Dynamic Stiffness Formulation for a Micro Beam using Timoshenko-Ehrenfest and Modified Couple Stress Theories with Applications

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Abstract

Several models within the framework of continuum mechanics have been proposed over the years to solve the free vibration problem of micro beams. Foremost amongst these are those based on non-local elasticity, classical couple stress, gradient elasticity and modified couple stress theories. Many of these models retain the basic features of the Bernoulli-Euler or Timoshenko-Ehrenfest theories but they introduce one or more material scale length parameters to tackle the problem. The work described in this paper deals with the free vibration problems of micro beams based on the dynamic stiffness method, through the implementation of the modified couple stress theory (MCST) in conjunction with the Timoshenko-Ehrenfest theory. The main advantage of the MCST is that unlike other models, it uses only one material length scale parameter to account for the smallness of the structure. The current research is accomplished first by solving the governing differential equations of motion of a Timoshenko-Ehrenfest micro beam in free vibration in closed analytical form. The dynamic stiffness matrix of the beam is then formulated by relating the amplitudes of the forces to those of the corresponding displacements at the ends of the beam. The theory is applied using the Wittrick-Williams algorithm as solution technique to investigate the free vibration characteristics of Timoshenko-Ehrenfest micro beams. Natural frequencies and mode shapes of several examples are presented and the effects of the length scale parameter on the free vibration characteristics of Timoshenko-Ehrenfest micro beams are demonstrated.

Keywords: Micro beams, Modified couple stress theory, Dynamic stiffness method, Vibration, Timoshenko-Ehrenfest theory, Wittrick-Williams algorithm

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1. Introduction

Within the confines of continuum mechanics, there are many applications of microbeams, particularly in the design and developments of micro-electromechanical systems (Attia et al., 1998; Li et al., 2003; Pei et al., 2004; Moeenfard and Ahmadian, 2013). For such applications, classical beam theories are inapplicable because these theories ignore the microstructure dependent size effect. There are numerous experimental observations (Fleck et al., 1994; Ma and Clarke, 1995; Stolken and Evans, 1998; Chong and Lam, 1999; Chong et al., 2001; Lam et al., 2003) which strongly support that the size effect in a micro-structure matters and its influence can be significant. The historical development of micro continuum mechanics took place more than a century ago, and the current state of the art owes a huge debt to early pioneers such as Cosserat and Cosserat (1909), Toupin (1962), Mindlin and Tiersten (1962), Mindlin (1963, 1964), Eringen (1966, 1972, 1983) and Koiter (1964). Foremost amongst the micro continuum mechanical theories which survived the test of time and continues to receive attention even to this day are the non-local elasticity theory originally proposed by Eringen (1966, 1972, 1983) and later further developed by Reddy (2007) and Challamel and Wang (2008); the classical couple stress theory introduced by Toupin (1962), Mindlin and Tiersten (1962), Koiter (1964) and further used by Anthoine (2000); the gradient elasticity theory in connection with non-local elasticity presented by Mindlin (1963, 1964) and subsequently applied by Lam (2003), Kong et al. (2009), Askes and Aifantis (2011) and Akgoz and Civalek (2012); and the modified couple stress theory (MCST) developed by Yang et al. (2002) and further exploited by Park and Gao (2006) and Attia and Emam (2018). A survey paper by Thai et al. (2017) elucidates the developments of micro mechanics continuum models and the paper provides considerable insights into the topic. Based on these micro continuum theories, the static and dynamic behavious of micro beams with varying degrees of complexities has been investigated by many researchers in recent years (Kong et al., 2008; Fathalilou et al., 2014; Liang et al., 2014; Challamel et al., 2015; Ma et al., 2008; Reddy, 2011; Darijini and Mohammadabadi, 2014; Ansari et al., 2015; Noori et al., 2016; Yayli, 2017; Mustapha, 2020; Essen, 2020). For instance, within the chosen background theory of Bernoulli-Euler beam, Kong et al. (2008) and Fathalilou et al. (2014) used couple stress theory whereas Liang et al. (2014), Challamel et al. (2015) and Yayli (2017) used strain-gradient based non-local elasticity theory to investigate the free vibration behavior of microbeams. The subsequent developments for the free vibration analysis of micro-beams followed quite intensely when the Timoshenko-Ehrenfest theory (Timoshenko, 1921, 1922; Elishakoff, 2020a, 2020b), which extends the

Bernoulli-Euler beam theory by accounting for the effects of shear deformation and rotatory inertia, was implemented in the micro beam theory (Ma et al., 2008; Reddy, 2011; Ansari et al., 2015). (It has come to the notice of the authors in recent years that Ehrenfest contributed in a substantial way to the enhancement of the Bernoulli-Euler beam theory to include the effects of shear deformation and rotatory inertia. In fact, the frequency equation of the so-called Timoshenko beam was jointly developed by Timoshenko and Ehrenfest before Timoshenko published his classic papers (Timoshenko, 1921, 1922) and the origin of the theory was recently researched by Elishakoff (2020a, 2020b). The contribution of Ehrenfest was acknowledged by Timoshenko, see his footnote in Timoshenko (1921). Recently Noori et al. (2016) published an interesting paper in which they developed higher order micro beam model to carry out the free vibration analysis. However, one of their conclusions "*The natural frequencies of beams with free and clamped edges employing Bernoulli-Euler model are identical, which is doubtful physically."* is frankly debatable. Noori et al. (2016) appears to have overlooked that the beam with free-free boundary condition will have rigid-body modes of zero frequencies which might have been discounted by them when drawing this conclusion. Natural frequencies corresponding to the elastic modes for the free-free and clamped-clamed cases are equal, but such a comparison needs qualification based on the order of the natural frequencies.

In the current work the Timoshenko-Ehrenfest theory in conjunction with the MCST which has the advantage of using only one material length scale parameter (Yang et al., 2002; Ma et al., 2008; Reddy, 2011; Mustapha, 2020) is a fundamental consideration to develop the dynamic stiffness theory and then carry out the free vibration analysis of the Timoshenko-Ehrenfest micro beam. In this context, it should be noted that linear small deflection theory has been used and any consideration of non-linear effects is outside the scope of this paper but interested readers are referred to (Farokhi et al., 2013; Ghayesh et al., 2013; Asghari et al., 2010; Wang et al., 2013; Simsek, 2014) for further research on the subject.

Thus, the main purpose of this paper is to develop for the first time the dynamic stiffness matrix of a micro beam using Timoshenko-Ehrenfest theory and MCST and then carry out its free vibration analysis. The general procedure for dynamic stiffness theory development and its application can be found in the work of Banerjee (1997, 2015), amongst others. In the current work, bending or flexural vibration is given precedence in the development of the dynamic stiffness theory because axial vibration is not affected by the small length parameter when the MCST is used (Ma et al., 2008; Reddy, 2011; Mustapha, 2020). The investigation is

carried out in following steps. First the potential and kinetic energies of a micro beam are formulated using Timoshenko-Ehrenfest theory and MCST. The former accounts for the shear deformation and rotatory inertia whereas the latter accounts for the material scale length parameter. Thus, the formulation starts with the fundamental premises of modified couple stress theory to obtain the expressions for the kinetic and potential energies and then by using Hamilton's principle the governing differential equations of motion in free vibration and natural boundary conditions of a micro beam based on Timosheno-Ehrenfest theory and MCST are derived. For harmonic oscillation, the governing differential equations are solved in closed analytical form. Next, the dynamic stiffness matrix of the micro beam based on the modified couple stress theory and Timoshenko-Ehrenfest hypothesis is formulated by relating the amplitudes of forces and moments to those of the displacements and rotation at the ends of the micro beam element. Finally, the well-established Wittrick-Williams algorithm (Wittrick and Williams, 1971; Williams and Wittrick, 1983; Williams, 1993) is applied to the resulting dynamic stiffness matrix to compute the natural frequencies and mode shapes of some selected examples of micro beams. The effects of the small length scale parameter as well as those of shear deformation and rotatory inertia on the results are demonstrated and some results are validated using published results. It is significant to note that the micro beam theory using MCST in combination with the Timoshenko-Ehrenfest theory gives rise to three displacement variables (excluding the axial displacement (u)) at each node of the beam in bending or flexure. These are bending or flexural displacement (v), bending or flexural rotation (θ) and the first derivative of the bending or flexural displacement (v'). Unfortunately, the published literature on the free vibration of Timoshenk-Ehrenfest micro beams does not give due recognition to the displacement component v' in any practical or introspective manner. A secondary purpose of this paper is to redress this disparity.

2. Theory

2.1 Fundamental Preliminaries

The modified couple stress theory used here in the derivation of the governing differential equations of motion in free vibration of a micro beam in conjunction with the Timoshenko-Ehrenfest model, was developed by Yang et al. (2002) with the underlying postulate that the strain energy is a function of both the strain tensor, which is conjugated to stress tensor, and importantly, the curvature tensor which is conjugated to couple stress tensor. Based on this

postulate, the strain or potential energy U of a linearly deformed elastic body can be written in tensorial notation as (Park and Gao, 2006; Kong et al., 2008; Reddy, 2011)

$$U = \frac{1}{2}\int_V (\boldsymbol{\sigma}:\boldsymbol{\varepsilon} + \mathbf{m}:\boldsymbol{\chi})\, dv \Leftrightarrow \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right) dv \tag{1}$$

where σ_{ij} and ε_{ij} (i and j are the usually adopted dummy notations with i, j =1, 2, 3) are the components of the stress ($\boldsymbol{\sigma}$) and strain ($\boldsymbol{\varepsilon}$) tensors and m_{ij} and χ_{ij} signify the components of the deviatoric part of the symmetric couple stress tensor ($\mathbf{m}$) and symmetric curvature tensor ($\boldsymbol{\chi}$), respectively and the integration is carried out over the entire volume of the elastic body.

The expressions for σ_{ij}, ε_{ij}, m_{ij} and χ_{ij} in the usual notations are given by (Kong et al., 2008; Ma et al., 2008)

$$\boldsymbol{\sigma} = \bar{\lambda} tr(\boldsymbol{\varepsilon})\mathbf{I} + 2G\boldsymbol{\varepsilon} \Leftrightarrow \sigma_{ij} = \bar{\lambda} tr\left(\varepsilon_{ij}\right)\delta_{ij} + 2G\varepsilon_{ij} \tag{2}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \Leftrightarrow \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

$$\mathbf{m} = \frac{El^2}{(1+\nu)}\boldsymbol{\chi} = 2Gl^2\boldsymbol{\chi} \Leftrightarrow m_{ij} = 2Gl^2\chi_{ij} \tag{4}$$

$$\boldsymbol{\chi} = \frac{1}{2}[\nabla\boldsymbol{\theta} + (\nabla\boldsymbol{\theta})^T] \Leftrightarrow \chi_{ij} = \frac{1}{2}\left(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i}\right) \tag{5}$$

where $\bar{\lambda}$ and G (often denoted by λ and μ) are Lame's constant, $\mathbf{I}$ is the unit or identity matrix (so that its elements δ_{ij} is 1 when $i = j$ and 0 when $i \neq j$), u_i, u_j are the components of the displacement vector $\mathbf{u}$, θ_i and θ_j are the components of rotation vector $\boldsymbol{\theta}$, and l is the material length scale parameter which identifies the effect of the couple stress as described by Yan et al. (2002). It should be noted that Chong et al. (2001) successfully determined the small length scale parameter l by experiment. For homogeneous and isotropic material, Lame's constants $\bar{\lambda}$ and G appearing in Eqs. (2) and (4) can be found in standard texts in elasticity. These are:

$$\bar{\lambda} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)} \tag{6}$$

where E is the Young's modulus, ν is the Poisson's ratio and G is the shear modulus of modulus of rigidity of the material.

2.2. Derivation of the governing differential equations

In a rectangular right-handed Cartesian coordinate system, Figure 1 shows a uniform micro beam of length *L* in a greatly exaggerated scale with the *X*-axis coinciding with the centroidal axis of the beam. The *Y*-axis is upward and perpendicular to the *X*-axis representing the direction of bending or flexural displacement of the beam. The *Z*-axis, which is not shown, is perpendicular to the plane of the paper represents the neutral axis of the beam.

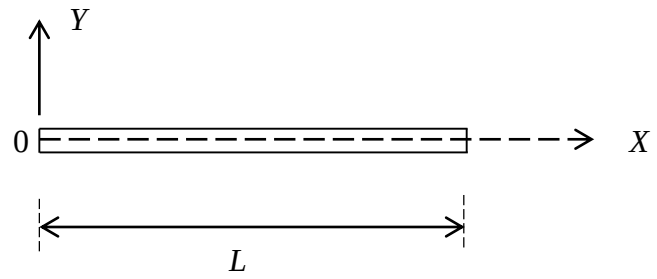

Figure. 1. Coordinate system of a microbeam

Based on the preliminaries of MCST (Yang et al., 2002; Ma et al., 2008; Reddy, 2011; Mustapha, 2020) briefly described in section 1.1, the strain or potential energy *U* of a linearly deformed elastic body like the micro beam shown in Figure 1 can now be formed whereas the kinetic energy can be routinely expressed using the transverse and rotational components of a point lying on the neutral axis of the beam. Using the kinetic and potential energies formulated in this way, Hamilton's principle is applied here to derive the governing differential equations of motion of a Timoshenko-Ehrenfest micro beam undergoing free vibration.

Referring to Figure 1, let *v* be the bending or flexural displacement and θ be the bending rotation of the Timoshenko-Ehrenfest micro beam. The generalized displacement field of Equations (3) and (5) then reduces to much simplified forms, as follows.

$$u_1 = u(x,t) - y\theta(x,t),\ u_2 = v(x,t), \quad u_3 = 0 \tag{7}$$

The bending rotation $\theta(x,t)$ in Equation (7) above is essentially the angle of rotation of the cross-section of the beam about the *Z*-axis which according to Timoshenko-Ehrenfest beam theory is given by

$$\theta(x,t) = \frac{\partial v(x,t)}{\partial x} - \gamma_{xy} \tag{8}$$

where γ_{xy} is the shearing strain (ε_{12} in tensorial notation) which in fact is the slope arising from the transverse shear force that must be deducted from the total slope $\frac{\partial v(x,t)}{\partial x}$ of the flexural axis to give the bending slope or the angle of rotation $\theta(x,t)$ of the cross-section.

Equation (3) with the help of Equations (7) and (8), gives the expression for the strains as

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{\partial u}{\partial x} - y\frac{\partial \theta}{\partial x},\ \varepsilon_{12} = \ \varepsilon_{21} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\ \gamma_{yx} = \frac{1}{2}(\frac{\partial v}{\partial x} - \theta),\ \varepsilon_{22} = \varepsilon_{23} = \varepsilon_{13} = 0 \quad (9)$$

The components of rotations based to the displacement field defined in Equation (7) are given by

$$\theta_1 = \theta_x = \theta_2 = \theta_y = 0,\ \theta_3 = \theta_z = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \theta\right) \quad (10)$$

With help of Equations (5) and (10), the components of the curvature tensor $\boldsymbol{\chi}$ can now be obtained as

$$\chi_{13} = \chi_{xz} = \frac{1}{2}\frac{\partial \theta_z}{\partial x} = \frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \theta}{\partial x}\right),\quad \chi_{11} = \chi_{22} = \chi_{33} = \chi_{12} = \chi_{23} = 0 \quad (11)$$

The components of classical stress tensor $\boldsymbol{\sigma}$ and the deviatoric part of the couple stress tensor **m** from Equations (2) and (4) are respectively, given by

$$\sigma_{11} = \sigma_{xx} = E\left(\frac{\partial u}{\partial x} - y\frac{\partial \theta}{\partial x}\right),\ \sigma_{12} = \tau_{xy} = G\left(\frac{\partial v}{\partial x} - \theta\right),\ \sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{13} = 0 \quad (12)$$

$$m_{13} = m_{xz} = 2Gl^2\chi_{13} = \frac{1}{2}Gl^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \theta}{\partial x}\right),\ m_{11} = m_{22} = m_{33} = m_{12} = m_{23} = 0 \quad (13)$$

Assuming linear small deflection theory, the expression for the potential energy U of the micro beam can be obtained by substituting Equations (9) – (13) into Equation (1) to give

$$U = \frac{1}{2}\int_0^L \int_A (\varepsilon_{11}\sigma_{11} + 2\varepsilon_{12}\sigma_{12} + 2\chi_{13}m_{13})\, dAdx$$

$$= \frac{1}{2}\int_0^L \int_A \left\{E\left(\frac{\partial u}{\partial x} - y\frac{\partial \theta}{\partial x}\right)^2 + kG\left(\frac{\partial v}{\partial x} - \theta\right)^2 + \frac{1}{4}Gl^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \theta}{\partial x}\right)^2\right\} dAdx$$

$$= \frac{1}{2}\int_0^L \left[EA\left(\frac{\partial u}{\partial x}\right)^2 + EI\left(\frac{\partial \theta}{\partial x}\right)^2 + kAG\left(\frac{\partial v}{\partial x} - \theta\right)^2 + \frac{GAl^2}{4}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \theta}{\partial x}\right)^2\right] dx \quad (14)$$

where A is the area and I is the second moment of area of the micro beam cross-section.

The expression of kinetic energy (T) is given by

$$T = \frac{1}{2}\int_0^L \int_A \rho\left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right] dA dx \tag{15}$$

where ρ is density of the micro beam material and t is time.

Substitution of Equation (7) into Equation (15) gives

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \int_A \rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] dA dx \\ &= \frac{1}{2}\rho A \int_0^L \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] dx + \frac{1}{2}\rho I \int_0^L \left(\frac{\partial \theta}{\partial t}\right)^2 dx \end{aligned} \tag{16}$$

The expressions for kinetic (T) and potential (U) energies developed above can now be used to derive the governing differential equations of motion of a micro beam undergoing free natural vibration by applying Hamilton's principle.

Hamilton's principle states

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \tag{17}$$

where t_1 and t_2 are the time interval in the dynamic trajectory, and δ is the usual variational operator.

Substituting T and U from Equations (16) and (14) into Equation (17) and then integrating by parts and collecting terms give the governing differential equations and natural boundary conditions as follows.

Governing differential equations:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{18}$$

$$EI\frac{\partial^2 \theta}{\partial x^2} + kAG\left(\frac{\partial v}{\partial x} - \theta\right) + \frac{AGl^2}{4}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^3 v}{\partial x^3}\right) = \rho I \frac{\partial^2 \theta}{\partial t^2} \tag{19}$$

$$kAG\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) - \frac{AGl^2}{4}\left(\frac{\partial^3 \theta}{\partial x^3} + \frac{\partial^4 v}{\partial x^4}\right) = \rho A \frac{\partial^2 v}{\partial t^2} \tag{20}$$

Natural boundary conditions

Axial force: $$f = -EA\frac{\partial u}{\partial x} \tag{21}$$

Shear force: $$s = -kAG\left(\frac{\partial v}{\partial x} - \theta\right) + \frac{GAl^2}{4}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^3 v}{\partial x^3}\right) \tag{22}$$

Bending Moment: $$m = -EI\frac{\partial \theta}{\partial x} - \frac{GAl^2}{4}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) \tag{23}$$

Higher order moment: $$\bar{m} = -\frac{GAl^2}{4}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) \tag{24}$$

Clearly the axial motion is uncoupled from the bending or flexural motion, but importantly, the free vibration behaviour in axial motion is not affected by the material scale length parameter (l), unlike the case with bending of flexural vibration, see Equations (18)-(20). Therefore, attention here is focused on bending or flexural vibration only when developing the dynamic stiffness method because the axial vibration being unaffected by material scale length parameter, is covered by classical theory that can be found in stand texts.

However, it should be recognized that for a Timoshenko-Ehrenfest micro beam in bending or flexural motion, there are three displacement components v, θ and v' that arise will lead to a sixth order system whereas in classical Timoshenko-Ehrenfest beam theory, the displacement field consists of v and θ only (without the v') resulting in a fourth order system. This difference was observed by earlier investigators (Ma et al., 2008; Reddy, 2011; Mustapha, 2020), but apparently, these investigators did not pay much attention to the additional displacement parameter v', particularly when obtaining the results. One of the purposes of this investigation is to consider the new displacement component v' both in the theory as well as in the results.

2.3. Formulation of the dynamic stiffness matrix

We assume harmonic oscillation with circular or angular frequency ω rad/s for the Timoshenko-Ehrenfest micro beam undergoing free bending or flexural motion so that

$$v = Ve^{i\omega t}; \quad \theta = \Theta e^{i\omega t} \tag{25}$$

where V and Θ are the amplitudes of the bending displacement and bending rotation, respectively and $i = \sqrt{-1}$.

Substituting Equation (25) into Equations (19) and (20) and introducing the non-dimensional length $\xi = x/L$ led to the following ordinary differential equations

$$\left(\frac{AGl^2}{4L^3}D^3 + \frac{kAG}{L}D\right)V + \left[\frac{1}{L^2}\left(EI + \frac{AGl^2}{4}\right)D^2 + \rho I\omega^2 - kAG\right]\Theta = 0 \tag{26}$$

$$\left(-\frac{AGl^2}{4L^4}D^4 + \frac{kAG}{L^2}D^2 + \rho A\omega^2\right)V - \left(\frac{AGl^2}{4L^3}D^3 + \frac{kAG}{L}D\right)\Theta = 0 \tag{27}$$

It is now possible to eliminate either V or Θ from Equations (26) and (27) to arrive at the following sixth order ordinary differential equation which is identically satisfied by both V and Θ.

$$(a_1D^6 + a_2D^4 + a_3D^2 + a_4)H = 0 \tag{28}$$

where

$$H = V \text{ or } \Theta \tag{29}$$

and

$$\left.\begin{aligned} a_1 &= \lambda^2 s^2 \\ a_2 &= \lambda^2 b^2 r^2 s^2 - 4[\lambda^2 + 2(1+\nu)r^2] \\ a_3 &= -[\lambda^2 b^2 s^2 + 8(1+\nu)b^2 r^2 (r^2 + s^2)] \\ a_4 &= 8(1+\nu)b^2 r^2 (1 - b^2 r^2 s^2) \end{aligned}\right\} \tag{30}$$

with

$$\lambda = \frac{l}{L}; \quad b^2 = \frac{\rho A\omega^2 L^4}{EI_e}; \; r^2 = \frac{I}{AL^2}; \quad s^2 = \frac{EI}{kAGL^2}; \quad \nu = \frac{E}{2G} - 1 \tag{31}$$

It should be noted that the differential equation for free vibration of a conventional beam of normal length using classical theory whether Bernoulli-Euler or Timoshenko-Ehrenfest theory results in a fourth order differential equation as opposed to the sixth order differential equation for a Timoshenko-Ehrenfest micro beam given by Equation (28).

By assuming the solution in the form $H = e^{p\xi}$ where p is a constant, yet to be determined, the characteristic or auxiliary equation of the differential equation, Equation (28) can be expressed as

$$a_1 p^6 + a_2 p^4 + a_3 p^2 + a_4 = 0 \tag{32}$$

The polynomial equation, Equation (32) can be reduced to a cubic and then solved analytically using standard procedure (Pipes and Harvill, 2014). By taking the square root of each of the three roots of the cubic, which could be either real or complex, the six roots r_j $(j = 1, 2, \cdots, 6)$ of the characteristic or auxiliary equation Equation (32) can be computed leading to the solutions of the differential equation, Equation (28) as:

$$V(\xi) = \sum_{j=1}^{6} R_j e^{r_j \xi}; \quad \Theta(\xi) = \sum_{j=1}^{6} Q_j e^{r_j \xi} \tag{33}$$

where R_j and Q_j $(j = 1, 2, \cdots, 6)$ are two sets of different constants which can be related to each other, by using Equations (26) and (27). The relationship between $R_j$and $Q_j$is obtained as:

$$Q_j = \alpha_j R_j \tag{34}$$

where

$$\alpha_j = \frac{8(1+\nu) r^2 [r_j^2 + b^2 s^2] - \lambda^2 s^2 r_j^2}{r_j \left[8(1+\nu) r^2 + \lambda^2 s^2 r_j^2 \right]} \tag{35}$$

The constant vectors **Q** and **R** can be written as:

$$\mathbf{Q} = [Q_1 \; Q_2 \; Q_3 \; Q_4 \; Q_5 \; Q_6]^T; \quad \mathbf{R} = [R_1 \; R_2 \; R_3 \; R_4 \; R_5 \; R_6]^T \tag{36}$$

where the upper suffix *T* denotes a transpose.

The amplitude of the bending or flexural rotation in terms of R_j from Equations (33) and (34) is

$$\Theta = \frac{1}{L} \sum_{j=1}^{6} \alpha_j \, R_j e^{r_j \xi} \tag{37}$$

The first derivative of the amplitude of the bending or flexural displacement is

$$V' = \frac{dV}{dx} = \frac{1}{L} \frac{dV}{d\xi} = \frac{1}{L} \sum_{j=1}^{6} r_j \, R_j e^{r_j \xi} \tag{38}$$

The amplitudes of the shear force (S), bending moment (M) and higher order moment ($\bar{M}$) are obtained in terms of R_j using Equations (22)-(25) and Equations (31), (33), (37) and (38) and after making some algebraic manipulation to give

$$S = \frac{EI}{L^3}\sum_{j=1}^{6}\mu_j R_j e^{r_j\xi};\ \ M = -\frac{EI}{L^2}\ \sum_{j=1}^{6}\tau_j R_j e^{r_j\xi}\ ;\ \ \bar{M} = -\frac{EI}{L^2}\sum_{j=1}^{6}\sigma_j R_j e^{r_j\xi} \tag{39}$$

where

$$\left.\begin{aligned}\mu_j &= \frac{\lambda^2 r_j^2}{8r^2(1+\nu)}\left[\alpha_j + r_j\right] - \frac{r_j}{s^2} + \frac{\alpha_j}{s^2}\\ \tau_j &= \frac{\lambda^2 r_j}{8r^2(1+\nu)}\left[\alpha_j + r_j\right] + \alpha_j r_j\\ \sigma_j &= \frac{\lambda^2 r_j}{8r^2(1+\nu)}\left[\alpha_j + r_j\right]\end{aligned}\right\} \tag{40}$$

The dynamic stiffness matrix of the Timoshenko-Ehrenfest micro beam can now be formulated by applying the boundary conditions in algebraic form for both displacements and forces at the ends (nodes) of the beam. The procedure essentially eliminates the constants R_j (j =1, 2 ,3….6) from Equations (33) and (39) to construct the required dynamic stiffness matrix, providing the force-displacement relationship at the nodes. Referring to the sign convention for positive shear force, bending moment and higher order moment shown in Figure 2, the boundary conditions for displacements and forces are illustrated in Figure 3. These are:

$$\text{At } \xi = 0:\ V = V_1;\ \ \Theta = \Theta_1;\ \ V' = V_1';\ \ S = S_1;\ \ M = M_1;\ \bar{M} = \bar{M}_1 \tag{41}$$

$$\text{At } \xi = 1:\ V = V_2;\ \ \Theta = \Theta_2;\ \ V' = V_2';\ \ S = -S_2;\ \ M = -M_2;\ \bar{M} = \overline{-M}_2 \tag{42}$$

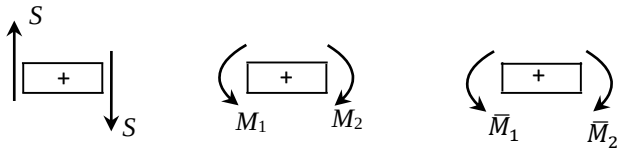

Figure 2. Sign convention for positive shear force S, bending moment M and higher order moment $\bar{M}$

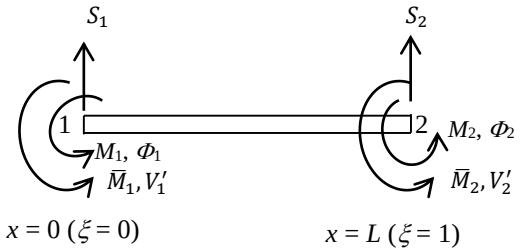

Figure 3. Boundary conditions for bending displacement, bending rotation, first derivative of bending displacement, shear force, bending moment and higher order moment.

Substituting Equations (41) and (42) into Equations (33), (37) and (38), gives the following matrix equation relating the displacements of the two ends with the constants R_j (j =1, 2, 3,..6).

$$\begin{bmatrix} V_1 \\ \Theta_1 \\ V_1' \\ V_2 \\ \Theta_2 \\ V_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1/L & \alpha_2/L & \alpha_3/L & \alpha_4/L & \alpha_5/L & \alpha_6/L \\ r_1/L & r_2/L & r_3/L & r_4/L & r_5/L & r_6/L \\ e^{r_1} & e^{r_2} & e^{r_3} & e^{r_4} & e^{r_5} & e^{r_6} \\ \alpha_1 e^{r_1}/L & \alpha_2 e^{r_2}/L & \alpha_3 e^{r_3}/L & \alpha_4 e^{r_4}/L & \alpha_5 e^{r_5}/L & \alpha_6 e^{r_6}/L \\ r_1 e^{r_1}/L & r_2 e^{r_2}/L & r_3 e^{r_3}/L & r_4 e^{r_4}/L & r_5 e^{r_5}/L & r_6 e^{r_6}/L \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} \tag{43}$$

or

$$\boldsymbol{\delta} = \mathbf{BR} \tag{44}$$

Similarly substituting Equations (41) and (42) into Equation (39) gives the following matrix equation relating the forces at the two ends with the constants R_j (j =1, 2, 3,.. 6).

$$\begin{bmatrix} S_1 \\ M_1 \\ \bar{M}_1 \\ S_2 \\ M_2 \\ \bar{M}_2 \end{bmatrix} = \begin{bmatrix} T_3\mu_1 & T_3\mu_2 & T_3\mu_3 & T_3\mu_4 & T_3\mu_5 & T_3\mu_6 \\ -T_2\tau_1 & -T_2\tau_2 & -T_2\tau_3 & -T_2\tau_4 & -T_2\tau_5 & -T_2\tau_6 \\ -T_2\sigma_1 & -T_2\sigma_2 & -T_2\sigma_3 & -T_2\sigma_4 & -T_2\sigma_5 & -T_2\sigma_6 \\ -T_3\mu_1 e^{r_1} & -T_3\mu_2 e^{r_2} & -T_3\mu_3 e^{r_3} & -T_3\mu_4 e^{r_4} & -T_3\mu_4 e^{r_5} & -T_3\mu_4 e^{r_6} \\ T_2\tau_1 e^{r_1} & T_2\tau_2 e^{r_2} & T_2\tau_3 e^{r_3} & T_2\tau_4 e^{r_4} & T_2\tau_5 e^{r_5} & T_2\tau_6 e^{r_6} \\ T_2\sigma_1 e^{r_1} & T_2\sigma_2 e^{r_2} & T_2\sigma_3 e^{r_3} & T_2\sigma_4 e^{r_4} & T_2\sigma_5 e^{r_5} & T_2\sigma_6 e^{r_6} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} \tag{45}$$

or

$$\mathbf{F} = \mathbf{DR} \tag{46}$$

where

$$T_2 = \frac{EI}{L^2}; \quad T_3 = \frac{EI}{L^3} \tag{47}$$

By eliminating the constant vector **R** from Equations (44) and (46), **F** and **δ** can now be related to give the dynamic stiffness matrix relationship of the Timoshenko-Ehrenfest micro beam as

$$\mathbf{F} = \mathbf{K}\,\boldsymbol{\delta} \tag{48}$$

where

$$\mathbf{K} = \mathbf{D}\,\mathbf{B}^{-1} \tag{49}$$

is the required 6×6 frequency-dependent dynamic stiffness matrix. It should be noted that the resulting dynamic stiffness matrix **K** of Equation (49) will be always symmetric and real as observed in an earlier investigation (Banerjee, 2003), with imaginary part of each element being zero although the matrices **B** and **D** are complex and asymmetric. The real part of the expanded dynamic stiffness matrix giving the relationship between the amplitudes of the forces to those of the displacements can be expressed in the following way.

$$\begin{bmatrix} S_1 \\ M_1 \\ \bar{M}_1 \\ S_2 \\ M_2 \\ \bar{M}_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{12} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{13} & k_{23} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{45} & k_{46} \\ k_{15} & k_{25} & k_{35} & k_{45} & k_{55} & k_{56} \\ k_{16} & k_{26} & k_{36} & k_{46} & k_{56} & k_{66} \end{bmatrix} \begin{bmatrix} V_1 \\ \Theta_1 \\ V_1' \\ V_2 \\ \Theta_2 \\ V_2' \end{bmatrix} \tag{50}$$

The above frequency dependent dynamic stiffness matrix **K** can now be used to compute the natural frequencies and mode shapes of either an individual Timoshenko-Ehrenfest micro beam, or an assembly of them for different boundary conditions. A reliable and accurate method of solving the eigenvalue problem is to apply the Wittrick-Williams algorithm (Wittrick and Williams, 1971) which is well suited for the dynamic stiffness method applications. The algorithm uses the Sturm sequence property of the dynamic stiffness matrix and ensures that no natural frequencies of the structure analysed are missed. The detailed working procedure of the algorithm is not given here because it has featured in literally hundreds of papers in the literature. However, interested readers are referred to the original work of Wittrick and Williams (Wittrick and Williams, 1971) for a detailed insight.

3. Numerical results and discussion

The dynamic stiffness matrix developed above is applicable to free vibration analysis of a Timoshenko-Ehrenfest micro beam in bending or flexural motion, but the dynamic stiffness matrix in axial vibration (Howson et al., 1983) which is uncoupled from bending or flexural vibration can also be incorporated. However, it is well known that the small length parameter (l) does not affect the axial vibration characteristics of a micro beam when using the MCST (Ma et al., 2008; Reddy, 2011; Mustapha, 2020). Therefore, leaving aside the axial vibration, only the bending or flexural vibration is considered here when computing the numerical results.

The first set of results was computed for an individual micro beam of length L with rectangular cross-section of width b and depth (or thickness) h, for three different boundary conditions, namely, the Simply supported-Simply supported (S-S), Clamped- Free (C-F) and Clamped-Clamped (C-C). The data used in the analysis are taken from (Reddy, 2011; Mustapha, 2020) and are given as follows.

Young's modulus E=1.44 GPa, density ρ = 1220 kg/m^3, Poisson's ration ν = 0.38, the shear correction or shape factor k = 0.833333, $h = 17.6\times10^{-6}$ m, $b = 2h$ and $L = 20h$.

The first five nondimensional natural frequencies $\lambda_i = \omega_i L^2\sqrt{\rho A/EI}$ (i = 1, 2, ..5) of the micro beam for the S-S, C-F and C-C boundary conditions are respectively shown in Tables 1-3 for various values of the size-dependent non-dimensional material length scale parameter l/h (or l/L). Note that the value of l must not be literally zero in the data, but a negligibly small value close to zero, e,g. $l = h\times10^{-6}$ can be used to avoid numerical overflow. From a computational accuracy standpoint, this is all right, bearing in mind that l = 0 corresponds to the case of classical Bernoulli-Euler or Timoshenko-Ehrenfest beam for which results can be found in standard texts.

Comparative results for the first three natural frequencies of the Timoshenko—Ehrenfest micro beam for S-S boundary condition have been published by Reddy (2011) and are shown in Table 1 alongside the results computed using the present theory. Clearly, the results computed using the present theory are in excellent agreement with those of Reddy (2011).

Table 1. Non-dimensional natural frequencies of a simply supported (S-S) Timoshenko-Ehrenfest micro beam.

l/h	*l/L*	Non-dimensional natural frequency $\lambda_i = \omega_i L^2\sqrt{\rho A/EI}$							
		λ_1		λ_2		λ_3		λ_4	λ_5
		Present theory	Reddy (2011)	Present theory	Reddy (2011)	Present Theory	Reddy (2011)	Present theory	Present theory
0.0	0.0	9.8696	9.86	39.478	39.32	88.827	88.02	157.91	246.74
0.2	0.01	10.693	10.68	42.773	42.60	96.240	95.36	171.09	267.35
0.4	0.02	12.852	12.84	51.409	51.20	115.67	114.61	205.63	321.31
0.6	0.03	15.808	15.79	63.229	62.97	142.27	140.97	252.91	395.18
0.8	0.04	19.195	19.18	76.781	76.47	172.76	171.18	307.12	479.88
1.0	0.05	22.824	22.80	91.295	90.92	205.41	203.54	365.17	570.58

The natural frequencies for C-F boundary condition of the micro beam are shown in Table 2. The first three natural frequencies for this problem for *l/h* =0.0 and 1.0 published recently by Mustapha (2020) who used finite element method, but importantly, accounted for the shear deformation and rotatory inertia effects as in the present case, agreed very well with the above results, but are not shown for brevity, the disagreement being less than 5%.

Table 2. Non-dimensional natural frequencies of a cantilever (C-F) Timoshenko-Ehrenfest micro beam.

l/h	*l/L*	Non-dimensional natural frequency $\lambda_i = \omega_i L^2\sqrt{\rho A/EI}$				
		λ_1	λ_2	λ_3	λ_4	λ_5
0.0	0.0	3.5159	22.035	61.698	120.90	199.86
0.2	0.01	3.8095	23.875	66.848	130.99	216.54
0.4	0.02	4.5786	28.692	80.341	157.44	260.25
0.6	0.03	5.6313	35.291	98.817	193.64	320.10
0.8	0.04	6.8383	42.854	119.99	235.13	388.70
1.0	0.05	8.1308	50.955	142.68	279.58	388.70

For completeness, the natural frequencies of the Timoshenko-Ehrenfest micro beam for the Clamped-Clamped (C-C) boundary condition were also computed, and they are shown in Table 3. The authors could not find suitable comparable results in the literature for this case.

Representative mode shapes of the Timoshenko-Ehrenfest micro beam for S-S, C-F and C-C boundary conditions with $\frac{l}{h} = 0.6$ are illustrated in Figs. 4-6, respectively showing bending displacement (V), bending rotation (Θ) and the first derivative of the bending displacement (V'). To be consistent with the units, the bending rotation (Θ) and the first derivative of the bending displacement (V') were multiplied by the width (b) of the Timoshenko-Ehrenfest micro beam when plotting the mode shapes. It is clear from the mode shapes that for lower order modes, the bending rotation (Θ) and the first derivative of the bending displacement (V') are almost indistinguishable, but for higher order modes, some differences in their values exist.

Table 3. Non dimensional natural frequencies of a Timoshenko-Ehrenfest micro beam with clamped-clamped (C-C) boundary condition.

$\frac{h}{l}$	$\frac{l}{L}$	$\lambda_i = \omega_i L^2\sqrt{\rho A/EI}$				
		λ_1	λ_2	λ_3	λ_4	λ_5
0.0	0.0	3.44003	21.7116	59.7297	114.126	182.965
0.1	0.005	3.58437	22.2007	61.0406	116.640	187.039
0.2	0.01	3.80188	23.5471	64.7489	123.747	198.501
0.3	0.015	4.13881	25.6324	70.4685	134.672	216.041
0.4	0.02	4.56890	28.2857	77.7324	148.484	238.120
0.5	0.025	5.06813	31.3587	86.1119	164.343	263.328
0.6	0.03	5.61808	34.7348	95.2749	181.584	290.556
0.7	0.035	6.20530	38.3241	104.972	199.715	318.10
0.8	0.04	6.81946	42.0683	115.022	218.371	348.021
0.9	0.045	7.45426	45.9157	125.291	237.289	377.180
1.0	0.05	8.104108	49.83725	135.6843	256.2571	406.159

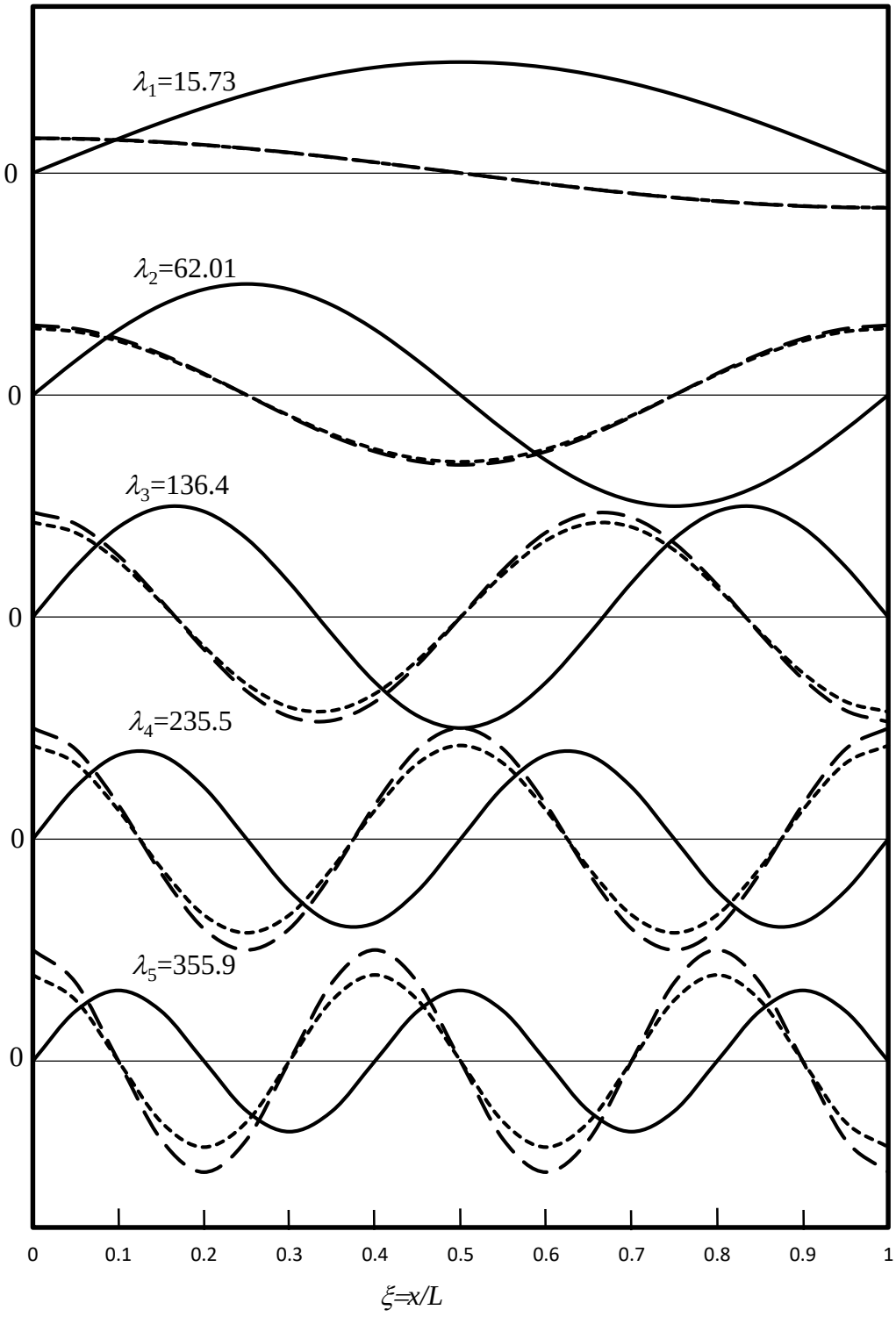

Figure 4. The mode shapes for Timoshenko-Ehrenfest micro beam with S-S boundary condition for $l/h = 0.6$

——— V; - - - - - - - - - - $b\Theta$; — — — — — — — — bV'

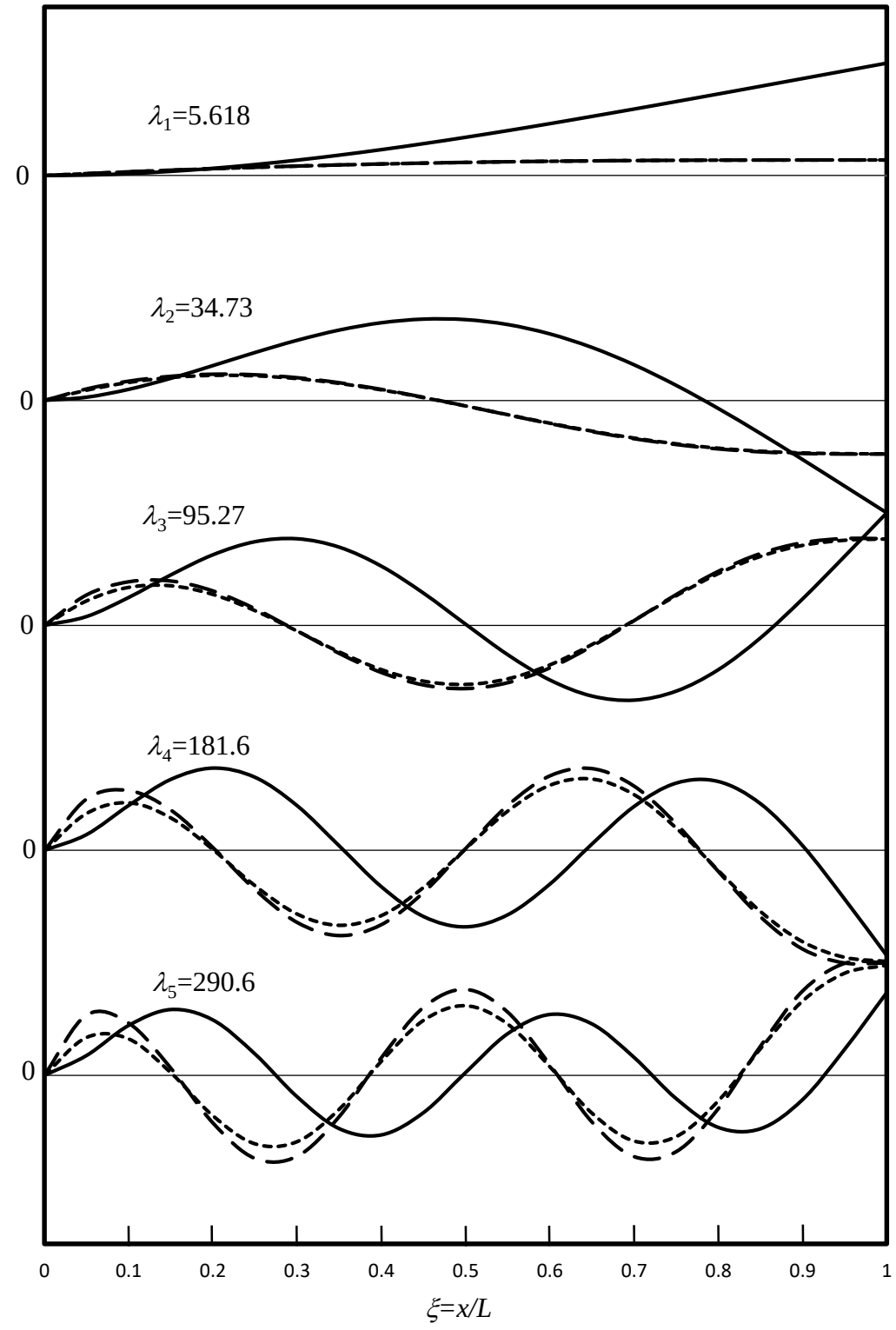

Figure 5. The mode shapes for Timoshenko-Ehrenfest micro beam with C-F boundary condition for l/h = 0.6.

——— V; - - - - - - - - - $b\Theta$; — — — — — — — bV'

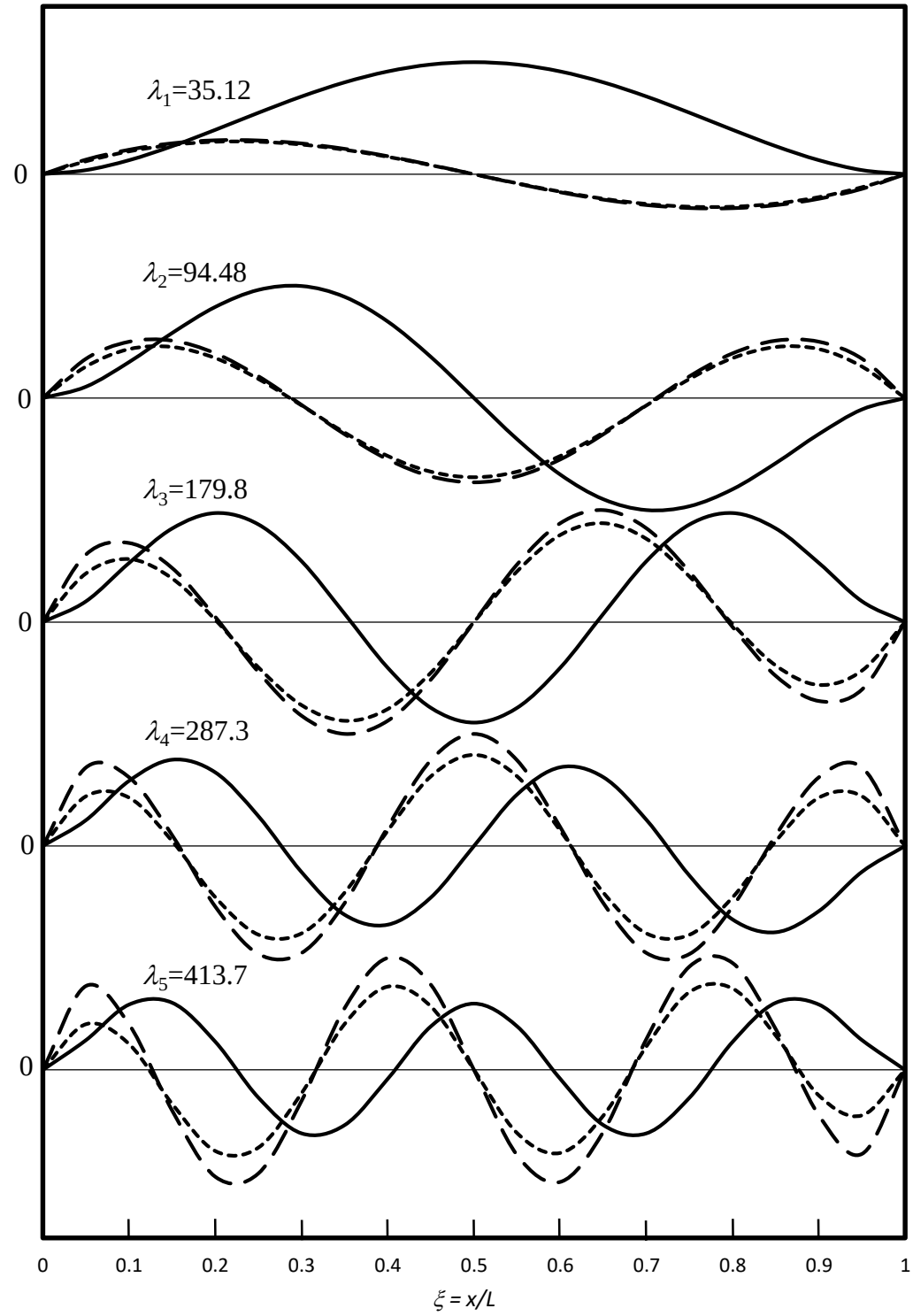

Figure 6. The mode shapes for Timoshenko-Ehrenfest micro beam with C-C boundary condition for l/h = 0.6.

——— V; - - - - - - - - - $b\Theta$; — — — — — — — — bV'

The final set of results was computed for a stepped micro beam with cantilever boundary condition shown in Figure 7, by using the above dynamic stiffness theory. Each of the two segments of the stepped micro beam has a solid circular cross-section with diameters $d_1 = 25 \times 10^{-6}$ m and $d_2 = 15 \times 10^{-6}$ m, respectively with the corresponding lengths of the two segments are taken as $L_1 = 100 \times 10^{-6}$ m and $L_2 = 125 \times 10^{-6}$ m so that the total length $L = 225 \times 10^{-6}$ m. The Young's modulus (E), density (ρ), Poisson's ratio (ν) of material and shear correction factor k are taken to be the same as the previous examples.

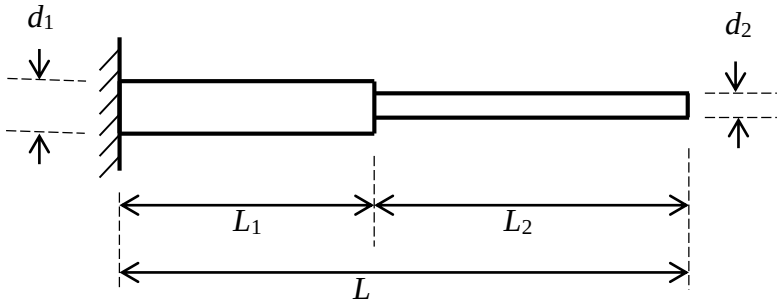

Figure 7. A stepped micro beam with cantilever boundary condition.

Table 4 shows the first five natural frequencies of the stepped micro beam for different values of $\frac{l}{L}$ (or $\frac{l}{d_2}$) ratios using both Timoshenko-Ehrenfest and Bernoulli-Euler dynamic theories. It should be noted that the results for $\frac{l}{L} = 0$ corresponds to classical Bernoulli-Euler and/or Timoshenko-Ehrenfest theories which do account for the small-scale effect. These results for $\frac{l}{L} = 0$ when using Bernoulli-Euler and Timoshenko-Ehrenfest dynamic theories were further checked using some well-established computer programs (Howson et al., 1983; Anderson et al., 1986). Clearly, for higher values of $\frac{l}{L}$ (or $\frac{l}{d_2}$) and for higher natural frequencies, the differences in the results using the two theories are pronounced, but importantly, the effects of shear deformation and rotatory inertia are more pronounced in micro beams when compared to classical beams. The Timoshenko-Ehrenfest theory gives lower values than that of the Bernoulli-Euler theory as expected. The percentage difference between the two theories in the third natural frequency for different values of $\frac{l}{L}$ are shown in Figure 8.

Table 4. Natural frequencies of a stepped cantilever micro beam.

l/L	l/d_2	Natural frequency f_i (MHz)					
		f_1		f_2		f_3	
		Timoshenko-Ehrenfest Theory	Bernoulli-Euler Theory	Timoshenko-Ehrenfest Theory	Bernoulli-Eule Theory	Timoshenko-Ehrenfest Theory	Bernoulli-Euler Theory
0.0	0.0	0.09774	0.09844	0.34744	0.35775	0.915654	0.98067
0.01	0.15	0.10160	0.10232	0.36309	0.37346	0.96420	1.08860
0.02	0.30	0.11169	0.11255	0.40473	0.41658	1.08460	1.16300
0.03	0.45	0.12577	0.12683	0.46465	0.47897	1.24960	1.34640
0.04	0.60	0.14242	0.14370	0.53595	0.55366	1.43880	1.55980

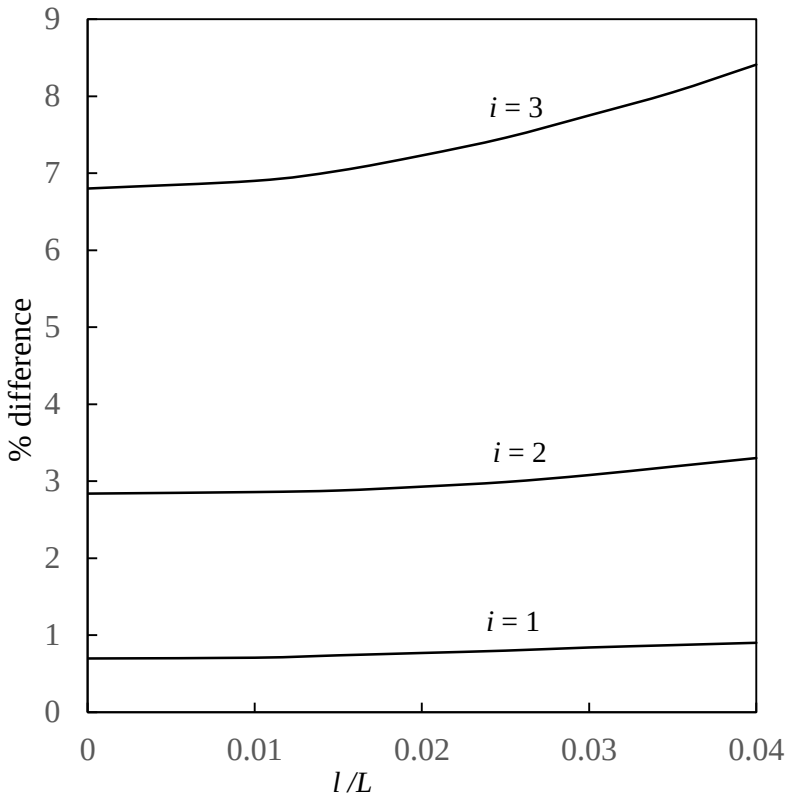

Figure 8. The effect of small length parameter on the percentage difference in the first three natural frequencies of the stepped micro beam shown in **Figure 7** when using Bernoulli-Euler and Timoshenko-Ehrenfest theories.

4. Conclusions

Using the Timoshenko-Ehrenfest beam theory and the modified couple stress model, the dynamic stiffness matrix of a micro beam is developed. The ensuing dynamic stiffness matrix which relates the amplitudes of the forces to those of the corresponding displacements of the harmonically vibrating Timoshenko-Ehrenfest micro beam is applied in conjunction with the Wittrick-Williams algorithm to compute the natural frequencies and mode shapes for different boundary conditions of some illustrative examples. A substantial part of the results is validated against published results. To demonstrate the versatility of the method, a stepped micro beam is also analysed. One of the displacement fields which is the first derivative of the flexural displacement appearing in the governing differential equations and natural boundary conditions has not been given due recognition in the literature. The current investigation has addressed this issue and provided the influence of this displacement on results. The results given in the paper can be used as benchmark to validate finite element and other approximate methods.

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