_1 \vee \cdots \vee aa_n, \end{aligned}$$

for every atom a. As $\mathbf{A}$ is finite, if $x \in A$, then we have $x = 1$ if and only if $b \leqslant x$, for every atom b, hence (1) and (2) are equivalent. Since $\mathbf{A}$ is symmetric, (2) and (3) are clearly equivalent, so we are done. □

Theorem 4.29. *Almost none of the labelled symmetric structures in* FSIAS *with no 3-cycles have complex algebras in* ISA.

Proof. Let $U := \{1, \ldots, n\}$ with $n \geqslant 3$, let $e \in U$, and assume that $a, b \in U \setminus \{e\}$ with $a \neq b$. The proportion of ternary relations T such that $\mathbf{Cm}(\langle U; T, \mathrm{id}_U, e\rangle) \in \mathsf{ISA}$ with no 3-cycles in which $[a, b, c]$ is consistent, for some $c \in U \setminus \{e\}$, is $1 - 1/4 = 3/4$, since the only possible choices for c are a and b when there are no consistent 3-cycles. Thus, the fraction of relations where this holds for all choices of a and b is $(3/4)^{\binom{n-1}{2}}$, so the fraction of labelled members of IAS with no 3-cycles that are symmetric where the condition (3) from Lemma 4.28 holds in its complex algebra is $(3/4)^{\binom{n-1}{2}}$. Clearly, $(3/4)^{\binom{n-1}{2}} \to 0$, hence almost none of these labelled structures are in ISA, which is what we wanted to show. □

Corollary 4.30. *Almost none of the labelled members of* FSIAS *have complex algebras in* RRA.

The proof of Proposition 4.6(1) in Maddux [**67**] does not extend to these algebras, so we cannot use Proposition 4.2 to extend this result to unlabelled structures.

4.5. Open problems

In Section 4.2, we saw that almost all members of NA are symmetric and integral members of RA_n, for each fixed $n \geqslant 3$. However, as we mentioned above in Section 4.4, it is not known whether or not almost all of these structures are strongly, qualitatively, or even feebly representable. Thus, we restate the following problem from Maddux [**68**] and Hirsch and Hodkinson [**36**].

Problem 11. Determine whether almost all members of NA are members of FRA, QRA, or RRA.

Earlier, in Section 4.3, we also saw that the class FAS has a (first-order) 0–1 law. However, this result does not extend to the class of all finite members of NA. Thus, the problem of determining whether or not such a law exists suggests itself. Further, one may also look at the subclasses of NA that we defined above.

Problem 12. Determine whether or not NA, FRA, QRA, or RRA have 0–1 laws.

Something ends, something begins

The study of relation-type algebras began as an offshoot of the calculus of relations, and has since developed into one of the most widely studied areas of algebraic logic. Much like many other classes of algebras that are associated with some form of logic, the subvariety lattices of these algebras have attracted some interest from researchers. Probabilistic topics have attracted interest, and tie into an important open problem. This thesis investigates the intersection of these two areas and two notions of representability, feeble and qualitative representability, that were introduced recently.

Chromatic algebras can be defined as ones with all or no cycles of a given size. The study of the feeble and qualitative representability of Ramsey algebras and algebras with only 3-cycles is completed in this thesis, and the qualitative representability of Lyndon algebras are the only remaining case for these notions of representability. The strong representability of Ramsey algebras and Lyndon algebras remain significant open problems.

The lattice of subvarieties of the variety of relation algebras has three atoms, the covers of two of which have been completely classified. The problem of finding all covers of the remaining atom is one of the main open problems in relation algebra. Up to cardinality, we solved the corresponding problem for nonassociative relation algebras, semiassociative relation algebras, feebly representable algebras, and qualitatively representable algebras. Our methods do not translate to relation algebras or strongly representable relation algebras, but are quite interesting in their own right.

We showed that almost all finite nonassociative relation algebras are symmetric integral relation algebras, and derived a simple formula counting these structures. We showed that the atom structures of nonassociative relation algebras have a 0–1 law, and discussed the problem of showing that almost all algebras are (some kind of) representable. The problem remains open, but these results may be useful in solving it.

It is the hope of the author that this work shines a light on the appeal of feeble and qualitative representability and sparks interest in these intriguing open problems, some of which have perhaps been neglected for too long.

Appendix A: Finite algebras that generate covers of A_3 in $\mathbf{\Lambda}_{\mathsf{SA}}$

Here we give a list of atom tables (and consistent cycles) of finite algebras in SA that generate covers of A_3. Here the set of atoms is of the form $\{0, \ldots, n\}$ with $n \in \mathbb{N}$, and 0 is always the identity element; this determines its products, so we omit them. Each entry of these tables lists the atoms that are below the corresponding product. These structures were found using code that was written and run by Tomasz Kowalski. This list is not an exhaustive list of the structures that were found.

$\cdot$	1	2	3
1	$0, 2$	$1, 2, 3$	$2, 3$
2	$1, 2, 3$	$0, 1$	1
3	$2, 3$	1	$0, 1$

$$[0, 0, 0], [0, 1, 1], [0, 2, 2], [0, 3, 3],$$
$$[1, 1, 2],$$
$$[1, 2, 2], [1, 2, 3],$$
$$[1, 3, 3].$$

$\cdot$	1	2	3	4
1	$0, 3$	$2, 3, 4$	$1, 2, 3, 4$	$2, 3, 4$
2	$2, 3, 4$	$0, 1, 2$	1	1
3	$1, 2, 3, 4$	1	$0, 1$	1
4	$2, 3, 4$	1	1	$0, 1$

$$[0, 0, 0], [0, 1, 1], [0, 2, 2], [0, 3, 3], [0, 4, 4],$$
$$[1, 1, 3],$$
$$[1, 2, 2], [1, 2, 3], [1, 2, 4],$$
$$[1, 3, 3], [1, 3, 4],$$
$$[1, 4, 4],$$
$$[2, 2, 2].$$

$\cdot$	1	2	3	4	5
1	$0,4$	$4,5$	$3,4,5$	$1,2,3,4,5$	$2,3,4,5$
2	$4,5$	$0,3$	2	1	1
3	$3,4,5$	2	$0,1$	1	1
4	$1,2,3,4,5$	1	1	$0,1$	1
5	$2,3,4,5$	1	1	1	$0,1$

$$
\begin{gathered}
[0,0,0],[0,1,1],[0,2,2],[0,3,3],[0,4,4],[0,5,5],\\
[1,1,4],\\
[1,2,4],[1,2,5],\\
[1,3,3],[1,3,4],[1,3,5],\\
[1,4,4],[1,4,5],\\
[1,5,5],\\
[2,2,3].
\end{gathered}
$$

$\cdot$	1	2	3	4	5	6
1	$0,3,5$	$5,6$	$1,3,4,5,6$	$3,4,5,6$	$1,2,3,4,5,6$	$2,3,4,5,6$
2	$5,6$	$0,3,4$	2	2	1	1
3	$1,3,4,5,6$	2	$0,1$	1	1	1
4	$3,4,5,6$	2	1	$0,1$	1	1
5	$1,2,3,4,5,6$	1	1	1	$0,1$	1
6	$2,3,4,5,6$	1	1	1	1	$0,1$

$$
\begin{gathered}
[0,0,0],[0,1,1],[0,2,2],[0,3,3],[0,4,4],[0,5,5],[0,6,6],\\
[1,1,3],[1,1,5],\\
[1,2,5],[1,2,6],\\
[1,3,3],[1,3,4],[1,3,5],[1,3,6],\\
[1,4,4],[1,4,5],[1,4,6],\\
[1,5,5],[1,5,6],\\
[1,6,6],\\
[2,2,3],[2,2,4].
\end{gathered}
$$

$\cdot$	1	2	3	4	5	6	7
1	$0,4,6$	$3,6,7$	$2,3,4,5,6,7$	$1,3,4,5,6,7$	$3,4,5,6,7$	$1,2,3,4,5,6,7$	$2,3,4,5,6,7$
2	$3,6,7$	$0,2,3,4,5$	$1,2$	2	2	1	1
3	$2,3,4,5,6,7$	$1,2$	$0,1$	1	1	1	1
4	$1,3,4,5,6,7$	2	1	$0,1$	1	1	1
5	$3,4,5,6,7$	2	1	1	$0,1$	1	1
6	$1,2,3,4,5,6,7$	1	1	1	1	$0,1$	1
7	$2,3,4,5,6,7$	1	1	1	1	1	$0,1$

$$[0,0,0],[0,1,1],[0,2,2],[0,3,3],[0,4,4],[0,5,5],[0,6,6],[0,7,7],$$
$$[1,1,4],[1,1,6],$$
$$[1,2,3],[1,2,6],[1,2,7],$$
$$[1,3,3],[1,3,4],[1,3,5],[1,3,6],[1,3,7],$$
$$[1,4,4],[1,4,5],[1,4,6],[1,4,7],$$
$$[1,5,5],[1,5,6],[1,5,7],$$
$$[1,6,6],[1,6,7],$$
$$[1,7,7],$$
$$[2,2,2],[2,2,3],[2,2,4],[2,2,5].$$

$\cdot$	1	2	3	4	5	6	7	8
1	$0,3,5,7$	$3,4,7,8$	$1,2,3,4,5,6,7,8$	$2,3,4,5,6,7,8$	$1,3,4,5,6,7,8$	$3,4,5,6,7,8$	$1,2,3,4,5,6,7,8$	$2,3,4,5,6,7,8$
2	$3,4,7,8$	$0,2,3,4,5,6$	$1,2$	$1,2$	2	2	1	1
3	$1,2,3,4,5,6,7,8$	$1,2$	$0,1$	1	1	1	1	1
4	$2,3,4,5,6,7,8$	$1,2$	1	$0,1$	1	1	1	
5	$1,3,4,5,6,7,8$	2	1	1	$0,1$	1	1	
6	$3,4,5,6,7,8$	2	1	1	1	$0,1$	1	
7	$1,2,3,4,5,6,7,8$	1	1	1	1	1	$0,1$	
8	$2,3,4,5,6,7,8$	1	1	1	1	1	1	$0,1$

$$[0,0,0],[0,1,1],[0,2,2],[0,3,3],[0,4,4],[0,5,5],[0,6,6],[0,7,7],[0,8,8]$$
$$[1,1,3],[1,1,5],[1,1,7]$$
$$[1,2,3],[1,2,4],[1,2,7],[1,2,8]$$
$$[1,3,3],[1,3,4],[1,3,5],[1,3,6],[1,3,7],[1,3,8]$$
$$[1,4,4],[1,4,5],[1,4,6],[1,4,7],[1,4,8]$$
$$[1,5,5],[1,5,6],[1,5,7],[1,5,8]$$
$$[1,6,6],[1,6,7],[1,7,8]$$
$$[1,7,7],[1,7,8]$$
$$[1,8,8]$$
$$[2,2,2],[2,2,3],[2,2,4],[2,2,5],[2,2,6].$$

An example of some of Tomasz Kowalski's code (in C) is shown below.

```
#include <stdio.h>
#include <sys/types.h>
#include <time.h>
#define LN 11
#define MAX max_cycles()

int cycletable[LN][LN][LN];
int invtable[LN] = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10};
int related[LN][LN];
int last_found[LN][LN];
int inv(int i)
{
    return invtable[i];
}
int two_to(int pow)
{
    int val = 1;
    int i;
    for(i = 0; i < pow; i++)
val <<= 1;
    return(val);
}
int max_cycles()
{
    int i,k,val=0;
    for(i=1; i<LN; i++)
val=val+(i*(LN-i));
    return val;
}
int compose_atoms(int a, int b)
/* composes atoms numbered a and b */
{
    int c, val=0;
    for(c=0; c<LN; c++)
if(cycletable[a][b][c] > 0)
    val = val | two_to(c);
    return val;
}
void a_priori_cycles()
```

```
{
    int i,j;
    for(i=0; i<LN; i++)
    {
cycletable[i][0][i] = 1;
cycletable[0][i][i] = 1;
        for(j=0; j<LN; j++)
    if(i==inv(j))
cycletable[i][j][0] = 1;
    }
}
int exclude_cycleset(int i, int j, int k)
{
    cycletable[i][j][k] = 0;
    cycletable[inv(i)][k][j] = 0;
    cycletable[inv(j)][inv(i)][inv(k)] = 0;
    cycletable[j][inv(k)][inv(i)] = 0;
    cycletable[inv(k)][i][inv(j)] = 0;
    cycletable[k][inv(j)][i] = 0;
}
int include_cycleset(int i, int j, int k)
{
    cycletable[i][j][k] = 1;
    cycletable[inv(i)][k][j] = 1;
    cycletable[inv(j)][inv(i)][inv(k)] = 1;
    cycletable[j][inv(k)][inv(i)] = 1;
    cycletable[inv(k)][i][inv(j)] = 1;
    cycletable[k][inv(j)][i] = 1;
}
void fixed_cycles()
{
    int i, j;
    for(i=3; i<LN; i++)
for(j=3; j<LN; j++)
    include_cycleset(i,j,1);
    for(i=2; i<LN-2; i++)
include_cycleset(2,2,i);
    for(i=1; i<LN/2; i++)
include_cycleset(1,1,LN-2*i);
    for(i=0; i<(LN-2)/4; i++)
include_cycleset(1,2,LN-4*i-2);
```

```
    for(i=0; i<(LN-1)/4; i++)
include_cycleset(1,2,LN-4*i-1);
}
int lex_smaller(int a, int b, int c, int x, int y, int z)
{
  int ctrl=0;
  if(a<x)
    ctrl=1;
  else if(a==x && b<y)
    ctrl=1;
  else if(a==x && b==y && c<=z)
    ctrl=1;
  return ctrl;
}
int minimal_triple(int i, int j, int k)
{
    int ctrl=0;
    if(lex_smaller(i,j,k, inv(i),k,j) &&
       lex_smaller(i,j,k, inv(j),inv(i),inv(k)) &&
       lex_smaller(i,j,k, j,inv(k),inv(i)) &&
       lex_smaller(i,j,k, inv(k),i,inv(j)) &&
       lex_smaller(i,j,k, k,inv(j),i))
ctrl=1;
    return ctrl;
}
/*
int allowed_triple(int i, int j, int k)
{
    int ctrl=1;
    if(i > 2 && j > 2 && k > 1)
ctrl = 0;
    return ctrl;
}
*/
int quick_next_cycletable()
{
    int i, ctrl = 1;
    for(i=1; i<4 && ctrl == 1; i++)
if(minimal_triple(1,2,i) == 1)
    if(cycletable[1][2][i] == 0)
    {
```

```
include_cycleset(1, 2, i);
ctrl=0;
    }
    else /* if(cycletable[i][j][k] == 1 */
    {
exclude_cycleset(1, 2, i);
    }
}
int next_cycletable()
{
    int i,j,k, ctrl = 1;
    for(i=1; i<3 && ctrl == 1; i++)
for(j=1; j<3 && ctrl == 1; j++)
    for(k=1; k<LN && ctrl == 1; k++)
if(minimal_triple(i,j,k) == 1)
    if(cycletable[i][j][k] == 0)
    {
include_cycleset(i, j, k);
ctrl=0;
    }
    else /* if(cycletable[i][j][k] == 1 */
    {
exclude_cycleset(i, j, k);
    }
}
int integral()
{
    int i,j,u, ctrl=1, found=0;
    for(i=0; i<LN && ctrl == 1; i++)
for(j=0; j<LN && ctrl == 1; j++)
{
    for(u=0; u<LN && found==0; u++)
if(cycletable[i][j][u] > 0)
    found=1;
    if(found==1)
found=0;
    else
ctrl=0;
}
    return ctrl;
}
```

```
int semi_associative()
{
    int v,w,x,y,z,u, ctrl=1, found=0;
    for(v=0; v<LN && ctrl == 1; v++)
for(w=0; w<LN && ctrl == 1; w++)
    for(x=0; x<LN && ctrl == 1; x++)
for(y=0; y<LN && ctrl == 1; y++)
    for(z=0; z<LN && ctrl == 1; z++)
if(cycletable[v][w][x] > 0 &&
   cycletable[x][y][z] > 0 &&
   (v!=z || w!=inv(y)))
{
    for(u=0; u<LN && found==0; u++)
if(cycletable[v][u][z] > 0)
    found=1;
    if(found==1)
found=0;
    else
ctrl=0;
}
    return ctrl;
}
int no_equiv()
{
    int i, j, ctrl=1, found=0;
    for(i=1; i<LN && ctrl == 1; i++)
    {
for(j=1; j<LN && found == 0; j++)
    if(cycletable[i][inv(i)][j]>0)
found=1;
if(found==1)
    found=0;
else
    ctrl=0;
    }
    return ctrl;
}
int noncommutative()
{
    int i, j, k, found=0;
    for(i=1; i<LN && found == 0; i++)
```

```
     for(j=i+1; j<LN && found == 0; j++)
    for(k=1; k<LN && found == 0; k++)
if(cycletable[i][j][k] != cycletable[j][i][k])
    found=1;
    return found;
}
int transitive()
{
    int i,j,k, ctrl=1;
    for(i=1; i<LN && ctrl == 1; i++)
for(j=1; j<LN && ctrl == 1; j++)
    for(k=1; k<LN && ctrl == 1; k++)
if(related[i][j] == 1 &&
   related[j][k] == 1 &&
   related[i][k] == 0)
    ctrl=0;
    return ctrl;
}
int diagonal()
{
    int i,j;
    for(i=0; i<LN; i++)
for(j=0; j<LN; j++)
    if(i==j)
related[i][j]=1;
    else
related[i][j]=0;
}
int next_relation()
{
    int i,j, ctrl = 1;
    for(i=1; i<LN && ctrl == 1; i++)
for(j=i+1; j<LN && ctrl == 1; j++)
    if(related[i][j] == 0)
    {
related[i][j] = 1;
related[j][i] = 1;
ctrl=0;
    }
    else
    {
```

```
related[i][j] = 0;
related[j][i] = 0;
    }
}
int converse_compatible()
{
    int i,j,ctrl=1;
    for(i=1; i<LN && ctrl==1; i++)
if(i != inv(i))
    for(j=1; j<LN && ctrl==1; j++)
if(related[j][i]==1 && related[inv(j)][inv(i)]==0)
    ctrl=0;
    return ctrl;
}
int composition_compatible()
{
    int v,w,x,y,z,u, ctrl=1, found=0;
    for(v=0; v<LN && ctrl == 1; v++)
for(w=0; w<LN && ctrl == 1; w++)
    for(u=0; u<LN && ctrl == 1; u++)
if(cycletable[v][w][u] == 1)
    for(z=0; z<LN && ctrl == 1; z++)
if(related[u][z]==1)
{
    for(x=0; x<LN && found == 0; x++)
for(y=0; y<LN && found == 0; y++)
    if(cycletable[x][y][z] == 1 &&
       related[v][x]== 1 &&
       related[w][y]== 1)
found=1;
    if(found==1)
found=0;
    else
ctrl=0;
}
    return ctrl;
}
int bounded_morphism()
{
    if(converse_compatible() && composition_compatible())
return 1;
```

```
    else
return 0;
}
int nontrivial_subalgebras()
{
    int i,j, before=0, after=0, ctrl=0;
    diagonal();
    while(ctrl==0)
    {
next_relation();
if(transitive() && bounded_morphism())
{
    for(i=0; i<LN; i++)
for(j=0; j<LN; j++)
    last_found[i][j]=related[i][j];
    ctrl=1;
}
    }
    before=related[LN-2][LN-1];
    next_relation();
    after=related[LN-2][LN-1];
    if(before>after)
ctrl=0;
    return ctrl;
}
void cycletable_to_screen()
{
    int i,j,k;
    for(i=0; i<LN; i++)
for(j=0; j<LN; j++)
    for(k=0; k<LN; k++)
if(cycletable[i][j][k]>0 && minimal_triple(i,j,k))
    printf("[%d, %d, %d]\n",i,j,k);
    printf("-------------------\n");
}
void full_cycletable_to_screen()
{
    int i,j,k;
    for(i=0; i<LN; i++)
for(j=0; j<LN; j++)
    for(k=0; k<LN; k++)
```

```
if(cycletable[i][j][k]>0)
    printf("(%d, %d, %d) (%d, %d, %d) (%d, %d, %d)\
 (%d, %d, %d) (%d, %d, %d) (%d, %d, %d)\n",
   i,j,k, inv(i),k,j, j,inv(k),inv(i),
   inv(j),inv(i),inv(k), inv(k),i,inv(j), k,inv(j),i);
    printf("-------------------\n");
}
void table_to_screen()
{
    int i,j;
    for(i=0; i<LN; i++)
    {
for(j=0; j<LN; j++)
    printf("%5d", compose_atoms(i,j));
printf("\n");
    }
    printf("\n");
}
void print_partition_class(int i)
{
    int j;
    printf("{ ");
    for(j=1; j<LN; j++)
if(last_found[i][j] == 1)
    printf("%d ", j);
    printf("}\n");
}
void print_partition()
{
    int i,k,done=0;
    for(i=1; i<LN; i++)
    {
for(k=1; k<i && done==0; k++)
    if(last_found[k][i]==1)
done=1;
if(done==1)
    done=0;
else
    print_partition_class(i);
    }
    printf("\n");
```

```
}
int not_last_table()
{
    int i,j,k,found=0;
    for(i=1; i<LN; i++)
for(j=i; j<LN; j++)
    for(k=j; k<LN; k++)
if(cycletable[i][j][k]==0)
    found=1;
    return found;
}
int clear_cycletable()
{
    int i,j,k;
    for(i=1; i<LN; i++)
for(j=1; j<LN; j++)
    for(k=1; k<LN; k++)
cycletable[i][j][k] = 0;
}
int first_cycletable(int c)
{
  int i,j,k,count=0;
  for(i=1; i<LN && count<c; i++)
      for(j=1; j<LN && count<c; j++)
  for(k=1; k<LN && count<c; k++)
      if(minimal_triple(i,j,k)==1)
      {
  include_cycleset(i,j,k);
  count = count++;
      }
}
int main()
{
    time_t t1,t2;
    (void) time(&t1);
    a_priori_cycles();
    fixed_cycles();
    include_cycleset(1,2,3);
    include_cycleset(3,4,5);
    include_cycleset(5,6,7);
    if(semi_associative() == 1
```

```
&& nontrivial_subalgebras() == 0)
    {
table_to_screen();
cycletable_to_screen();
    }
    else
    {
table_to_screen();
cycletable_to_screen();
  printf("\n Dupa\n");
    }
    (void) time(&t2);
    printf("\n Time elapsed: %d seconds\n", (int) t2-t1);
}
/*
int main()
{
    time_t t1,t2;
    (void) time(&t1);
    int found=0;
    a_priori_cycles();
    fixed_cycles();
    while(not_last_table() && found < 1)
    {
quick_next_cycletable();
if(semi_associative() == 1 &&
   nontrivial_subalgebras() == 0)
{
    table_to_screen();
    cycletable_to_screen();
    found++;
}
    }
    (void) time(&t2);
    printf("\n Time elapsed: %d seconds\n", (int) t2-t1);
    printf(" Number of algebras found: %d\n", found);
}
*/
```

Appendix B: Code for finding qualitative representations

Below we give SageMath code for finding qualitative representations of algebras. The atom set, identity atom set, and consistent triples are the inputs; here we use $\mathbf{C}_5$ from Table 5 of Jipsen [**43**]. This code was used was intended to be used for the algebras in Tables 4 and 5 from [**43**], but was never run for long enough to terminate. Line breaks were added to fit the code within the page.

```
# Input section.
# Here we need the start of AL and IL to match; use natural numbers for atoms
  and make sure identity atoms come first in your list.
# I think this only works for integral algebras at the moment.

A = {1,2,3,4,5} # Atom set.
I = {1} # Identity triples.
C = {(1,1,1),(1,2,2),(1,3,3),(1,4,4),(1,5,5),(2,1,2),(2,2,1),(2,2,4),(2,2,5),
     (2,3,5),(2,4,1),(2,4,4),(2,4,5),(2,5,2),(2,5,3),(2,5,4),(2,5,5),(3,1,3),
     (3,2,4),(3,3,1),(3,3,2),(3,3,3),(3,3,4),(3,3,5),(3,4,2),(3,4,3),(3,4,4),
     (3,4,5),(3,5,3),(3,5,4),(3,5,5),(4,1,4),(4,2,2),(4,2,3),(4,2,4),(4,2,5),
     (4,3,3),(4,3,4),(4,3,5),(4,4,2),(4,4,3),(4,4,4),(4,4,5),(4,5,1),(4,5,2),
     (4,5,3),(4,5,4),(4,5,5),(5,1,5),(5,2,1),(5,2,4),(5,2,5),(5,3,2),(5,3,3),
     (5,3,4),(5,3,5),(5,4,1),(5,4,2),(5,4,3),(5,4,4),(5,4,5),(5,5,2),(5,5,3),
     (5,5,4),(5,5,5)}
AL = list(A) # Indexed list of atoms.
IL = list(I) # Indexed list of identity atoms.
CL = list(C) # Indexed list of cycles.
a = len(AL) # Number of atoms.
i = len(IL) # Number of identity atoms.
c = len(CL) # Number of cycles.
lown = a # Minimum number of nodes needer for a qualitative representation.
maxn = 3*(c-3*a+2) # Maximum number of nodes needed.

# Set initial values for nodes and search status.
found = 0
n = lown
# Start looking for networks!
while found == 0 & n <= maxn:
```

```
# Calculate the number of off diagonal elements of N^2 up to symmetry.
nod = int((n**2-n)/2)
# Create an index set for functions defined on the diagonal elements.
if i > 1:
    dtemp = list(range(i**n))
    dfunc = [ZZ(x).digits(base=i, padto=n)[::-1] for x in dtemp]
if i==1:
    dtemp = list(range(n))
    for j in list(range(n)):
        dtemp[j]=0
    dfunc = [dtemp]
# Create an index set for symmetric functions on the off-diagonal elements.
if a > 1:
    odtemp = list(range(a**nod))
    odfunc = [ZZ(x).digits(base=a, padto=nod)[::-1] for x in odtemp]
if a == 1:
    odtemp = list(range(a))
    for j in list(range(a)):
        odtemp[j]=0
    odfunc = [odtemp]
# Create an index set for combining these index sets.
bigindex = list(range((i**n)*(a**nod)))
for j in list(range((i**n)*(a**nod))):
    bigindex[j]=[int(j)//int(a**nod),mod(j,a**nod)]
# Create lists of diagonal elements of N^2.
d = list(range(n))
for j in list(range(n)):
    d[j]=[j,j]
# Create a list of off-diagonal elements of N^2 (up to symmetry).
od = list(range(nod))
for j in list(range(n-1)):
    od[j]=[0,j+1]
for j in list(range(n-2)):
    for k in list(range(n-j-2)):
        od[int((j+1)*n-(j+1)*(j+2)//2+k)] = [j+1,j+k+2]
# Combine the previous two lists to get all elements of N^2 (up to symmetry).
dom = d+od
# Create a list of elements of N^3.
Ntemp = list(range(n**3))
Ntup = [ZZ(x).digits(base=n, padto=3)[::-1] for x in Ntemp]
# Set initial index for search through functions.
```

```
    j = 0


    while j < len(bigindex):
        func = dfunc[bigindex[Integer(j)][0]]+odfunc[bigindex[Integer(j)][1]]
        LN3 = set(range(0))
        for k in list(range(n^3)):
            a1 = AL[func[dom.index([min(Ntup[k][0],Ntup[k][1]),
                                                    max(Ntup[k][0],Ntup[k][1])])]]
            a2 = AL[func[dom.index([min(Ntup[k][1],Ntup[k][2]),
                                                    max(Ntup[k][1],Ntup[k][2])])]]
            a3 = AL[func[dom.index([min(Ntup[k][0],Ntup[k][2]),
                                                    max(Ntup[k][0],Ntup[k][2])])]]
            LN3 = LN3.union({(a1,a2,a3)})
        if LN3 == C:
            found = 1
            workingnodes = n
            workingindex = j
            j = len(bigindex)
        if LN3 != C:
            j = j+1
    # If no network is found at this step we increment the number of nodes.
    if found == 0 & n < maxn:
        n = n+1

# Report the result of the search.
if found == 0:
    print('No cigar.')
if found == 1:
    winningfunc = dfunc[bigindex[workingindex][0]]+
                                            odfunc[bigindex[workingindex][1]]
    for k in list(range(len(winningfunc))):
        winningfunc[k] = AL[winningfunc[k]]
    print('There is a network with',workingnodes,'nodes.')
    print('The labels are as follows:',winningfunc)
```

The following modification of this code looks for representations of chromatic algebras, and takes the number of atoms and defining subset of $\{1, 2, 3\}$ as inputs. This code was used to construct some small networks while working on the material presented in Chapter 2. Line breaks were added to fit the code within the page.

```
# For finding qualitative representations of E_n(X); the inputs are n and X.
X = {1,3}
a = 3 # Number of atoms, not the number of diversity atoms.


# First we construct the universe and the set of consistent triples.
# 1 will always be the identity and {1,...,n} will be the universe.
A = {1} # This will be the universe.
for i in range(a-1):
    A = A.union({i+2})

C = set(range(0)) # This will be the set of all consistent triples.
t = 0 # This will be the total number of consistent diversity triangles.

# First we add in all the identity triples.
for i in range(a):
    C = C.union({(1,i+1,i+1),(i+1,1,i+1)})

# Add in all the equilateral triples if 1 is in X.
if X == X.union({1}):
    t = t+a-1
    for i in range(a-1):
        C = C.union({(i+2,i+2,i+2)})

# Add in all the isosceles triples if 2 is in X.
if X == X.union({2}):
    t = t+(a-1)*(a-2)
    for i in range(a-1):
        for j in range(a-1):
            if i != j:
                C = C.union({(i+2,i+2,j+2),(i+2,j+2,i+2),(j+2,i+2,i+2)})

# Add in all the scalene triples if 3 is in X.
if X == X.union({3}):
    t = int(t+(a-1)*(a-2)*(a-3)/6)
    for i in range(a-1):
```

```
        for j in range(a-1):
            for k in range(a-1):
                if i != j and i !=k and j != k:
                    C = C.union({(i+2,j+2,k+2)})


# Now we look for a suitable network.
# First we find the maximum and minimum numbers of points a network can have.
maxn = 3*(t-1)
minn = 0
while minn*(minn-1)*(minn-2)/3 < t:
    minn = minn+1

# Now we use a while loop to search for networks.
found = 0 # This variable will be used to keep track of whether we've found
            a network.
if (a==1)|(a==2):
    found = 1
n = minn # This will be the number of points in the network.
while found == 0 & n < maxn:
    ed = n*(n-1)/2 # The number of edges (by symmetry and integrality).
    od = list(range(ed))  # We calculate a list of lists of the form [i,j]
         with i < j to extract labels from the labelling function.
    for i in list(range(n-1)):
        od[i]=[0,j+1]
    for j in list(range(n-2)):
        for k in list(range(n-j-2)):
            od[int((j+1)*n-(j+1)*(j+2)//2+k)] = [j+1,j+k+2]
    nf = n**ed # The number of atomic networks to consider.
    # Labellings are like length base n numbers with at most ed digits.
    # This is how will enumerate the functions without storing a massive
      array like in my old code.
    index = 0
    while index < nf:
        T = set(range(0)) # This will be the set of triples in the network.
        for i in range(a-1): # We add in all the identity triples, which
                                   must be present.
            T = T.union({(1,i+2,i+2),(i+2,1,i+2)})
        rem = int(a*(i/a-floor(n/a)))
        quo = floor(n/a)
        fun = list(range(ed))
```

```
        fun[ed-1]=rem+1
        for i in range(ed-1):
            rem = int(a*(quo/a-floor(quo/a)))
            quo = floor(quo/a)
            fun[ed-2-i]=rem+1
        for i in range(n):
            for j in range(n):
                for k in range(n):
                    if i!= j & i != k & j != k:
                        T = T.union({(fun[od.index([min(i,j),max(i,j)])],
            fun[od.index([min(j,k),max(j,k)])],fun[od.index([min(i,k),max(i,k)])])})
        if T == C:
            found = 1
            winningindex = index
            winningfun = fun
        else:
            index = index+1
    if found == 0:
        n = n+1
if found == 0:
    print('No cigar.')
if (found == 1)&((a==1)|(a==2)):
    print('There is a small network that is easy to find')
if (found == 1)&(a!=1)&(a!=2):
    print(winningindex,winningfun)
```

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