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Review

Advances in the Theory of Compact Groups and Pro-Lie Groups in the Last Quarter Century †

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Abstract: This article surveys the development of the theory of compact groups and pro-Lie groups, contextualizing the major achievements over 125 years and focusing on some progress in the last quarter century. It begins with developments in the 18th and 19th centuries. Next is from Hilbert's Fifth Problem in 1900 to its solution in 1952 by Montgomery, Zippin, and Gleason and Yamabe's important structure theorem on almost connected locally compact groups. This half century included profound contributions by Weyl and Peter, Haar, Pontryagin, van Kampen, Weil, and Iwasawa. The focus in the last quarter century has been structure theory, largely resulting from extending Lie Theory to compact groups and then to pro-Lie groups, which are projective limits of finite-dimensional Lie groups. The category of pro-Lie groups is the smallest complete category containing Lie groups and includes all compact groups, locally compact abelian groups, and connected locally compact groups. Amongst the structure theorems is that each almost connected pro-Lie group G is homeomorphic to $\mathbb{R}^I \times C$ for a suitable set I and some compact subgroup C. Finally, there is a perfect generalization to compact groups G of the age-old natural duality of the group algebra $\mathbb{R}[G]$ of a finite group G to its representation algebra $R(G,\mathbb{R})$, via the natural duality of the topological vector space $\mathbb{R}^I$ to the vector space $\mathbb{R}^{(I)}$, for *any* set I, thus opening a new approach to the Hochschild-Tannaka duality of compact groups.

Keywords: topological group; Lie group; compact group; pro-Lie group; Lie algebra; duality; Tannaka duality; Pontryagin duality; LCA group

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1. Introduction

Certain areas of mathematical research draw their particular fascination from the fact that they are based between two principal domains of mathematics, such as algebra and topology. Between these two, we find algebraic topology and topological algebra. An observer looking at mathematics from a distance may wonder if these two fields differ much. The language itself points out the difference: Topological algebra is a specialty located in algebra, the art of calculating—adding and multiplying, while using the tools of geometry, and manipulating the concept of continuity adds an extra attraction.

Groups emerged in 1770 in the work on permutation groups of Joseph-Louis Lagrange (1736–1813) and in 1799 in the context of solving quintic equations in the work of Paolo Ruffini (1765–1822). Groups in their abstract form can to be traced back to Augustin-Louis Cauchy (1789–1857), Niels Henrik Abel (1802–1829), and Évariste Galois (1811–1832), when groups were formative in the development of abstract algebra. Galois was, in fact, the first to use the word group (*groupe* in French). The beginnings of *Topology* reach back hundreds of years; however, as August Ferdinan Möbius (1790–1868) said, it was Jules Henri Poincaré (1854–1912) who "gave topology wings" in several articles, the first of which appeared in 1895. (Johann Benedict Listing (1808–1882) introduced the (German) term *Topologie* in 1847.) *Topology* as an independent area had not yet crystallized, though *Geometry* was quite present, when Felix Klein (1849–1925) and Sophus Lie (1842–1899) (and followers, such as Friedrich Engel (1861–1941) and Wilhelm Karl Joseph Killing (1847–1923)) founded the area of what later became named *Lie groups*. Algebra, geometry, and analysis were thoroughly mixed into the genesis of Lie group theory.

2. Hilbert's Fifth Problem and Locally Compact Groups

In 1900, David Hilbert (1862–1943) gave his famous address to the International Congress of Mathematicians in Paris. In an apparently unforgettable fashion, it foreshadowed crucial developments of mathematical research in the 20th century. Hilbert formulated 23 open problems leading to groundbreaking research in the 20th century. By that time, *topology* was present in the minds of mathematicians, although it may not have reached the heights it would attain in the course of the century. Yet, enough was available to Hilbert for him to formulate, for instance, his famous Fifth Problem:

If a group is defined on a euclidean manifold in such a way that multiplication and inversion are continuous functions, can it be given the structure of a **differentiable** *manifold so that the continuous group operations are in fact differentiable?*

This would make it a group of the kind that Lie had created in a visionary way. In modern parlance, Hilbert posed the question:

Is a locally euclidean topological group a Lie group?

He envisioned a positive answer. However, it would take a little over half a century to confirm his vision.

Yet, this half century advanced the research of topological groups enormously. The most consequential steps were:

(i) the discovery of fundamental properties of *compact groups* by Hermann Weyl (1885–1955) and his doctoral student Fritz Peter (1899–1949) in 1927;
(ii) the discovery that every *locally compact group* has a (left) invariant measure by Alfréd Haar (1885–1933) in 1932; and
(iii) the discovery in 1934 of the duality between the *category of (discrete) abelian groups* and the *category of compact abelian groups* by Lev Semyonovich Pontryagin (1908–1988), rounded off in 1935 with the extension to arbitrary *locally compact abelian groups* by Egbert van Kampen (1908–1942), and by André Weil (1906–1998) in 1938, who also established that *a complete topological group with a Haar measure has to be* locally compact. (See References [1–3]. For a discussion of Pontryagin Duality outside the class of locally compact abelian groups, see Reference [4] and its references. For a category theory proof of Pontryagin Duality, see Reference [5].)

Inasmuch as these milestones were set up close to Lie groups, they are naturally linked to topological groups whose underlying topological spaces (for the most part) are connected. It was recognized early on that, in a topological group G, the connected component G_0 of the identity is a closed normal subgroup which is mapped into itself by any continuous endomorphism of G. (We recall that such a subgroup is called *fully characteristic*.) Obviously, therefore, it is very special. Indeed, in any Lie group (real or complex), the benefit drawn from the presence of the Lie algebra of G invented by Sophus Lie reaches as far as G_0, and not the tiniest bit beyond. One is tempted to remark that G_0 *supports all the (traditional) geometry of group theory*.

The observation that real Lie algebras are attached to topological groups in a more general sense was first anticipated by Richard Lashof (1922–2010) in 1957 for locally compact groups. More recently, as explained and illustrated in our book (Reference [6]), this was extended to a much wider class of topological groups. It is natural to ask how much of the structure of a topological group G is supported by the elementary concept of

$\mathrm{Hom}(\mathbb{R}, G)$, the space of morphisms of topological groups $\mathbb{R} \to G$, also called *one-parameter subgroups* of G.

That half-a-century of developments of topological groups went alongside an astounding unfolding of topology. However, there was a second impact on the domain of topological groups. This advance emerged from algebra itself, more specifically, from GALOIS THEORY. As a typical example, the algebraic completion A of a field F is the directed colimit of all finite extensions $(K{:}F)$. Inevitably, the Galois group $G(A{:}F)$ is the projective limit of the finite Galois groups $G(K{:}F)$. A projective limit G of a directed inverse system of finite groups automatically carries a group topology making it a compact totally disconnected topological group. Here, *totally disconnected* means exactly that G_0 is a singleton subgroup. This example clearly illustrates the fact that this group theory, belonging to the home of pure classical algebra, uncontroversially leads to a class of topological groups located opposite to the type of connected topological groups which have arisen, historically, out of Lie theory. Yet, the link between the two disparate classes of topological groups was, from the very beginning, the fact that:
every topological group G gives rise to a connected topological subgroup G_0, its identity component*, and, by contrast, the totally disconnected quotient group $G_t = G/G_0$.*

The complete solution of Hilbert's Fifth Problem arrived in 1952 (9 years after the death of Hilbert), when Andrew Mattei Gleason (1921–2008), Deane Montgomery (1909–1992), and Leo Zippin (1905–1995) settled it with a positive answer. This effort was crowned by the fundamental discovery in 1953 by Hidehiko Yamabe (1923–1960) that:
in a topological group G whose component factor group G_t is compact, any compact identity neighborhood of G contains a closed normal subgroup N, such that the factor group G/N is a Lie group, indeed, precisely one of those Lie groups, which had so fascinated Hilbert in 1900. The compactness of the factor group $G_t = G/G_0$ led to the standard terminology that a topological group having this property is called *almost connected*.

Yamabe's major contribution to the solution of Hilbert's Fifth Problem was soon followed by an immensely influential paper [7], by Kenkichi Iwasawa (1917–1998), on the structure of locally compact groups.

One way of expressing the theorem of Yamabe was to say that:
every almost connected group is a projective limit of Lie groups.
(Projective limits are discussed and explained, e.g., in References [1,6].)

This fact caused much of the work on *locally compact groups* in the second half of last century to be focused on *projective limits of Lie groups*. In this endeavor, it is truly very helpful that the projective limit presentation of an *almost connected locally compact group G*, in terms of its Lie group quotients G/N, has limit maps $G \to G/N$ that are particularly well behaved because each of them is a *proper* morphism, i.e., it is a closed continuous map such that the inverse image of each compact set is compact.

A substantial step in a general structure theory of locally compact totally disconnected groups occurred in the 1990s, when George A. Willis innovated the theory by introducing concepts, such as *tidy subgroups* and *scaling functions* [8,9].

The second half of the twentieth century saw a substantial amount of research on what has become known as *Abstract Harmonic Analysis*. This subject, outside the scope of this survey, was built on the realization by André Weil that, using Haar measure, Fourier series and Fourier integrals are a special case of a construction on locally compact groups. The standard references are References [2,10,11], but also see Reference [12].

3. Pro-Lie Groups: From Connected to Almost Connected Ones

One should be aware of the fact that not every locally compact group is a projective limit of Lie groups, as $\mathrm{SL}(2, \mathbb{Q}_p)$, the group of p-adic 2 by 2 matrices of determinant 1, illustrates for any prime p.

However, within topological group theory, in this immense activity of the 20th century on the projective limit representation of locally compact (and, in particular, compact) groups, it was almost overlooked that a topological group G which has a filter basis $\mathcal{N}$ of closed normal subgroups N such that, firstly,

(1) *G/N is a Lie group for all $N \in \mathcal{N}$*

and, secondly,

(2) *the natural map $G \to \lim_{N\in\mathcal{N}} G/N$ is an isomorphism*

certainly does not force G to be locally compact. In fact, under these circumstances, G is locally compact if and only if $\mathcal{N}$ contains a compact member. The simplest examples failing to be locally compact are groups, such as $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$, with their product topologies. Indeed, the second of these examples illustrates the fact that we are touching a subject that is understood with the concepts of basic linear algebra over the real or complex field (or, indeed, any locally compact field). Given the Axiom of Choice, we know that every vector space V over $\mathbb{R}$ has a basis, equivalently, that it is a direct sum $\bigoplus_{j\in J}\mathbb{R}_j$ of some family of copies $\mathbb{R}_j \cong \mathbb{R}$ of $\mathbb{R}$, denoted by $\mathbb{R}^{(J)}$. The vector space $\mathrm{Hom}(V,\mathbb{R})$ of all linear forms of V is (naturally isomorphic to) the Cartesian product $\mathrm{Hom}(\bigoplus_{j\in J}\mathbb{R}_j,\mathbb{R}) \cong \prod_{j\in J}\mathrm{Hom}(\mathbb{R}_j,\mathbb{R}) \cong \mathbb{R}^J$ of copies of $\mathbb{R}$. Since $\mathbb{R}$ has a natural topology, this is true for $\mathbb{R}^J$ with its *Tychonov* topology or product topology—and that is locally compact if and only if J is a finite set. So, with $J = \mathbb{N}$, the topological vector space $\mathbb{R}^{\mathbb{N}}$ is the first one to break this barrier. Topological vector spaces which are isomorphic to $\mathbb{R}^J$ for some set J are called *weakly complete vector spaces*. There is no problem in extending this terminology to vector spaces over the complex ground field $\mathbb{C}$.

It has become customary to call a topological group satisfying (1) and (2) above a *pro-Lie group*. Their systematic study coincides neatly with the beginning of the twenty-first century. The simplest examples are the weakly complete vectors spaces themselves. They are even closer to elementary vector spaces than one spontaneously thinks. Indeed, if $W \cong \mathbb{R}^J$ is a weakly complete vector space, then the vector space $\mathrm{Hom}_{\mathrm{continuous}}(W,\mathbb{R})$ of all continuous linear forms on W is isomorphic to $\mathbb{R}^{(J)}$, and a slightly more detailed consideration shows that this is the background of a perfect *duality between the category of real vector spaces and that of weakly complete vector spaces*. This rather elementary duality is discussed in detail in the first edition of Reference [1] in 1998 and in the first monograph of Reference [6] to have a systematic study of pro-Lie groups in 2007.

Here, the natural question arises how the concepts of a pro-Lie group and that of the historically fundamental one of a manifold based Lie group differ. The concept of a manifold had developed at that time vastly, being now based on locally convex topological vector spaces. Accordingly, the concept of a Lie group had developed deeply into the domain of infinite dimensional manifolds [13]. Nevertheless, from Reference [14], we know precisely how the two concepts are related:

Theorem 1. *A pro-Lie group is a Lie group if and only if it is locally contractible.*

Here, a topological group G is called *locally contractible*, if some identity neighborhood U can be homotopically contracted to a point in G, and it is called 1-*connected* if $\pi_1(G)$ is singleton. In the spirit of Lie theory from any viewpoint, it is fascinating that local contractibility of a 1-connected pro-Lie group can be detected purely from the Lie algebra $\mathfrak{g}$ of G: Every pro-finite dimensional Lie algebra $\mathfrak{g}$ has a maximal (semi-)direct summand $\mathfrak{s}$ being a product of some collection of simple finite dimensional Lie algebras. Indeed,
a 1-connected pro-Lie group G is locally contractible iff, apart from a finite number of these factors, each of the factors is isomorphic to the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ (the group of 2 by 2 real matrices of determiant 1).

The weakly complete real vector spaces provide an exemplarily simple class of pro-Lie groups beyond traditional Lie groups. In Reference [6], the authors proved the fairly deep theorem, saying that:
a connected pro-Lie group G contains a closed subspace E and a compact subgroup C such that E is homeomorphic to some weakly complete real vector space and the function

$$(e,c) \mapsto ec : E \times C \to G \text{ is a homeomorphism.}$$

We might say, so far so good, for connected pro-Lie groups. However, the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ of countably infinitely many generators supports a nondiscrete pro-Lie topology with rather bizarre properties. (This is described in Proposition 2 in Chapter 5 on abelian pro-Lie groups in Reference [6].) So, one ventures outside *connected* pro-Lie groups with trepidation. Even basic issues are settled only very partially, exemplified by the question: when is a quotient of a pro-Lie group a pro-Lie group? (See, e.g., Reference [6], Chapter 4, Theorem 4.28.)

It is, therefore, astonishing how much positive information has been gathered on pro-Lie groups, even if they fail to be connected.

Our monograph [6] presents a reasonably comprehensive theory of connected pro-Lie groups. While classical Lie theory is used intensively, the technical difficulties to bring them to bear on the general situation are often painfully complex on the technical level.

At the opposite end, we face totally disconnected pro-Lie groups. By definition, such a group G is a projective limit of Lie group quotients G/N. The pro-Lie algebra map $\mathfrak{L}(G) \to \mathfrak{L}(G/N)$ induced by the quotient morphism is surjective. (See Reference [6], 4.21.) However, $\mathfrak{L}(G) = \{0\}$, since G is totally disconnected. So, the Lie algebra of the Lie group G/N vanishes. Therefore, it is discrete. Accordingly, G is a projective limit of discrete quotients. Therefore, it is called *prodiscrete*. In the domain of locally compact groups, *prodiscrete groups* are generally considered still tractable. This applies certainly to the realm of compact groups where they are traditionally known as *profinite groups* and are treated extensively in the monograph literature. (See, e.g., Reference [15].) By contrast, one would have to admit, however, that no coherent structure or representation theory exists for prodiscrete groups, in general, outside the locally compact domain.

So, there arise obvious questions which link connectivity and prodiscreteness.

Problem 1. *Let G be a pro-Lie group. Is there a neighborhood of G_0 whose structure is reasonably well understood, at least topologically?*

Perhaps more explicitly (and optimistically):

Problem 2. *Let G be a pro-Lie group. Is there a closed totally disconnected subgroup H of G such that the subgroup G_0H is open?*

The consequences of such pieces of information would be far reaching. In the case of a locally compact group G, indeed, there exists a totally disconnected compact subgroup D such that G_0D is open. So, the answers for both Problem 1 and Problem 2 are affirmative if G is locally compact. Conclusive answers are not available if G fails to be locally compact, but partial answers to these questions were provided after the appearance of Reference [6] by the authors in Reference [16], and in a survey in Reference [17], including the following result:

Theorem 2. *Let G be an almost connected pro-Lie group. Then, every compact subgroup is contained in a maximal one and all of these are conjugate. There is a closed subspace homeomorphic to a weakly complete vector space E in G such that, for each maximal compact subgroup C, the function*

$$(e, c) \mapsto ec : E \times C \to G$$

is a homeomorphism.

The proof in Reference [16], in 2011 (after the appearance of Reference [6]), provides additional information on the way that E is constructed. A shorter, but perhaps more easily recalled, formulation is the following:

Corollary 1. *Any almost connected pro-Lie group is homeomorphic to $\mathbb{R}^J \times C$, for some set J and a compact subgroup C of G.*

It should be emphasized that this theorem gives a definitive insight into the *topological* structure of an almost connected pro-Lie group modulo the known structure of a compact group, as detailed in Reference [1]. Indeed,
a compact group C is homeomorphic to $C_0 \times C/C_0$, where C/C_0 is either finite or is homeomorphic to a power $\{0,1\}^J$ of the two element space for a suitable set J. (See Reference [1], 10.40.)
The compact connected group C_0 itself is a semidirect product of the closed commutator group C_0' and a compact connected abelian subgroup $A \cong C/C_0'$. (See Reference [1], 9.39.)

The compact semisimple commutator subgroup is described explicitly in Reference [1], 9.19, where it is argued that it is not too far from a product of a possibly large family of compact connected simple Lie groups.

For the pro-Lie group-theoretical understanding of the abelian connected compact group A, we also have explicit knowledge, namely the *Resolution Theorem* (Reference [1], 8.20), which specifies a profinite subgroup Δ of A and a continuous open surjective homomorphism $\Delta \times \mathfrak{L}(A) \to A$ for the Lie algebra $\mathfrak{L}(A)$ of A. Here, the Lie algebra $\mathfrak{L}(A)$ is none other than a weakly complete real vector space. In particular, these pieces of information together with Theorem 2 above yield the following rather complete information of the topology of an almost connected pro-Lie group:

Theorem 3. The Topology of Almost Connected Pro-Lie Groups: *Any infinite almost connected pro-Lie group is homeomorphic to a pro-Lie group of the form*

$$\mathbb{R}^I \times S \times A \times F,$$

where F is either finite or $\mathbb{Z}(2)^J$; here, I and J are sets, $\mathbb{Z}(2)$ is the two-element group, where S is a compact connected group that agrees with its commutator subgroup S' and is, modulo a central profinite subgroup, a Cartesian product of compact connected simple Lie groups, and, where, finally, A is a compact connected abelian group.

It may be helpful here to recall a consequence of Pontryagin Duality, namely that *the category of all compact connected abelian groups is dual to the (vast) category of all torsion-free abelian groups.*

The history of locally compact groups has illustrated that an insight into the structure of *abelian* locally compact groups preceded the solution of Hilbert's 5th Problem. In this spirit, we have had some success in getting the basics of a structure theory of *abelian* pro-Lie groups formulated. (See Reference [6], 5.20).) Indeed, we proved the following result.

Theorem 4. Main Structure Theorem of Abelian Pro-Lie Groups: *Any abelian pro-Lie group G is the direct sum $E \oplus H$ of closed subgroups, where E is isomorphic to $\mathbb{R}^J$, for a set J, and H has the following properties:*

(i) *H_0 is compact and is the unique largest compact connected subgroup;*
(ii) *every compact subgroup of G is contained in H;*
(iii) *the totally disconnected quotient groups $G_t = G/G_0$ and $H_t = H/H_0$ are isomorphic; and*
(iv) *The union $\mathrm{comp}(G)$ of all compact subgroups of G is a fully characteristic closed subgroup of G that is contained in H, and*

$$G_0 + \mathrm{comp}(G) = E \oplus \mathrm{comp}(G)$$

is a fully characteristic closed subgroup G_1 of G such that every monothetic subgroup of G/G_1 is isomorphic to the discrete group $\mathbb{Z}$.

The factor group $G/\,\mathrm{comp}(G)$ does not contain any nonsingleton compact subgroup, and the Main Structure Theorem implies immediately that its identity component is a weakly complete real vector space isomorphic to $\mathbb{R}^J$ and is a direct summand.

The factor group G/G_1 is a totally disconnected abelian pro-Lie group without any nontrivial compact subgroup whose structure remains largely uncharted and mysterious.

Indeed, *A. Weil's Lemma* on the Classification of Monothetic Subgroups of Locally Compact Groups (Reference [1], 7.43) was extended by the authors to pro-Lie groups in the following fashion:

Theorem 5. Weil's Lemma for Pro-Lie Groups: *Let $E = \mathbb{Z}$ or $E = \mathbb{R}$ and $X\colon E \to G$ a morphism of topological groups into a pro-Lie group. Then, exactly one of the following statement holds:*

(i) $r \mapsto X(r) : E \to X(E)$ *is an isomorphism of topological groups;*
(ii) $\overline{X(E)}$ *is compact.*

As a consequence, *if a pro-Lie group G has no nontrivial compact subgroups, then every monothetic subgroup is isomorphic to $\mathbb{Z}$ as a topological group.*

In all of topological group theory, the subclass of commutative topological groups is usually considered a test class which is representative of the status of information provided by current research. This is exemplified by information provided for locally compact abelian groups (often called *LCA-groups*) and, similarly, by all the information on real topological vector spaces made available by functional analysis.

It was, therefore, natural to raise the issue of duality for abelian pro-Lie groups in Reference [6], pp. 237ff.

Notably, satisfactory results emerged for *almost connected* abelian pro-Lie groups, and some interesting general additional aspects were pointed out in Reference [6] (5.36, 5.40, 5.41). In particular, it was observed in Reference [6] (Comments to 14.15) that an abelian pro-Lie group G may fail to be reflexive. (Here, a topological abelian group is called *reflexive*, if the natural morphism $G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.) Overall, one might consider the structure theory of abelian pro-Lie groups still as regrettably incomplete. Some aspects that we do know are collected in the following theorem.

Theorem 6. The Structure of Almost Connected Abelian Pro-Lie Groups: *Let G be an almost connected abelian pro-Lie group. Then,* $\mathrm{comp}(G)$ *is a compact subgroup, and*

(i) $G \cong \mathbb{R}^J \times \mathrm{comp}(G)$. *In particular, each weakly complete* real *vector space is reflexive.*
(ii) *The annihilator of G_0 in $\widehat{G}$ is* $\mathrm{comp}\,\widehat{G}$.

Now, assume that G is an abelian pro-Lie group which is algebraically generated by a compact subset. Let $G_1 = G_0 + \mathrm{comp}(G)$ be the fully characteristic subgroup of G introduced in Theorem 3(iv). Then, G_1 is locally compact, and G/G_1 is a Polish space (i.e., it is completely metrizable and separable) if and only if $G \cong \mathbb{R}^m \times \mathrm{comp}(G) \times \mathbb{Z}^n$, for nonnegative integers m and n.

More details can be found in Reference [6], including a version of a universal covering theorem which, in Reference [1], was called a ''Resolution Theorem''.

Wayne Lewis noted recently that the Resolution Theorem suggests introducing into the study of LCA groups a more systematic use of the *adele ring*

$$\prod_{p\,\mathrm{prime}}^{\mathrm{local}} (\mathbb{Q}_p, \mathbb{Z}_p) \times \mathbb{R}\,,$$

thus relating the structure theory of LCA groups to algebraic number theory. (The term *idele* was introduced by Claude Chevalley (1909–1984) and is an abbreviation of "ideal element". The term *adele* stands for "additive idele".)

Theorem 5 confirms the impression that we can regard the condition of being *almost connected* in the theory of pro-Lie groups as very satisfactory, but that we do not have a comprehensive theory of totally disconnected abelian pro-Lie groups, in general. The recent study of Reference [18] on locally compact totally disconnected abelian groups G satisfying $G = \mathrm{comp}(G)$ deals with this subject, as well as illustrates the fact that not even the presence of a wealth of compact open subgroups provides for structural simplicity.

While we noted that each locally compact group G having a pro-Lie group as identity component G_0 is largely determined by a profinite dimensional Lie algebra $\mathfrak{L}(G)$, nevertheless, we observed that $\mathfrak{L}(G)$ has no effect on the totally disconnected portion $G_t = G/G_0$ of G.

One recent branch in the research on locally compact groups provides noteworthy connections between locally compact groups and topology without having such restrictions. Indeed, the set of all closed subgroups of any locally compact group G always supports a compact topology making that set into a compact Hausdorff space $\mathfrak{Ch}(G)$, called the *Chabauty space* of G. (The names of Leopold Vietoris (1891–2002) or James Michael Gardner Fell (1923–2016) would have been just as appropriate as that of Claude Chabauty (1910–1990).) The example of the circle group $G = \mathbb{T}$ shows that the Chabauty space may have pathological aspects even in the compact connected case. On the other hand, this tool appears to come in handy for totally disconnected locally compact groups G, as the following example shows:

For any locally compact group G, the function $g \mapsto \overline{\langle g\rangle} : G \to \mathfrak{Ch}(G)$ *is continuous iff G is totally disconnected.*

In this sense, the operators $\mathfrak{L}$ and $\mathfrak{Ch}$ are opposite in their prospect as tools for the structure theory of G. (See Reference [19].)

4. Linear Algebra Meets Pro-Lie Group Theory

In Reference [20], in 1939, Tadeo Tannaka (1908–1986) formalized the process of reconstructing a compact group from the systematically structured class of finite dimensional linear representations. This approach he proved to be a way of generalizing Pontryagin's duality of the categories of *abelian* compact, respectively, discrete groups to a noncommutative situation. This led to vast generalizations in the abstract world of category theory. (See Reference [21].) On the other hand, at a very early point in his book [22], Gerhard Paul Hochschild (1915–2010) formalized very concretely the idea that the real vector space $R(G,\mathbb{R})$ of coefficient functions of finite dimensional linear representations of a compact group G is not only a commutative algebra, but also a coalgebra and, indeed, a symmetric *Hopf algebra*. He specified the conditions under which the spectrum of a symmetric Hopf algebra is a compact group G whose Hopf algebra $R(G,\mathbb{R})$ is isomorphic to the given one. This produces a duality between the category of compact groups and a category of *certain* symmetric Hopf algebras. The connection between $R(G,\mathbb{R})$ and the linear representations indicates an existing equivalence of Hochschild's duality with Tannaka's.

We have proposed a topological group algebra $\mathbb{R}[G]$ of any compact group G. This allows us to produce a certain category of *topological* symmetric Hopf algebra which is *equivalent* to the category of compact groups via $G \mapsto \mathbb{R}[G]$. This links us with Hochschild-Tannaka duality through the fact that $R(G,\mathbb{R})$ and $\mathbb{R}[G]$ are natural duals of each other as symmetric Hopf algebras in their respective domains of plain vector spaces and topological vector spaces.

From the very beginning of the study of pro-Lie groups, it was clear that one would have to consider pro-Lie algebras. One of the difficult problems with which Sophus Lie found himself confronted was the question of whether, for each Lie algebra $\mathfrak{g}$, one could find a Lie group G whose Lie algebra was (isomorphic to) $\mathfrak{g}$. A satisfactory answer became known in the history of Lie theory as Lie's *Third Fundamental Theorem*. In the development of the theory of pro-Lie groups, it seemed conceptually fitting to find a response to a more comprehensive question. At that point in the history of topological groups, one had a good hold of category theory, and it was understood that the Lie algebra functor $\mathfrak{L}$ from the category $\mathcal{G}$ of pro-Lie groups to the category of profinite-dimensional Lie algebras $\mathcal{L}$ has a right adjoint Γ. Thus, for every morphism $f\colon \mathfrak{g} \to \mathfrak{L}(G)$, there is a unique morphism $f'\colon \Gamma(\mathfrak{g}) \to G$, producing a natural isomorphism

$$f \mapsto f'\colon \mathcal{L}(\mathfrak{g}, \mathfrak{L}(G)) \to \mathcal{G}(\Gamma(\mathfrak{g}), G)).$$

In particular, the right adjoint functor $\mathfrak{L}$ preserves all limits, so, if G is a projective limit of finite dimensional Lie groups, then $\mathfrak{L}(G)$ is a projective limit of finite dimensional Lie algebras, i.e., a *profinite dimensional Lie algebra*. (See Reference [23].) As an immediate elementary consequence, the real topological vector space underlying $\mathfrak{L}(G)$ is a projective limit of finite dimensional vector spaces. This returns us to the fact that one had to discuss at a comparatively early stage in References [1,6] that the projective limit property, indeed, characterized a real or complex topological vector space to be weakly complete. The insight that the category of $\mathbb{K}$-vector spaces, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is dual to the category of weakly complete topological $\mathbb{K}$-vector spaces was explicitly elucidated both in References [1,6]. We note here with some circumspection that, for $\mathbb{K} = \mathbb{R}$, the duality between real vector spaces, on the one hand, and weakly complete topological real vector spaces, on the other, may be regarded a special case for abelian pro-Lie groups of Pontryagin duality (also see Reference [1], A7.10).

It is clear that pro-Lie group theory and elementary linear algebra are tied together from the beginning. However, when the first systematic study of pro-Lie groups [6] was compiled, another avenue leading from ‘‘elementary linear algebra’’ directly to pro-Lie groups had not yet been observed, even though its mathematical underpinning would have been available. This avenue leads from weakly complete topological vector spaces to weakly complete associative topological algebras. That associative unital algebras would appear in the vicinity of groups and their linear representation theory is perhaps not surprising, given the history of representation theory and module theory. It is perhaps astonishing that the concept of weakly complete algebras appeared so late.

Indeed, a *weakly complete unital algebra A* is an associative algebra whose addition and scalar multiplication are that of a weakly complete vector space and whose multiplication is associative and continuous and has an identity. Let us denote the multiplicative group of invertible elements by A^{-1}. At first glance, and in light of the numerous types of associative unital algebras that functional analysis deals with in the representation theory of topological groups, the following fact may come as a surprise:

Theorem 7. *Every weakly complete unital algebra A is a projective limit of finite dimensional unital quotient algebras.*

In other words, a weakly complete associative algebra is automatically *profinite dimensional*.

The essence of the above result was first observed by Bogfiellmo, Dahmen, and Schmeding [24]. For more on this theorem, see Reference [1], A7.32–A7.43.

These facts require absolutely no additional hypothesis apart from the fact that the algebra topology is the weakly complete one. We recorded that the categories $\mathcal{V}$ of vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and the category $\mathcal{W}$ of weakly complete topological vector spaces are dual. This suggests that Theorem 7 is just one step away from a purely algebraic result. Indeed, let us reconsider the categories $\mathcal{V}$ and $\mathcal{W}$ and, for each of the two, the occasionally tricky concept of its tensor product $\otimes_{\mathcal{V}}$, respectively, $\otimes_{\mathcal{W}}$. (The basic properties of $\otimes_{\mathcal{W}}$ were first studied in the Master’s thesis (Diplomarbeit) in 2007 of Raphael Dahmen.) The most significant property of this pair of tensor products is its compatibility with duality:

$$(V_1 \otimes_{\mathcal{V}} V_2)^* \cong V_1^* \otimes_{\mathcal{W}} V_2^* \text{ and } (W_1 \otimes_{\mathcal{W}} W_2)' \cong W_1' \otimes_{\mathcal{V}} W_2'.$$

With the aid of the tensor product, the multiplication of a weakly complete algebra A may now be expressed as a $\mathcal{W}$-morphism $m\colon A \otimes_{\mathcal{W}} A \to A$ subject to the commutativity of a diagram expressing associativity (Reference [1], Definition A3.63a), and the identity element 1 of the algebra may be expressed by a morphism $u\colon \mathbb{K} \to A$, $u(t) = t \cdot 1$, subject to a commutative diagram (cf. loc. cit.). Now, the dual object $m'\colon A' \to A' \otimes_{\mathcal{V}} A'$, together with $u'\colon A' \to \mathbb{K}$, represents a coassociative *coalgebra* with *coidentity*. So, all such coalgebras, being purely algebraic objects in the category $\mathcal{V}$, are *locally finite* in the sense that every element is contained in a finite dimensional subcoalgebra. In other words, each associative counital coalgebra is a directed colimit of finite dimensional subcoalgebras, or, once again reformulated, each counital coassociative coalgebra in $\mathcal{V}$ is a projective colimit of finite dimensional subalgebras. This result is referred to as the "CARTIER Lemma", and also as "The Fundamental Theorem of Coalgebras". (See Michaelis, in Reference [25].) Now, we see that Theorem 7 is the dual of the Cartier Lemma.

An almost immediate consequence of Theorem 7 is the following.

Theorem 8. Fundamental Theorem of Weakly Complete Algebras: *Let A be a weakly complete unital algebra. The group of units (that is, multiplicatively invertible elements), A^{-1}, is an almost connected pro-Lie group. It is dense in A, and the exponential function $\exp\colon A \to A^{-1}$ converges everywhere and defines the exponential function of the pro-Lie group A^{-1} if A is considered as a Lie algebra with respect to the bracket $[a, b] = ab - ba$.*

The Fundamental Theorem of Weakly Complete Algebras yields an assignment $A \mapsto A^{-1}$, which is clearly functorial, mapping the category $\mathcal{WA}$ of weakly complete unital algebras into the category $\mathcal{L}$ of pro-Lie groups. Its left adjoint functor $G \to \mathbb{K}[G] : \mathcal{L} \to \mathcal{WA}$ assigns to a pro-Lie group G its *group algebra* (over the groundfield $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). In the case of $\mathbb{K} = \mathbb{R}$ the duality yields an isomorphism $\mathbb{R}[G]' \cong R(G, \mathbb{R})$ with the topological dual $\mathbb{R}[G]'$ of the weakly complete group algebra $\mathbb{R}[G]$ and the ring of representative functions $R(G, \mathbb{R}) \subseteq C(G, \mathbb{R})$, familiar notably in the representation theory of compact groups. (See Reference [1], Chapter 3, Definition 3.3.) The group algebra $\mathbb{K}[G]$ was discussed in detail in References [26,27] and in the book of Reference [6]. In a natural way, $\mathbb{K}[G]$ is, in fact, a *symmetric Hopf algebra*. Here, a Hopf algebra is simultaneously an associative unital algebra and an associative counital coalgebra linked in a compatible fashion. It is a *symmetric* Hopf algebra if it further includes a "symmetry", an involutory self-map, acting in a similar way as "inversion" makes a semigroup into a group.

For compact groups, this concept, the equivalence of the category of compact groups with a certain category of weakly complete symmetric Hopf algebras, via duality, eventually leads us to the conclusive form of the *Hochschild-Tannaka Duality* of the category of compact groups and a certain subcategory of the category of purely algebraic symmetric Hopf algebras. (The interested reader will find this discussed in Reference [1], Chapter 3: Part 3, pp. 90–12, and in Appendix 3 on Category Theory: Section on Commutative Monoidal Categories and their Monoids, Part 5: Symmetric Hopf Algebras over $\mathbb{R}$ and C, pp. 856–862, and, finally, in Appendix 7: Weakly Complete Topological Vector Spaces, Subsection on: Weakly Complete Unital Algebra, pp. 936–941.)

It must be noted here that, for a Hopf algebra A with multiplication $m\colon A \otimes A \to A$ and identity $u\colon \mathbb{K} \to A$, comultiplication $c\colon A \to A \otimes A$, and coidentity $k\colon A \to \mathbb{K}$, we call an element $a \in A$ *group-like* if $c(a) = a \otimes a$ and $k(a) = 1$, and *primitive* if $c(a) = a \otimes 1 + 1 \otimes a$. Then, an additional general structural feature is to be added to Theorem 8:

Theorem 9. Fundamental Theorem of Weakly Complete Hopf Algebras: *If A is a weakly complete symmetric Hopf algebra, then the set $\mathbb{G}(A)$ of group-like elements is a closed subgroup of A^{-1} and, thus, is a pro-Lie subgroup of A^{-1}.*

The set $\mathbb{P}(A)$ of primitive elements is a closed Lie subalgebra of A_{Lie} and is the Lie algebra of $\mathbb{G}(A)$, and its exponential function $\exp_{\mathbb{G}(A)}\colon \mathbb{P}(A) \to \mathbb{G}(A)$ is induced by the exponential function of A.

This applies, in particular, to the group algebra $\mathbb{R}[G]$ of each compact group G, where we have:

Theorem 10. The Group Algebra Theorem for Compact Groups: *A compact group G may be identified with the subgroup $\mathbb{G}(\mathbb{R}[G])$ of group-like elements in the group algebra $\mathbb{R}[G]$, and its Lie algebra $\mathfrak{L}(G)$ may be identified with the Lie subalgebra of $\mathbb{P}(\mathbb{R}[G])$ of primitive elements of $\mathbb{R}[G]$, and, finally, its exponential function $\exp\colon \mathfrak{L}(G) \to G$ is then induced by the exponential* *function of the weakly complete algebra $\mathbb{R}[G]$. The cocommutative weakly complete symmetric Hopf algebra $\mathbb{R}[G]$ is dual to the commutative symmetric Hopf algebra $R(G, \mathbb{R})$. (See Reference [26].)*

The essential feature of a Lie group G is its Lie algebra $\mathfrak{g}$, which is at the heart of its algebraic structure. In analogy to the way that leads from groups to group algebras, there is a traditional path leading from Lie algebras to associative algebras. It has been observed recently that the functor from the category of weakly complete unital algebras to the category of profinite dimensional Lie algebras which associates with a weakly complete unital algebra A the Lie algebra A_{Lie} whose underlying vector space is that underlying A with the Lie bracket $[a, b] = ab - ba$. Since A is profinite dimensional, so is the weakly complete Lie algebra A_{Lie}. The assignment $A \mapsto A_{\mathrm{Lie}}$ is a functor from the category $\mathcal{WA}$ of weakly complete associative unital algebras to the category $\mathcal{FL}$ of profinite dimensional Lie algebras. The left adjoint $\mathbf{U}\colon \mathcal{FL} \to \mathcal{WA}$ yields for a profinite dimensional Lie algebra $\mathfrak{g}$ the weakly complete unital associative algebra $\mathbf{U}(\mathfrak{g})$. (See References [27,28].)

Theorem 11. The Enveloping Algebra Theorem: *Let $\mathfrak{g}$ be a profinite dimensional Lie algebra and $U(\mathfrak{g})$ its traditional enveloping algebra over $\mathbb{K}$. Then, $\mathbf{U}(\mathfrak{g})$ is a weakly complete unital associative symmetric Hopf algebra containing the classical enveloping algebra $U(\mathfrak{g})$ as a dense sub-Hopf algebra. The weakly complete algebra $\mathbf{U}(\mathfrak{g})$ has an exponential function $\exp\colon \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^{-1}$.*

The Lie subalgebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of primitive elements contains naturally a copy of $\mathfrak{g}$ which generates $\mathbf{U}(\mathfrak{g})$ algebraically and topologically as an algebra. $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ is the Lie algebra of the pro-Lie group of group-like elements $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$.

While the classical enveloping algebra does not contain any nonidentity group-like elements, the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ contains within the pro-Lie group $\mathbf{U}(\mathfrak{g})^{-1}$ the group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of group-like elements, which, in turn, contains the group $\Gamma(\mathfrak{g}) = \overline{\langle \exp \mathfrak{g} \rangle}$ that is attached to $\mathfrak{g}$ by Lie's Third Theorem, and the exponential function $\exp\colon \mathfrak{g} \to \Gamma(\mathfrak{g})$ is induced by the exponential of the weakly complete algebra $\mathbf{U}(\mathfrak{g})$.

5. Postscript

After a brief review of 100 years of history of Lie groups and locally compact groups, we have tried to emphasize the widening of the horizon from the landscape of classical Lie group and locally compact group theory to pro-Lie groups. Apart from an emphasis to include functorial thinking into the study of topological groups, this enlargement of scope is strengthened by the viewpoint that Lie group theory deals in essence with topological groups G having a Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp : \mathfrak{L}(G) \to G$ that crucially determines the structure of G via the Lie algebra structure of $\mathfrak{L}(G)$. Functorial thinking tells us how far we have to go from the long standing classical field of finite dimensional Lie algebras and connected (or at least almost connected!) Lie groups, and, so, we shall unquestionably arrive at pro-Lie groups.

The prime testing ground for pro-Lie group theory remains the field of compact groups. At the beginning of their history, decades ago, it was detected that they were pro-Lie groups automatically by their representation theory. Now, they tell us how far we can go with a clear structure theory of pro-Lie groups past the boundaries imposed by connectivity. In that process, we redetect the significance of "almost connected" groups, namely those G whose space $G_t = G/G_0$ of connected components is compact. In the realm of locally compact groups, Hidehiko Yamabe had justly drawn attention to almost connected locally compact groups for which one could demonstrate that they were pro-Lie groups.

A second testing ground for any theory of topological groups is the class of commutative ones. As far as pro-Lie groups are concerned, this territory is largely uncharted. Yet, once more, the subterritory of almost connected abelian pro-Lie groups is crystal clear: it comprises all groups which are direct products $E \times C$, where E is (the additive group of) a so-called "weakly complete" real topological vector space. These topological vector spaces are also the ones that are underlying the Lie algebras of all pro-Lie groups. So, they play a significant role in pro-Lie theory on both the group and the algebra level. How complicated are they?

The answer was simple since the beginning of their presence a quarter of a century ago: They are simply the duals of ordinary real vector spaces V, together with the topology that these inherit from their nature as function spaces $E = \mathrm{Hom}(V, \mathbb{R}) \stackrel{\text{def}}{=} V^* \subseteq \mathbb{R}^V$ in the form of the topology of pointwise convergence, or, equivalently expressed, the topology induced by the Tychonov product topology of $\mathbb{R}^V$. Traditionally, this topology on E is called the "weak-* topology", which led to the terminology of *weakly complete vector spaces*. Their truly basic nature is emphasized by the fact that the topological dual $E' = \mathrm{Hom}_{\mathrm{continuous}}(E, \mathbb{R}) \subseteq C(E, \mathbb{R})$ is naturally isomorphic to V, that is $V \cong V^{*\prime}$, and that, likewise, $E \cong E'^*$. Compactly phrased, the categories $\mathcal{W}$ of *weakly complete vector spaces* and the category $\mathcal{V}$ of (ordinary) real vector spaces are dual. This interplay pertains, therefore, to *elementary linear algebra*. Moreover, the quotient map $\mathbb{R} \mapsto \mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ induces an isomorphism:

$$V^* = \mathrm{Hom}(V, \mathbb{R}) \cong \mathrm{Hom}(V, \mathbb{T}) = \widehat{V} = \textit{Pontryagin Dual of } V.$$

Thus, a closer appropriate inspection shows that the duality between $\mathcal{V}$ and $\mathcal{W}$ is just another manifestation of Pontryagin Duality expressed as $V \cong \widehat{\widehat{V}}$ and $E \cong \widehat{\widehat{E}}$ (where an *ordinary vector space* V is equipped with its unique smallest locally convex topology).

The category $\mathcal{W}$ allows an immediate natural access from elementary linear algebra to the category $\mathcal{W}\!A$ of all weakly complete unital associative algebras. It is astonishing that each such algebra A provides an immediate connection to the world of pro-Lie groups insofar as A is a projective limit of finite dimensional algebras and as the group of units A^{-1} is a pro-Lie group whose Lie algebra $\mathfrak{L}(A^{-1})$ is the Lie algebra A_{Lie} defined on A by the Lie bracket, while their exponential function is the ordinary exponential function $\exp A_{\mathrm{Lie}} \to A^{-1}$, $\exp a = 1 + a + \frac{1}{2!} \cdot a^2 + \cdots$ defined on all of A. This opens up the general definition of a weakly complete group algebra $\mathbb{R}[G]$ of a pro-Lie group G and a weakly complete universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional Lie algebra $\mathfrak{g}$. Here, pro-Lie group theory meets weakly complete algebras in the form of appropriate group algebras and appropriate weakly complete enveloping algebras on the basis of a weakly complete symmetric Hopf algebra theory which we have described. Yet, even for compact groups, this opens up previously unnoticed connections to the classical Tannaka-Hochschild duality theory.

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