# Contributions to Uniform Diophantine Approximation

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## Abstract

This thesis reports on an investigation into the metric theory of uniform Diophantine approximation. The well-known theorems of Khinchin and Jarník are fundamental results in the theory of asymptotic Diophantine approximation, which is concerned with improving a corollary of Dirichlet's theorem (1842). By contrast, uniform Diophantine approximation investigates strengthening Dirichlet's theorem itself. Kleinbock and Wadleigh (2018) characterised the set of Dirichlet non-improvable numbers in terms of the growth of their continued fraction entries. They established a Lebesgue measure criterion for the size of the set of Dirichlet non-improvable numbers.

This thesis presents three main new results on uniform Diophantine approximation. In the first result, the size of the set of Dirichlet non-improvable numbers is compared with the size of the set of well-approximable numbers. It is proved that there are uncountably many more Dirichlet non-improvable numbers than well-approximable numbers by calculating the Hausdorff dimension of the difference set, which turns out to be the same dimension as that of the set of well-approximable numbers.

The second result is a contribution to the metric theory of continued fractions. The Jarník-Besicovitch set is concerned with the growth of one partial quotient whereas the set of Dirichlet non-improvable numbers is concerned with the growth of the product of consecutive partial quotients. The latter set properly contains the former set. It is proved that the difference set has positive Hausdorff dimension and hence is nontrivial.

The third new result is concerned with the generalised Hausdorff measure of the set of Dirichlet non-improvable numbers. We prove a Hausdorff measure dichotomy statement (zero or infinite) for the set of Dirichlet non-improvable numbers for non-essentially sub-linear dimension functions. Our result complements the result of Hussain, Kleinbock, Wadleigh and Wang (2018) who proved the aforementioned dichotomy statement for essentially sub-linear dimension functions.

Dedicated to my father, who always encouraged me to study and learn, my mother, a true academic completing her first book at 84 and my children, who all graduated from university, so gladdening my heart.

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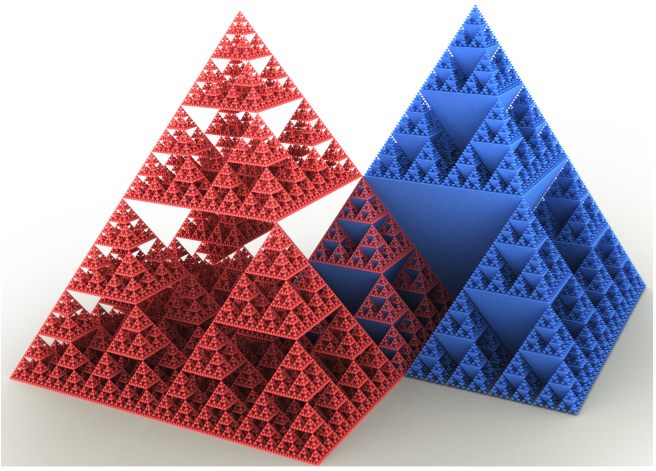
My thanks go to Nicole for her support and care, which allowed me the time and concentration to complete the thesis.



*The Sierpinski Pyramids are Cantor sets with fractal dimensions* $\frac{\log 5}{\log 2} = 2.32193\ldots$

# Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgement in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

The contents of Section 4.4 has been submitted for publication and the preprint is available on the arXiv preprint server [10]:
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The contents of chapter 5 have been published in [2]:
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The contents of chapter 6 have been published in [1]:
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# Contents

# Notation

To simplify the presentation, we start by fixing some notation.

The Vinogradov symbols $\gg$ and $\ll$ will be used: If $B \geq 0$, the notation $A \ll B$ means that there exists a positive constant $c$ such that $|A| \leq cB$. The constant $c$ may well depend on certain other parameters but the meaning will be plain from the context. $B \gg A$ means the same as $A \ll B$ but will be used only when $A$ and $B$ are both non-negative, see [52]. If $A \gg B$ and $A \ll B$ we write $A \asymp B$, and say that the quantities $A$ and $B$ are comparable, which means that $|A/B|$ is bounded between unspecified positive constants.

We use $\lambda(\cdot)$, $\dim_{\mathcal{H}}$ and $\mathcal{H}^s$ to denote the Lebesgue measure, Hausdorff dimension, and $s$-dimensional Hausdorff measure, respectively. For any subset $A$, we denote by $|A|$, the diameter of $A$, and by $\mu(A)$, we will mean the Gauss measure of $A$, unless otherwise specified.

We use 'i.m.' for *'infinitely many'*. We write $\mathbb{I}$ for the unit interval $[0, 1)$. Throughout this thesis, for any integer vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, $I_n$ denotes a *basic cylinder* of order $n$. We denote the integer part of a real number $x$, by $[x]$. We use $\lfloor x \rfloor$ to denote the floor of $x$, that is, the largest integer no larger than $x$. By $\|x\|$, we mean the nearest integer to $x$.

# Chapter 1

# Metric Diophantine Approximation

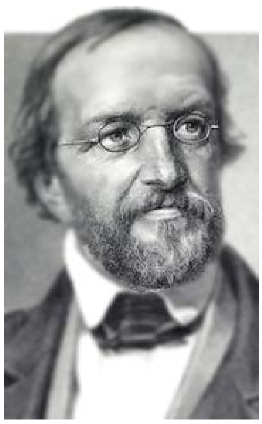
Diophantine approximation concerns answering a simple question: how well can a real number $x$ be approximated by a rational number $r = \frac{p}{q}$? A *qualitative* answer is provided by the fact that under the topology induced by the Euclidean metric, the rationals $\mathbb{Q}$ are *dense* in the reals $\mathbb{R}$, and so for a given $\zeta \in \mathbb{R}$ we can find infinitely many rational points contained in an arbitrarily small open interval around $\zeta$. In other words, any real number can be approximated by a rational number with any assigned degree of accuracy

A subset $S \subseteq \mathbb{R}$ is said to be *dense in* $\mathbb{R}$ if for every element $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists an $s \in S$ such that $|x - s| < \epsilon$. The set of rational numbers is dense in the set of real numbers.

A subset $T \subseteq \mathbb{R}$ is said to be *discrete in* $\mathbb{R}$ if there exists an $\epsilon > 0$ such that given an element $t \in T$, $\{x \in \mathbb{R} : |t - x| < \epsilon\} \cap T = \{t\}$. That is, each element $t \in T$ is isolated from other points in $T$ by a distance of at least $\epsilon$. The set of integers is discrete in the set of real numbers.

To obtain increasing precision in the rational approximation of elements of $\mathbb{R} \setminus \mathbb{Q}$, requires considering rationals $p/q$ with arbitrarily large denominators. This gives rise to the idea of relating the quality of approximation to the size of the denominator, $q$. Rationals with smaller denominators are simpler.

Seeking a *quantitative* answer leads to the theory of Diophantine approximation, which measures how well a real number can be approximated by a rational number with a *bounded* denominator. It is named after Hellenistic mathematician Diophantus of Alexandria (born between AD 201 and 215, and died around 84 years of age).



*Peter Gustav Dirichlet*
*(1805 – 1859)*

Metric Diophantine approximation refers to the measure theoretical study to which much of this thesis is devoted. Dirichlet's theorem (1842) is the starting point in the theory of metric Diophantine approximation. Dirichlet's original paper can be found in [14].

## 1.1 Dirichlet's Theorem

**Theorem 1.1.1 ([14], Dirichlet 1842)** *For any $x \in \mathbb{R}$ and $t > 1$, there exist integers $q \in \mathbb{Z} \setminus \{0\}$, $p \in \mathbb{Z}$ such that*

$$|qx - p| \leq 1/t \quad \text{and} \quad 1 \leq q < t. \tag{1.1}$$

Dirichlet's theorem is a *uniform* Diophantine approximation result as it guarantees a nontrivial integer solution for all $t$.

**Proof:** Assume that $x$ is nonnegative. With $[x]$ denoting the integer part of $x$ and $\{x\} = x - [x]$ denoting the fractional part of $x$, note that for any $x \in \mathbb{R}$ we have that $0 \leq \{x\} < 1$.

Write $N := [t]$. Consider the $N + 1$ real numbers

$$\{0x\}, \{x\}, \{2x\}, \ldots, \{Nx\} \tag{1.2}$$

and their distribution in the $N$ equal semi-open subintervals

$$I_j = \left[\frac{j}{N}, \frac{j+1}{N}\right), \quad j = 0, 1, 2, \ldots, N-1.$$

Observe that

$$[0, 1) = \bigcup_{j=0}^{N-1} I_j$$

and hence contains the $N + 1$ values in Eq. (1.2). It follows from the pigeonhole principle that some subinterval contains at least two distinct points, say $\{n_1 x\}$ and $\{n_2 x\}$ with $0 \leq n_1 < n_2 \leq N$. Since the intervals are of length $1/N$ and not closed at both ends, we have that

$$\begin{aligned} |\{n_2 x\} - \{n_1 x\}| &< \frac{1}{N}, \quad \text{that is} \\ |(n_2 x - [n_2 x]) - (n_1 x - [n_1 x])| &= (n_2 - n_1)\, x - ([n_2 x] - [n_1 x]) < \frac{1}{N}. \end{aligned} \tag{1.3}$$

Write $q = n_2 - n_1 \in \mathbb{N}$ and $p = [n_2 x] - [n_1 x] \in \mathbb{N}$. Since $0 \leq n_1 < n_2 \leq N$ we have $1 < q \leq N$ and by (1.3) we get

$$|qx - p| < \frac{1}{N}.$$

For negative $x$, the statement "$p = [n_2 x] - [n_1 x] \in \mathbb{N}$" is replaced by symmetry, with "$p = [n_2 x] - [n_1 x] \in -\mathbb{N}$" and so for $x \in \mathbb{R}$, we have shown that $p \in \mathbb{Z}$.

Now, from the last inequality, (1.1) readily follows. ∎

**Example 1.1.2** Observe that

$$\left|\pi - \frac{31}{10}\right| < \frac{1}{10} \text{ and } \left|\pi - \frac{314}{100}\right| < \frac{1}{100}$$

but then

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{7^2}, \quad \left|\pi - \frac{355}{113}\right| < \frac{1}{113^2}, \quad \text{and} \quad \left|\pi - \frac{103993}{33102}\right| < \frac{1}{33102^2}.$$

We may then ask, how many integer pairs $(p, q)$ are there such that

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{q^2}? \tag{1.4}$$

From inequality (1.4), a natural question is: what is the cardinality of the set

$$\left\{(p, q) \in \mathbb{Z}^2 : \left|x - \frac{p}{q}\right| < \frac{1}{q^2}\right\}$$

for any $x \in \mathbb{R}$? An important consequence of Dirichlet's theorem is the following global statement concerning the rate of rational approximation to any irrational number. This consequence was known before Dirichlet (see Legendre [39, pp. 18-19])

**Corollary 1.1.3 (Dirichlet's corollary / Legendre 1808)** *For any irrational $x \in \mathbb{R}$, there exist infinitely many (i.m.) $q \in \mathbb{N}$ such that*

$$|qx - p| < \frac{1}{q} \quad \text{for some } p \in \mathbb{Z}. \tag{1.5}$$

The generalisations of this corollary has been the subject of much study and this stream of research is sometimes referred to as the *asymptotic* Diophantine approximation theory.

## 1.2 Continued Fractions

Continued fractions are an important tool in answering questions such as posed in Section 1.1 and were used by Legendre in proving Dirichlet's corollary before there was Dirichlet's theorem.

Continued fractions derive naturally from the Euclidean algorithm.

$$355 = 113 \times 3 + 16 \qquad \text{and} \qquad 113 = 7 \times 16 + 1$$

which can be written as follows:

$$\frac{355}{113} = 3 + \frac{1}{\frac{113}{16}} \qquad \text{and} \qquad \frac{113}{16} = 7 + \frac{1}{16}.$$

Combining these steps we get

$$\frac{355}{113} = 3 + \frac{1}{7 + \frac{1}{16}}.$$

The last expression is called the *simple continued fraction* expansion of $\frac{355}{113}$, and we may write it in shorthand, as $[3; 7, 16]$.

Every real number $x \in \mathbb{R}$ has a *continued fraction expansion*,

$$x = a_0(x) + \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \tag{1.6}$$

which terminates or continues forever as $x$ is respectively rational or irrational. We write $x = [a_0(x); a_1(x), a_2(x), \ldots]$ for reasons of space. For each $n \geq 1$, the positive integers $a_n(x)$ are called the *partial quotients* of $x$. The fractions obtained by finite truncations in Eq. (1.6)

$$\frac{p_n(x)}{q_n(x)} := [a_0(x); a_1(x), \ldots, a_n(x)] \quad (n \geq 1),$$

$(p_n(x), q_n(x)$ coprime) are called the $n$th *convergents* of $x$.

**Remark 1.2.1** From Eq. (1.7), $p_n = p_n(x)$ and $q_n = q_n(x)$ are determined by the partial quotients $a_0 = a_0(x), a_1 = a_1(x), \ldots, a_n = a_n(x)$, so we may also write $p_n = p_n(a_0, a_1, \ldots, a_n), q_n = q_n(a_0, a_1, \ldots, a_n)$. When it is clear for which $x$ they are the expansion, we use $a_n, p_n, q_n$ for the obvious simplifications. ▪

**Proposition 1.2.2 ([54])** *With the conventional starting values*

$$(p_{-1}, q_{-1}) = (1, 0), \quad (p_0, q_0) = (0, 1),$$

*the sequences $\{p_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ can be generated by the following recursive relations*

$$\begin{aligned} p_{n+1} &= a_{n+1} p_n + p_{n-1}, \\ q_{n+1} &= a_{n+1} q_n + q_{n-1}. \end{aligned} \tag{1.7}$$

▪

**Proof:**

$$\begin{aligned} \begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Multiplying the matrices on the right gives the result. ■

We will study properties of continued fractions in the next chapter.

## 1.3 Improvements of Dirichlet's Corollary

Dirichlet's theorem and corollary provide a rate of approximation which works for all real numbers. A natural question now arises. Is it possible to do better? That is to say, can we replace the right-hand sides of (1.1) and (1.5) by faster decreasing functions of $t$ and $q$ respectively. This raises the question of the sizes of the corresponding sets. We first, in this section, survey results associated with the strengthening of Dirichlet's Corollary 1.1.3.

Hurwitz (1891) proved that Dirichlet's corollary cannot be improved beyond a constant by showing that the error of rational approximation $1/q^2$ can be replaced with no better than $1/\sqrt{5}q^2$.

**Theorem 1.3.1 ([27], Hurwitz 1891)** *Given any irrational number $\zeta$, there are infinitely many rational numbers $p/q$ in lowest terms such that*

$$\left|\zeta - \frac{p}{q}\right| < \frac{1}{kq^2} \tag{1.8}$$

*for $k \leq \sqrt{5}$ but false for $k > \sqrt{5}$.* ∎

For a full proof see [25, Theorem 193], [46, Theorems 6.1 and 6.2] or [47, Theorem 5.2]. Hurwitz's 1891 paper [27] and his 1894 paper [28] can be found in the references.

The insight into $k = \sqrt{5}$ being the best possible choice is evident when evaluating the golden ratio $\phi$ as a continued fraction

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}} = 1 + \frac{1}{\phi} = \frac{1+\sqrt{5}}{2}.$$

This is best possible in the sense that for the golden ratio $\frac{\sqrt{5}+1}{2}$ and for any arbitrary small $\epsilon > 0$ the inequality

$$\left|\frac{\sqrt{5}+1}{2} - \frac{p}{q}\right| < \frac{1}{\left(\sqrt{5}+\epsilon\right) q^2}.$$

holds for finitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$. In fact there is no need for the inequality to be symmetrical and in [47, page 129] we find the following two theorems.

**Theorem 1.3.2 (B. Segre 1946)** *For any real number $r \geq 0$, an irrational number $\zeta$ can be approximated by infinitely many rational numbers $p/q$ in such a way that*

$$-\frac{1}{\sqrt{1+4r}q^2} < \frac{p}{q} - \zeta < \frac{r}{\sqrt{1+4r}q^2}.$$ ∎

When $r = 1$, this is Hurwitz's Theorem. For $r \neq 1$, the expression is unsymmetrical. Indeed, this work is interesting in that it can be used to show that one side of Hurwitz's inequality can be strengthened without essentially weakening the other.

**Theorem 1.3.3 (R.M. Robinson 1947)** *Given $\epsilon > 0$, the inequality*

$$-\frac{1}{\left(\sqrt{5}-\epsilon\right) q^2} < \frac{p}{q} - \zeta < \frac{1}{\left(\sqrt{5}+1\right) q^2}$$

*has infinitely many solutions for $p/q$.* ■

By ignoring the golden ratio, the constant on the right-hand side in (1.8) can be replaced with $\sqrt{8}$. That is, let $x \in \mathbb{R} \setminus (\mathbb{Q} \cup \frac{1+\sqrt{5}}{2})$. Then there are infinitely many solutions for

$$\left| x - \frac{p}{q} \right| < \frac{1}{kq^2}$$

for $k \leq \sqrt{8}$ but this is false for $k > \sqrt{8}$. The constant can further be replaced with another constant by ignoring all those $x$ which are equivalent to $1 + \sqrt{2} = [2; 2, \ldots]$. Continuing in this way we get the Lagrange spectrum converging to the limit $1/3$.

$$\frac{1}{\sqrt{5}} > \frac{1}{\sqrt{8}} > \frac{5}{\sqrt{221}} > \cdots > \frac{233}{\sqrt{488597}} > \cdots \longrightarrow \frac{1}{3}.$$

### 1.3.1 Badly Approximable Numbers

The above discussion, regarding the existence of irrational numbers for which the inequality (1.5) does not give infinitely many solutions beyond a constant, leads to the notion of *badly approximable* numbers.

**Definition 1.3.4** A real number $x$ is *badly approximable* if and only if there exists a constant $c = c(x) > 0$ such that for all integers $p$ and $q > 0$

$$\left| x - \frac{p}{q} \right| > \frac{c\,(x)}{q^2}.$$

The set of badly approximable numbers will be denoted by **Bad**. ■

In view of Hurwitz' theorem we necessarily have that $0 < c(x) \leq 1/\sqrt{5}$. **Bad** is nonempty because the golden ratio is badly approximable.

Notice that the set of badly approximable numbers is invariant under integer translation. In fact, this will be the case for all the sets considered in this thesis. For that reason, we will often restrict our attention to the unit interval $[0, 1)$ and no generality is lost in doing this.

A beautiful property enjoyed by the badly approximable numbers is that $x$ is badly approximable if and only if the partial quotients $a_k$, in their continued fraction expansions, are bounded.

**Theorem 1.3.5** *An irrational $x = [a_0; a_1, a_2, \ldots]$ is in **Bad** if and only if there exists a constant $B \geq 1$ such that $a_k \leq B$ for every $k \in \mathbb{N}$.* ■

For the proof we refer to [34, Theorem 23]

**Example 1.3.6** This connection is accentuated by the fact that the golden ratio $\phi$ has continued fraction expansion given by $\phi = [1; 1, 1, \ldots]$. The golden ratio is also an example of a quadratic irrational. In fact,

all quadratic irrationals are badly approximable.

This is due to the fact that they have continued fraction expansions that are eventually periodic, and are thus bounded. ■

### 1.3.2 Well-Approximable Numbers

Now, consider the situation when the exponent on the right-hand side of inequality (1.5) is changed to a parameter $\tau \in \mathbb{R}$ and ask the size of the corresponding set of numbers:

**Definition 1.3.7** Let $\tau \geq 1$. Define, the set of $\tau$-approximable numbers

$$W(\tau) := \left\{ x \in [0, 1) : |qx - p| < \frac{1}{q^\tau} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

We refer to the elements in $W(\tau)$ as *$\tau$-approximable numbers.* ■

It follows from Dirichlet's corollary that

$$W(\tau) = [0, 1) \quad \text{for} \quad \tau \leq 1.$$

What is the situation when $\tau > 1$?

We take a moment to discuss a spectacular result of Roth (1955) applicable to all algebraic numbers. Recall that a real or complex number is algebraic if it is a solution of a polynomial equation with rational coefficients. All integers and rational numbers are algebraic, as are all roots of integer polynomials. There are real or complex numbers that are not algebraic, such as $\pi$ and $e$. These numbers are called transcendental numbers. While the set of complex numbers is uncountable, the set of algebraic numbers is countable and has Lebesgue measure zero as a subset of the complex numbers, and in this sense almost all complex numbers are transcendental.

Given an algebraic number, there is a unique monic polynomial (with rational coefficients) of least degree that has the number as a root. This polynomial is called its minimal polynomial. If its minimal polynomial has degree $n$, then the algebraic number is said to be of degree $n$.

**Theorem 1.3.8 (Roth 1955)** *Supose $\zeta$ is real and algebraic of degree $d \geq 2$. Then for each $\delta > 0$, the inequality*

$$|q\zeta - p| < \frac{1}{q^{1+\delta}}$$

*can have only finitely many solutions in coprime integers $p$ and $q$.* ■

Roth's result with exponent 1 is in some sense the best possible, because by (1.5), the statement would fail on setting $\delta = 0$. For details of this result, as was conjectured by Siegel, see the work by Klaus Friedrich Roth (29 October 1925 – 10 November 2015) covered in detail in chapter V of [51]. This result is nontrivial, highly influential, and it was cited in Roth's (1958) Fields Medal award.

To reiterate the main point of the above discussion: the exponent 1 in (1.5) is best possible in the sense that if we replace it by $\tau > 1$ then (1.5) is no longer valid for all irrationals.

**Definition 1.3.9** Real numbers are *well approximable* if and only if they are not badly approximable or, in other words, numbers when represented in their continued fraction expansions have unbounded partial quotients. ■

In contrast with badly approximable numbers, we can consider irrationals which are very well approximable by rational numbers. These very well-approximable numbers can be approximated by rationals to within a rate of $q^{-\tau}$ for some $\tau > 1$.

**Definition 1.3.10** An irrational number $\zeta$ is said to be *very well approximable* if it is contained in $W(\tau)$ for *some* $\tau > 1$; that is, if there exists some $\tau > 1$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{\tau+1}} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

We denote by $VWA$ the set of very well-approximable numbers in $[0, 1)$. Thus

$$VWA = \bigcup_{\tau>1} W(\tau).$$ ■

Given any fixed $\tau > 1$, it is relatively straightforward to construct numbers in $W(\tau)$ using the theory of continued fractions. However, Liouville was first to construct explicit examples of numbers that lie in *every* $W(\tau)$ for *all* $\tau > 1$, and the set of such numbers now bears his name.

**Definition 1.3.11 (Liouville numbers)** An irrational $\zeta$ is said to be a *Liouville number* if

$$\zeta \in \bigcap_{\tau>1} W(\tau).$$

We denote the set of Liouville numbers by $\mathcal{L}$. ■

Liouville's result was that these numbers are, in fact, transcendental, the first establishment of the existence of transcendental numbers. Liouville also showed the following explicit construction of such numbers.

**Example 1.3.12** A consequence of Liouville's Theorem (page 103 and 119 of [11]) is that the number

$$10^{-1!} + 10^{-2!} + 10^{-3!} + \dots$$

is a Liouville number. ■

Clearly, $\mathcal{L} \subset W(1+\theta)$ for arbitrary $\theta > 0$ and both badly approximable and Liouville numbers are quite rare. Indeed,

$$\lambda(\mathbf{Bad}) = \lambda(\mathcal{L}) = 0 \tag{1.9}$$

where $\lambda(X)$ is the Lebesgue measure of the set $X$.

The sets **Bad** and $VWA$ provide good points of reference as they represent two extremes of approximation. In the next section we ask the question of how large the sets **Bad** and $VWA$ are. From these results, we can comment on the size of the set of numbers in between the two concepts.

Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function such that $\psi(r) \to 0$ as $r \to \infty$. Throughout this thesis, we refer to $\psi$ as an *approximating function.* We now generalise the set $W(\tau)$ of $\tau$-approximable numbers to $\psi$-approximable numbers.

**Definition 1.3.13** A number is called *$\psi$-approximable* if it is a member of the set $W(\psi)$, where

$$W(\psi) := \{x \in [0,1) : |qx - p| < \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\}$$ ■

Historically, attention has been focused on determining the size of the set of $\psi$-approximable numbers. To this end, the first major result is by Khinchin (1924).

### 1.3.3 Khinchin's Theorem

A straightforward application of the Borel-Cantelli lemma from measure theory shows that if the function $\psi$ decreases rapidly enough so that the $\sum_{q=1}^{\infty} \psi(q)$ converges then the set of $\psi$-approximable numbers is of zero Lebesgue measure, that is,

$$\sum_{q=1}^{\infty} \psi(q) < \infty \quad \Longrightarrow \quad \lambda(W(\psi)) = 0.$$

A natural question now arises. What can we say about the the size of the set $W(\psi)$ when the measure sum, $\sum_{q=1}^{\infty} \psi(q)$, diverges? The following ground-breaking measure-theoretic theorem is fundamental to the metrical theory of Diophantine approximation.

**Theorem 1.3.14 (Khinchin 1924)** *Let $\psi$ be an approximating function. Then*

$$\lambda\left(W\left(\psi\right)\right) = \begin{cases} 0 & \text{if} \quad \displaystyle\sum_{q=1}^{\infty} \psi(q) < \infty \\ \\ 1 & \text{if} \quad \displaystyle\sum_{q=1}^{\infty} \psi(q) = \infty \quad \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

For the proof, we refer to [34, Theorem 32] and for a modern proof, as a consequence of an ubiquity argument, we refer to [3, 4].

Khinchin's theorem is a delicate statement and strengthens Dirichlet's corollary. The strengthening is only for almost every number: Khinchin is a metric (or metrical) statement, while Dirichlet is a global statement.

For example, by choosing $\psi(q) = 1/(q \log q)$, Khinchin's theorem implies that $\lambda\left(W\left(\psi\right)\right) = 1$ as $\sum_{q=1}^{\infty} \frac{1}{q \log q} = \infty$. Comparing with (1.5), we see that Dirichlet's corollary has been improved in this example.

By using Khinchin's theorem we can get a direct result for the Lebesgue measure of the set of badly approximable numbers, as we quoted in Eq. (1.9).

**Corollary 1.3.15** $\lambda\left(\mathbf{Bad}\right) = 0.$

**Proof:** Consider the function $\psi\left(q\right) = 1/q \log q$ and denote $\mathbb{I} = [0, 1)$. Observe that

$$\mathbf{Bad} \cap \mathbb{I} \subseteq \mathbb{I} \setminus W\left(\psi\right),$$

By Khinchin's theorem, $\lambda\left(W\left(\psi\right)\right) = 1$. Thus $\lambda\big(\mathbb{I} \setminus W\left(\psi\right)\big) = 0$ and so

$$\lambda\left(\mathbf{Bad} \cap \mathbb{I}\right) = 0. \qquad \blacksquare$$

This result shows that the set **Bad** is small in the Lebesgue measure sense. Similarly, it follows from Khinchin's thereom that the set of very well-approximable numbers $W\left(\tau\right)$ for $\tau > 1$, and consequently Liouville numbers are of Lebesgue measure zero.

The convergence part of Theorem 1.3.14 does not need the function $\psi$ to be decreasing and an easy consequence of the Borel-Cantelli lemma. Thus the divergence case constitutes the main substance of the theorem and that is where the assumption $\psi$ is decreasing comes into play. Regarding the divergence part, in his original paper Khinchin actually required that $q \mapsto q\psi\left(q\right)$ is monotonically decreasing. It was subsequently shown by Beresnevich, Dickinson and Velani [3] that this condition can be weakened to the assumption that $\psi(q)$ is decreasing. This condition cannot in general be relaxed as was shown by Duffin and Schaeffer (1941).

Duffin and Schaeffer [19] produced a counterexample showing that this monotonicity assumption is necessary. More precisely, they constructed a non-monotonic approximating function $\psi : \mathbb{N} \to \mathbb{R}_+$ for which the sum $\sum_{q=1}^{\infty} \psi(q)$ diverges but $\lambda(W(\psi)) = 0$.

In fact, Duffin-Schaeffer conjectured that the following set

$$W'(\psi) := \{x \in [0,1) : |qx - p| < \psi(q), \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \ \& \ (p,q) = 1\}$$

has full Lebesgue measure if the series $\sum_{q=1}^{\infty} \phi(q) \frac{\psi(q)}{q}$ diverges, where $\phi(q)$ denotes the Euler totient function. A set is described to be of full measure, if the complement is of zero measure. In July 2019, Dimitris Koukoulopoulos and James Maynard announced a proof of the conjecture [37].

**Theorem 1.3.16 (Koukoulopoulos-Maynard, 2020)** *The Duffin-Schaeffer conjecture is true.* ▪

Going back to Khinchin's theorem, it does not distinguish between null sets, that is, sets which have Lebesgue measure zero, for instance

$$\lambda(W(\tau)) = 0 \quad \text{for any} \quad \tau > 1.$$

For example, $\lambda(W(10)) = \lambda(W(100)) = 0$. Intuitively, $W(10)$ should be larger than $W(100)$. This leads us to a more refined notion of size capable of distinguishing between the null sets.

### 1.3.4 Hausdorff Measure and Dimension

For completeness we give a very brief introduction to Hausdorff measures and dimension. For further details we refer to the texts [6, 20, 50].

**Definition 1.3.17** The function $f$ is a *dimension function*, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, continuous function such that $f(r) \to 0$ as $r \to 0$.

We define the *diameter* $|U|$ of a nonempty set of $\mathbb{R}^n$ as, the greatest distance apart of pairs of points in $U$. Thus, $|U| = \sup\{|x - y| : x, y \in U\}$. ▪

Let $f$ be a dimension function and let $E \subset \mathbb{R}^n$. Then, for any $\rho > 0$ a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $|B_i| \le \rho$ such that $E \subset \bigcup_i B_i$ is called a $\rho$-cover of $E$. Let

$$\mathcal{H}^f_\rho(E) = \inf \sum_i f\big(|B_i|\big),$$

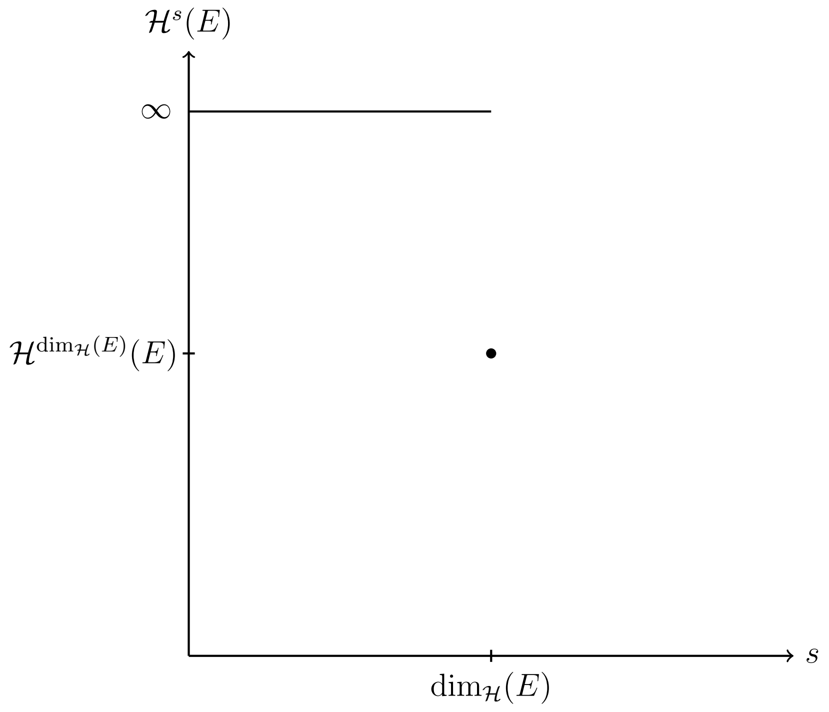
where the infimum is taken over all possible $\rho$-covers $\{B_i\}$ of $E$. It is easy to see that $\mathcal{H}^f_\rho(E)$ increases as $\rho$ decreases and so approaches a limit as $\rho \to 0$. This limit could be zero or infinity, or take a finite positive value. Accordingly, the *Hausdorff $f$-measure* $\mathcal{H}^f$ of $E$ is defined to be

$$\mathcal{H}^f(E) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(E).$$

Hausdorff $f$-measure is monotonic and countably sub-additive, and that $\mathcal{H}^f(\emptyset) = 0$. Thus it is an outer measure on $\mathbb{R}^n$.

In the case when $f(x) = x^s$ for some $s \geq 0$, we write $\mathcal{H}^s$ for $\mathcal{H}^f$. Furthermore, for any subset $E$ one can verify that there exists a unique critical value of $s$ at which $\mathcal{H}^s(E)$ jumps from infinity to zero, as $s$ increases. The value taken by $s$ at this discontinuity is referred to as the *Hausdorff dimension of $E$* and is denoted by $\dim_{\mathcal{H}} E$; that is,

$$\dim_{\mathcal{H}} E := \inf\{s \geq 0 : \ \mathcal{H}^s(E) = 0\}.$$




*Graph of Hausdorff measure $\mathcal{H}^s(E)$ against the exponent $s$*

At the critical exponent $s = \dim_{\mathcal{H}} E$, the quantity $\mathcal{H}^s(E)$ is either zero, infinite or strictly positive and finite. In the latter case; that is,

$$0 \ < \ \mathcal{H}^s(E) \ < \ \infty,$$

the set $E$ is said to be an $s$-set, see [20] for further details.

Since Hausdorff measure is defined in terms of the diameter of the covering sets, therefore, it is unchanged by restriction to closed, open or convex sets. It is also unchanged by translations and rotations but it is affected by scaling. When $s = n$, $\mathcal{H}^n$ is comparable (up to some scaling factor) with the $n$-dimensional Lebesgue measure.

Computing Hausdorff dimension of a set is typically accomplished in two steps: obtaining the upper and lower bounds separately. Upper bounds usually follow by finding a suitable covering argument. When dealing with a limsup set, one usually applies the Hausdorff measure version of the Borel-Cantelli lemma (see Lemma 3.10 of [6]) usually referred to as the Hausdorff-Cantelli lemma.

**Proposition 1.3.18** *Let $\{B_i\}_{i\geq 1}$ be a sequence of measurable sets in $\mathbb{R}^n$ and suppose that for some dimension function $f$,*

$$\sum_i f\big(\,|B_i|\,\big) \;<\; \infty.$$

*Then*

$$\mathcal{H}^f(\limsup_{i\to\infty} B_i) = 0$$

*where*

$$\limsup_{n\to\infty} B_n = \bigcap_{n=1}^{\infty}\bigcup_{k\geq n}^{\infty} B_k.$$

∎

Often it is straightforward to apply this lemma by using the natural covering of the set under consideration but sometimes it is extremely challenging to construct such a natural cover. This can be seen when working on Diophantine approximation on manifolds. For this direction of research we refer the reader to [6].

### 1.3.5 The Jarník-Besicovitch Theorem

We now discuss the role of Hausdorff measure and dimension in the theory of Diophantine approximation. In 1928, Jarník [32], proved, that $\dim_{\mathcal{H}}(\mathbf{Bad}) = 1$, using a Cantor set construction. In [22], Good described the state of play in 1941 as follows.

> "It seems that the only published work on the fractional dimensional theory of continued fractions is a paper by Jarník, in which the investigation is inspired by a problem of Diophantine approximation. Jarník is concerned with the set $E$ of continued fractions whose partial quotients are bounded, and with the sets $E_2, E_3, \ldots,$ where $E_a$ is the set of continued fractions whose partial quotients do not exceed $\alpha$. He proves that the dimensional number of $E$ is one, but he does not attempt to find the exact dimensional number of any of the sets $E_2, E_3, \ldots$".

In modern parlance this is about the set **Bad** and Theorem 1.3.5.

Since **Bad** has Lebesgue measure zero (see Corollary 1.3.15), it immediately follows that

$$\mathcal{H}^s(\mathbf{Bad}) = \begin{cases} 0 & \text{if} \quad s \geq 1 \\ \infty & \text{if} \quad 0 \leq s < 1. \end{cases}$$

The convergence part of Khinchin's theorem implies that for any $\tau > 1$ the set $W(\tau)$ is of Lebesgue measure zero. Intuitively, we would expect the size of $W(\tau)$ to decrease as the rate of approximation increases; that is, as $\tau$ increases.

We say a set $A \in \mathbb{R}^k$ has full Hausdorff dimension if $\dim_{\mathcal{H}} A = k$. The following classical result obtained by Jarník in 1929 [33] and independently (with different methods) by Besicovitch in 1934 [8] allows us to distinguish between sets of $\tau$-approximable numbers $W(\tau)$.

**Theorem 1.3.19 (Jarník-Besicovitch 1929/1934)** *For any real* $\tau \geq 1$

$$\dim_{\mathcal{H}} W(\tau) = \frac{2}{1+\tau}\ . \qquad \blacksquare$$

The theorem confirms our intuition: the size of $W(\tau)$, expressed in terms of Hausdorff dimension, decreases as $\tau$ increases. In particular, $\dim_{\mathcal{H}} W(9) = 1/5 > \dim_{\mathcal{H}} W(99) = 1/50$.

Moreover, it follows from the definition of Hausdorff dimension and the Jarník-Besicovitch theorem that

$$\mathcal{H}^s(W(\tau)) = \begin{cases} 0, & \text{if} \quad s > \dfrac{2}{1+\tau}; \\ \infty, & \text{if} \quad s < \dfrac{2}{1+\tau}. \end{cases}$$

As an immediate demonstration of this more delicate notion of size,

$$\dim_{\mathcal{H}}(\mathbf{Bad}) = 1 > 0 = \dim_{\mathcal{H}} \mathcal{L}.$$

**Example 1.3.20** Consider the approximating functions

$$\psi_1(q) = \frac{1}{q^9} \qquad \text{and} \qquad \psi_2(q) = \frac{1}{q^9 \log q}\ .$$

It follows from the Jarník-Besicovitch Theorem 1.3.19 that

$$\dim_{\mathcal{H}} W(\psi_1) = \dim_{\mathcal{H}} W(\psi_2) = \frac{1}{5}\ . \qquad \blacksquare$$

However, the Jarník-Besicovitch Theorem 1.3.19 says nothing about the $s$-dimensional Hausdorff measure $\mathcal{H}^s(W(\tau))$ at the critical exponent $s = \frac{2}{1+\tau} = \dim_{\mathcal{H}} W(\tau)$.

### 1.3.6 Jarník's Theorem

This shortcoming was subsequently addressed by the following theorem, due to Jarník [33], which reveals much more than the dimension of the sets $W(\tau)$. Jarník's theorem can be regarded as the general Hausdorff measure version of Khinchin's Theorem 1.3.14. It gives an elegant *zero-infinity* law for the set $W(\psi)$. We refer the reader to [3] for a modern proof.

**Theorem 1.3.21 ([33], Jarník 1931)** *Let $\psi$ be an approximating function and let $f$ be a dimension function such that $q^{-1}f(q) \to \infty$ as $q \to 0$ and $q^{-1}f(q)$ is decreasing. Then*

$$\mathcal{H}^f(W(\psi)) \;=\; \begin{cases} 0, & \text{if} \quad \displaystyle\sum_{q=1}^{\infty} qf\left(\frac{\psi(q)}{q}\right) < \infty; \\ \infty, & \text{if} \quad \displaystyle\sum_{q=1}^{\infty} qf\left(\frac{\psi(q)}{q}\right) = \infty \quad \textit{and } \psi \textit{ is monotonic.} \end{cases}$$ ∎

It is worth noting that when $\mathcal{H}^f$ is equivalent to one-dimensional Lebesgue measure Jarník's result does not apply. (This occurs for $s$-dimensional Hausdorff measure $\mathcal{H}^s$, when $s = 1$.) This is because the condition $q^{-1}f(q) \to \infty$ as $q \to 0$ excludes the possibility that $f(q) = q$. However, in this case Khinchin's theorem provides the relevant result. Hence $s \in (0,1)$ in the $\mathcal{H}^s$ version of the statement of Jarník's theorem.

We remark that the monotonicity assumption in Jarník's theorem once more seems vital. In fact, very little is known when this restriction is not imposed.

**Remark 1.3.22 (Historical improvements of Jarník's hypotheses.)** Analogous to Khinchin's original statement, in Jarník's original statement the additional hypotheses that $q^2\psi(q)$ is decreasing, $q^2\psi(q) \to 0$ as $q \to \infty$, and that $q^2 f(\psi(q))$ is decreasing were assumed. These conditions were proven unnecessary in [3]. Again, $\psi$ being monotonic is needed only for the divergence case. As with Khinchin's Theorem 1.3.14, the divergence part constitutes the main substance. ∎

Recall that in the case that $\mathcal{H}^f$ is the standard $s$-dimensional Hausdorff measure $\mathcal{H}^s$ (that is $f(q) = q^s$), it follows from the definition of Hausdorff dimension (see Section 1.3.4) that

$$\dim_{\mathcal{H}} W(\psi) \;=\; \inf\left\{ s : \sum_{q=1}^{\infty} q \left(\frac{\psi(q)}{q}\right)^s < \infty \right\}.$$

In view of this, Jarník's zero-infinity law not only implies the Jarník-Besicovitch theorem (Theorem 1.3.19), namely

$$\dim_{\mathcal{H}} W(\tau) = \frac{2}{1+\tau} \quad (\tau \geq 1),$$

but also that

$$\mathcal{H}^{2/(1+\tau)}(W(\tau)) \;=\; \infty \quad (\tau > 1).$$

However, it is much more powerful than this. The zero-infinity law further allows us to discriminate between sets with the same dimension and even the same $s$–dimensional Hausdorff measure.

**Example 1.3.23** With $\tau \geq 1$ and $0 < \epsilon_1 < \epsilon_2$ consider the approximating functions

$$\psi_{\epsilon_i}(q) \;:=\; q^{-(1+\tau)} \; (\log\, q)^{-\frac{1+\tau}{2}(1+\epsilon_i)} \qquad (i = 1, 2).$$

It is easily verified that for any $\epsilon_i > 0$,

$$\lambda(W(\psi_{\epsilon_i})) = 0, \quad \dim_{\mathcal{H}} W(\psi_{\epsilon_i}) = \frac{2}{1+\tau} \quad \text{and} \quad \mathcal{H}^{2/(1+\tau)}(W(\psi_{\epsilon_i})) = 0.$$

However, consider the dimension function $f$ given by

$$f(q) := q^{2/(1+\tau)}(\log q^{-1/(1+\tau)})^{\epsilon_1}.$$

Then

$$\sum_{q=1}^{\infty} q\, f\,(\psi_{\epsilon_i}(q)) \;\asymp\; \sum_{q=1}^{\infty} (q\,(\log q)^{1+\epsilon_i-\epsilon_1})^{-1}.$$

Hence, Jarník's zero-infinity law implies that

$$\mathcal{H}^f\,(W(\psi_{\epsilon_1})) \;=\; \infty \qquad \text{whilst} \qquad \mathcal{H}^f\,(W(\psi_{\epsilon_2})) \;=\; 0\;.$$

Thus the Hausdorff measure $\mathcal{H}^f$ does make a distinction between the sizes of the sets under consideration; unlike $s$-dimensional Hausdorff measure. ■

## 1.4 Recent Improvements of Dirichlet's Theorem

It is quite surprising that historically most metric theories on Diophantine approximation are intended to strengthen Dirichlet's Corollary 1.1.3 instead of Dirichlet's original Theorem 1.1.1. Since the theorem was proved by a simple pigeon-hole argument, there should be a large room for improvement.

Beyond some particular choices of $\psi$, little was known until recently, when Kleinbock and Wadleigh [36] considered improvements to Dirichlet's theorem by considering the set of $\psi$-Dirichlet improvable numbers.

**Definition 1.4.1** Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed and $t\psi(t) < 1$ for all $t \geq t_0$. A real number $x$ is said to be *$\psi$-Dirichlet improvable* if the system

$$|qx - p| < \;\psi(t) \;\; \text{and} \;\; |q| < t$$

has a nontrivial integer solution for all $t$ sufficiently large. Denote the collection of such points by $D(\psi)$. Elements of the complementary set, $D(\psi)^c$, will be referred to as *$\psi$-Dirichlet non improvable* numbers. ■

**Remark 1.4.2** The condition that $t\psi(t) < 1$ for all $t \geq t_0$ is natural in that we want an *improvement* to Dirichlet's theorem, and so $\psi(t) < 1/t$ is the natural condition. ■

## 1.5 Metrical Theory for $D(\psi)^c$

Kleinbock and Wadleigh observe [36, Theorem 1.7], that Dirichlet's theorem is sharp in the sense that $D(\psi)^c \neq \emptyset$ whenever $\psi$ is non-increasing and $t\psi(t) < 1$ for all $t \gg 1$. They also exhibit real numbers which are not $\psi$-Dirichlet under these conditions.

The main result of Kleinbock and Wadleigh [36] is a Lebesgue measure dichotomy statement for $D(\psi)^c$. To state their result and other results of this thesis, we introduce an auxiliary function

$$\Psi(t) := \frac{t\psi(t)}{1 - t\psi(t)} = \frac{1}{1 - t\psi(t)} - 1. \tag{1.10}$$

**Theorem 1.5.1 ([36], Theorem 1.8)** *Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be non-increasing, and suppose the function $t \mapsto t\psi(t)$ is non-decreasing and $t\psi(t) < 1$ for all $t > t_0$. Then*

$$\lambda\left(D(\psi)^c \cap [0,1]\right) = \begin{cases} 0, & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} < \infty; \\ 1, & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} = \infty. \end{cases} \tag{1.11}$$

■

**Example 1.5.2**

$$\lambda(D(\psi)^c) = \begin{cases} 0, & \text{if} \quad \psi(t) = \frac{1}{t}\left(1 - \frac{1}{\log t (\log\log t)^{2+\epsilon}}\right) \text{ for any } \epsilon > 0; \\ \text{full}, & \text{if} \quad \psi(t) = \frac{1}{t}\left(1 - \frac{1}{\log t (\log\log t)^{2}}\right). \end{cases}$$

■

### 1.5.1 ESL Dimension Functions

The generalised Hausdorff measure of the set $D(\psi)^c$ has been derived by Hussain, Kleinbock, Wadleigh and Wang in [29] for a class of dimension functions defined next.

**Definition 1.5.3 (ESL dimension function)** A dimension function $f$ is *essentially sub-linear* (ESL), if

$$\text{there exists } B > 1 \text{ such that } \limsup_{x \to 0} \frac{f(Bx)}{f(x)} < B. \tag{1.12}$$

■

We remark that the essentially sub-linear condition is equivalent to the doubling condition but with exponent $\alpha < 1$. (A function $f$ is called *doubling with exponent* $\alpha$ if $f(cx) \ll c^\alpha f(x)$ for all $x$ and all $c > 1$.)

As an example, the dimension functions of the form $f(r) = r^s$ for $s \in (0,1)$ are ESL dimension functions.

**Theorem 1.5.4 ([29], Theorem 1.6)** *Let $\psi$ be a non-increasing, positive function with $t\psi(t) < 1$ for all sufficiently large $t$. Let $f$ be an ESL dimension function. Then*

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0, & \text{if} \quad \sum\limits_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty, & \text{if} \quad \sum\limits_t tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

When $f(r) = r^s$ for $s \in (0,1)$ we have the following consequence of this theorem.

**Corollary 1.5.5 ([29], Theorem 1.4)** *Let $\psi$ be a non-increasing positive function with $t\psi(t) < 1$ for all large $t$ and $\Psi(t)$ as in Eq. (1.10). Then for any $0 \le s < 1$*

$$\mathcal{H}^s(D(\psi)^c) = \begin{cases} 0, & \text{if} \quad \sum\limits_t t\left(\frac{1}{t^2\Psi(t)}\right)^s < \infty; \\ \infty, & \text{if} \quad \sum\limits_t t\left(\frac{1}{t^2\Psi(t)}\right)^s = \infty. \end{cases} \tag{1.13}$$

So, the Hausdorff dimension of the set $D(\psi)^c$ is given by

$$\dim_{\mathcal{H}} D(\psi)^c = \frac{2}{2+\tau}, \quad \text{where } \tau = \liminf_{t\to\infty} \frac{\log\Psi(t)}{\log t}. \tag{1.14}$$

**Example 1.5.6** As an example,

$$\dim_{\mathcal{H}} D(\psi)^c = \frac{2}{2+\tau}, \quad \text{for } \psi(t) = \frac{1}{t} - \frac{r}{t^{\tau+1}} \ (r > 0, \tau > 0).$$

**Remark 1.5.7** The condition $s < 1$ is necessary. $\mathcal{H}^1$ is the Lebesgue measure case, which is the subject of Theorem 1.5.1. Note that the sum condition in Corollary 1.5.5 does not agree with the one in Theorem 1.5.1: indeed, when $s = 1$, the summand $\sum \frac{1}{t\Psi(t)}$ in Eq. (1.13) differs from the summand $\sum_t \frac{\log\Psi(t)}{t\Psi(t)}$ in Eq. (1.11) by a factor of $\log\Psi(t)$. This factor is not superfluous, as can be seen by taking

$$\Psi(t) = \log t\,(\log\log t)^2.$$

### 1.5.2 NESL Dimension Functions

As stated in its hypothesis, Theorem 1.5.4 holds for ESL dimension functions. Hence, naturally it is desirable to calculate the $\mathcal{H}^f$ measure of the set $D(\psi)^c$ for *non-essentially sub-linear* (NESL) dimension functions $f$, which we define next.

**Definition 1.5.8** A dimension function for which the condition given by Eq. (1.12) fails to hold, will be termed as a *non-essentially sub-linear* (NESL) dimension function. ■

The dimension functions of the form $f(x) = x$ or $f(x) = x\log(1/x)$ are examples of NESL dimension functions. Further examples of NESL dimension functions can be found in Example 4.1.2.

**Theorem 1.5.9** *Let $\psi$ be a non-increasing, positive function with $t\psi(t) < 1$ for all sufficiently large $t$. Let $f$ be an NESL dimension function. Then*

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0, & \text{if} \quad \sum_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty, & \text{if} \quad \sum_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$ ■

This theorem is proved in chapter 4.

As a consequence of this theorem, the Hausdorff dimension of the set $D(\psi)^c$ is given by

$$\dim_{\mathcal{H}} D(\psi)^c = \frac{2}{2+\tau}, \quad \text{where } \tau = \liminf_{t\to\infty} \frac{\log\Psi(t)}{\log t}. \tag{1.15}$$

**Remark 1.5.10** Note that the Hausdorff dimension of $D(\psi)^c$ is the same irrespective of ESL (see Eq. (1.14)) or NESL (see Eq. (1.15)) dimension functions but, of course, the Hausdorff measure criteria (sum conditions) are different. ■

Together with Theorem 1.5.1 and Theorem 1.5.4, Theorem 1.5.9 provides a complete Hausdorff measure theory for the set of Dirichlet non-improvable numbers.

We refer to chapter 4 for further description of ESL and NESL dimension functions. Note that the sum condition is in agreement with the sum condition given in Kleinbock and Wadleigh's Theorem 1.5.1, by choosing the NESL dimension function $f(x) = x$.

**Example 1.5.11** For the NESL dimension function $f(x) = x\log(1/x)$ we have,

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0, & \text{if} \quad \sum_t \frac{\log(t)\log(\Psi(t))}{t\Psi(t)} < \infty; \\ \infty, & \text{if} \quad \sum_t \frac{\log(t)\log(\Psi(t))}{t\Psi(t)} = \infty. \end{cases}$$ ■

## 1.6 Dirichlet Non-Improvable Numbers versus $\psi$-Approximable Numbers

Kleinbock and Wadleigh proved the following important $\psi$-Dirichlet improvability criterion which rephrases the $\psi$-Dirichlet improvability of $x$ in terms of the growth of the product of consecutive partial quotients.

**Lemma 1.6.1 ([36], Lemma 2.2)** Let $x \in [0,1) \setminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t\psi(t) < 1$ for all $t \geq t_0$ and $\Psi(t)$ as in Eq. (1.10). Then

(i) $x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \leq \Psi(q_n)/4$ for all sufficiently large $n$.

(ii) $x \in D(\psi)^c$ if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many $n$. ■

We reproduce this important proof in chapter 3.

As a consequence of Lemma 1.6.1 we have the inclusions

$$\mathcal{K}(3\Psi) \subset G_1(\Psi) \subset G(\Psi) \subset D(\psi)^c \subset G(\Psi/4), \tag{1.16}$$

where

$$G(\Psi) := \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) > \Psi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N}\Big\},$$
$$G_1(\Psi) := \Big\{x \in [0,1) : a_{n+1}(x) > \Psi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N}\Big\},$$
$$\mathcal{K}(\Psi) := \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}.$$

The set $G(\Psi)$ is the set of $\Psi$-Dirichlet non-improvable numbers and $\mathcal{K}(\Psi)$ is just the classical set of $1/q^2\Psi(q)$-approximable numbers $W(\psi(q) = 1/q^2\Psi(q))$. Although it is straightforward to prove that $\mathcal{K}(3\Psi) \subset G_1(\Psi)$ by using elementary properties of continued fractions, we prove this inclusion in Lemma 4.2.1.

The inclusions (1.16) are the key observations in proving the divergence part of the Hausdorff measure statement for $D(\psi)^c$ in Theorem 1.5.4. That is, Jarník's Theorem 1.3.21 readily gives the divergence statement for $\mathcal{K}(3\Psi)$. To be precise, from (1.16)

$$\mathcal{H}^s(\mathcal{K}(3\Psi)) = \infty \implies \mathcal{H}^s(G(\Psi)) = \infty.$$

When the sum criteria in Theorem 1.3.21, or the later improvement in Theorem 4.3.1 $\sum_t t\left(\frac{1}{t^2\Psi(t)}\right)^s$, diverges, both the sets $G(\Psi)$ and $\mathcal{K}(3\Psi)$ have full measure. However, since the inclusions (1.16) are proper, it is natural to expect that the set $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is nontrivial. From a measure theoretic point of view there is no new information as both the sets have the same $s$-dimensional Hausdorff measure. However, from a dimension point of view there is more to ask. We completely determine the Hausdorff dimension for the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$ for any $C > 0$.

**Theorem 1.6.2** *Let $\Psi : [1, \infty) \to \mathbb{R}_+$ be a non-decreasing function and $C > 0$. Then*

$$\dim_{\mathcal{H}} \Big( G(\Psi) \setminus \mathcal{K}(C\Psi) \Big) = \frac{2}{\tau + 2}, \quad \textit{where} \quad \tau = \liminf_{q \to \infty} \frac{\log \Psi(q)}{\log q}. \qquad \blacksquare$$

The term $\tau$ indicates how a function $\Psi$ grows near infinity and is known as the *lower order at infinity.* It appears naturally in determining the Hausdorff dimension of exceptional sets, when general distance functions are involved, see [17, 18].

This theorem implies that there are uncountably more $\Psi$-Dirichlet non-improvable numbers than the $\psi$-approximable numbers. We present the proof of this theorem in chapter 5.

## 1.7 Metrical Theory of Continued Fractions

The metrical theory of continued fractions, which focuses on investigating the properties of partial quotients for almost all $x \in [0, 1)$, is one of the important areas of research in the study of continued fractions and is closely connected with Diophantine approximation. The main connection is that the convergents of a real number $x$ are good rational approximates for $x$.

In fact, for any $\tau > 0$ the Jarník-Besicovitch set (or the set of $\tau$-approximable numbers $W(\tau)$)

$$\left\{ x \in [0, 1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^{\tau+2}} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\},$$

(so named because of Theorem 1.3.19) can be written in the following form,

$$\{ x \in [0, 1) : a_n(x) \geq q_n^{\tau}(x) \text{ for infinitely many } n \in \mathbb{N} \}, \tag{1.17}$$

by using Legendre's Theorem 2.3.4 and property ($P_3$) of the Lemma 2.3.3, which essentially rely on elementary properties of continued fractions. For a proof see Lemma 4.2.1. For further details about this connection we refer the reader to [22]. Thus the growth rate of the partial quotients reveals how well a real number can be approximated by rationals.

The Borel-Bernstein theorem [7, 9] is a fundamental result in the metrical theory of continued fractions. It gives an analogue of the Borel-Cantelli '0-1' law with respect to Lebesgue measure for the set of real numbers with large partial quotients. Substantial work has been done in improving the Borel-Bernstein theorem. For example, the estimation of the Hausdorff dimension of sets, when the partial quotients $a_n(x)$ obey different conditions, has been studied in [21, 22, 41].

Let $\Phi : \mathbb{N} \to (1, \infty)$ be an arbitrary function such that $\lim_{n \to \infty} \Phi(n) = \infty$. Define a generalised version of the set in (1.17) as

$$\mathcal{E}_1(\Phi) := \{ x \in [0, 1) : a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \}.$$

**Theorem 1.7.1 ([9], Borel-Bernstein)** *Let $\Phi : \mathbb{N} \to (1, \infty)$ be an arbitrary function such that* $\lim_{n\to\infty} \Phi(n) = \infty$. *Then*

$$\lambda\big(\mathcal{E}_1(\Phi)\big) = \begin{cases} 0, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{1}{\Phi(n)} < \infty, \\ 1, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{1}{\Phi(n)} = \infty. \end{cases}$$

The Borel-Bernstein theorem is a remarkably simple dichotomy result, yet it fails to distinguish between Lebesgue null sets. Lebesgue null sets arise as a result of rapidly increasing functions $\Phi$ for example if $\Phi(n) = n^{1+\eta}$, then the Lebesgue measure of corresponding set will be zero for any $\eta > 0$. To distinguish between Lebesgue null sets, Hausdorff dimension is one of the tools utilised. Wang and Wu [55] completely determined the Hausdorff dimension of the set $\mathcal{E}_1(\Phi)$. Before we state their result, we introduce necessary notation.

The continued fraction expansion defined earlier can be induced by the Gauss map $T : [0, 1) \to [0, 1)$ defined as

$$T(0) = 0, \quad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor := \frac{1}{x} \pmod 1, \quad \text{for } x \in (0, 1).$$

Each irrational number $x \in (0, 1)$ has a unique simple continued fraction expansion as follows $x := [a_1(x), a_2(x), a_3(x), \ldots]$ where $a_1(x) = \lfloor 1/x \rfloor$, $a_n(x) = \lfloor a_1(T^{n-1}(x)) \rfloor$ for $n \geq 2$ and the positive integers $a_n(x)$ are the $n$th partial quotients of $x$. We denote the derivative of $T$ by $T'$. (We shall speak more of the Gauss map in Section 2.2). Let $\mathsf{P}(\cdot)$ represent the pressure function as defined in Section 2.4.

**Theorem 1.7.2 ([55], Theorem 4.2)** *Let $\Phi : \mathbb{N} \to (1, \infty)$ be an arbitrary function. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \quad \textit{and} \quad \log b = \liminf_{n\to\infty} \frac{\log \log \Phi(n)}{n}.$$

*Then*

$$\dim_{\mathcal{H}} \mathcal{E}_1(\Phi) = \begin{cases} 1, & \text{if} \quad B = 1; \\ s_B := \inf\{s \geq 0 : \mathsf{P}(T, -s(\log B + \log |T'|)) \leq 0\}, & \text{if} \quad 1 < B < \infty; \\ \frac{1}{1+b}, & \text{if} \quad B = \infty. \end{cases}$$

Further, the dimensional number $s_B$ is continuous and its limit, as referred to next in Proposition 1.7.3 exists, as proved by Wang and Wu.

**Proposition 1.7.3 ([55], Lemma 2.6.)** *The parameter $s_B$ is continuous with respect to $B$, and*

$$\lim_{B\to 1} s_B = 1, \ \lim_{B\to\infty} s_B = 1/2.$$

Lemma 1.6.1 characterises a real number $x$ to be $\psi$-Dirichlet non improvable in terms of the growth of product of consecutive partial quotients. Kleinbock and Wadleigh proved a zero-one law for the Lebesgue measure of the set

$$\mathcal{E}_2(\Phi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n) \ \text{ for infinitely many } n \in \mathbb{N}\}.$$

**Theorem 1.7.4 ([36], Theorem 3.6)** *Let $\Phi : \mathbb{N} \to [1, \infty)$ be an arbitrary function such that $\lim_{n\to\infty} \Phi(n) = \infty$. Then*

$$\lambda\big(\mathcal{E}_2(\Phi)\big) = \begin{cases} 0, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} < \infty; \\ 1, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} = \infty. \end{cases}$$

■

We refer the reader to [26, Theorem 1.5] for a generalisation of Theorem 1.7.4. The Hausdorff dimension of $\mathcal{E}_2(\Phi)$ was recently calculated by Huang, Wu and Xu [26]. In fact, Huang, Wu and Xu considered the following general form of $\mathcal{E}_2(\Phi)$

$$\mathcal{E}_m(\Phi) := \left\{ x \in [0,1) : \begin{array}{c} a_n(x)a_{n+1}(x)\cdots a_{n+m-1}(x) \geq \Psi(n) \\ \text{for infinitely many } n \in \mathbb{N} \end{array} \right\}.$$

**Theorem 1.7.5 ([26], Theorem 1.7)** *Let $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim\limits_{n\to\infty} \Phi(n) = \infty$. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \ \text{ and } \ \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}.$$

*Then*

$$\dim_{\mathcal{H}} \mathcal{E}_m(\Phi) = \begin{cases} 1, & \text{if} \quad B = 1; \\ \inf\{s \geq 0 : \mathsf{P}\left(T, -f_m(s)\log B - s\log|T'|\right) \leq 0\} & \text{if} \quad 1 < B < \infty; \\ \frac{1}{1+b} & \text{if} \quad B = \infty; \end{cases}$$

*where $f_m$ is given by the following iterative formula*

$$f_1(s) = s, \quad f_{k+1}(s) = \frac{s f_k(s)}{1 - s + f_k(s)}, \ k \geq 1.$$

■

Note that the $\mathcal{E}_1(\Phi)$ is properly contained in $\mathcal{E}_2(\Phi)$. Since the inclusion is proper, this raises a natural question of the size of the set $\mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi)$. In other words, a natural question is to estimate the size of the set

$$\begin{aligned} \mathcal{F}(\Phi) &:= \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi) \\ &= \left\{ x \in [0,1) : \begin{array}{l} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}, \end{aligned}$$

in terms of Hausdorff dimension. We prove the following theorem in chapter 6.

**Theorem 1.7.6** *Let* $\Phi : \mathbb{N} \to (1, \infty)$ *be any function with* $\lim_{n\to\infty} \Phi(n) = \infty$. *Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \text{ and } \log b = \liminf_{n\to\infty} \frac{\log \log \Phi(n)}{n}.$$

*Then*

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) = \begin{cases} t_B := \inf\{s \geq 0 : \mathsf{P}\left(T, -s^2 \log B - s \log(|T'|\right) \leq 0\}, & \text{if } \ 1 < B < \infty; \\ \frac{1}{1+b}, & \text{if } \ B = \infty. \end{cases}$$

∎

Note that if we take $B = 1$ then from the definition of $\mathcal{F}(\Phi)$ we have $a_{n+1}(x) < 1$ which is not possible as all the partial quotients $a_{n+1}(x) \geq 1$. Therefore, $B$ is strictly greater than 1.

Chapter **2**

# Preliminaries and Auxiliary Results

In this chapter we group together some elementary results and various techniques which will be used in the course of proving our theorems.

## 2.1 Ergodicity and Mixing

There is a strong connection between Diophantine approximation and physical systems known as dynamical systems. Ergodic theory studies properties of deterministic dynamical systems, and the theory is based on general notions of measure theory. A central concern of ergodic theory is the behaviour of a dynamical system when it is allowed to run for a long time, and the resulting orbits or trajectories.

For the purposes of this thesis we will mostly be concerned with the dynamical systems of continued fractions (the Gauss map). We observe that the continued fraction expression of a real number and the Gauss map allow for the use of ideas from ergodic theory to prove certain metrical results. We commence with several definitions to set the scene.

Let $X$ be a set and $\mathcal{B}$ be a $\sigma$-algebra on subsets of $X$, which defines the subsets that will be measured. Together they are a *measurable space*.

**Definition 2.1.1** Let $(X, \mathcal{B})$ be a measurable space with a map $T : A \to A$. We say that $T$ preserves the measure $\mu$ (or $\mu$ is $T$-invariant) to mean that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$. ■

A measurable function is a function between the underlying sets of two measurable spaces that preserves the structure of the spaces: the preimage of any measurable set is measurable.

**Definition 2.1.2** Let $(X, \mathcal{B})$ be a measurable space. If $T$ is a measurable function from $X$ to itself and $\mu$ a probability measure on $(X, \mathcal{B})$ then we say that $T$ is $\mu$-ergodic or $\mu$ is an ergodic measure for $T$, if $T$ preserves $\mu$ and the following condition holds:

For any $A \in \mathcal{B}$ such that $T^{-1}(A) \subset A$ either $\mu(A) = 0$ or $\mu(A) = 1$. ■

In other words, there are no $T$-invariant subsets up to measure 0 (with respect to $\mu$).

**Definition 2.1.3** A measure-preserving dynamical system is defined as a probability space and a measure-preserving transformation on it. In more details, it is a system $(X, \mathcal{B}, \mu, T)$ with the following structure:

- $X$ is a set,
- $\mathcal{B}$ is a $\sigma$-algebra over $X$,
- $\mu : \mathcal{B} \to [0, 1]$ is a probability measure, so that $\mu(X) = 1$, and $\mu(\emptyset) = 0$,
- $T : X \to X$ is a measurable transformation which preserves the measure $\mu$. ■

**Definition 2.1.4** A transformation $T$ of a probability measure space $(X, \mu)$ is said to be (strong) mixing for the measure $\mu$ if for any measurable sets $A, B \subset X$ the following holds:

$$\lim_{n\to+\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$ ■

This notion of mixing is sometimes called strong mixing, as opposed to weak mixing which means that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left|\mu(T^{-n}A \cap B) - \mu(A)\mu(B)\right| = 0.$$

Strong mixing implies weak mixing. It is immediate that a mixing transformation is also ergodic (taking $A$ to be a $T$-stable subset and $B$ its complement). However, the converse is not true: there exist ergodic dynamical systems which are not weakly mixing, and weakly mixing dynamical systems which are not strongly mixing.

## 2.2 The Gauss Map and Gauss Measure

To consider Euclid's algorithm more closely, we consider $x \in \mathbb{I}$ and formalise the continued fraction algorithm in the form of a self-mapping of the unit interval. Define the *transformation* $T : [0, 1) \to [0, 1)$ by

$$T(0) = 0, \quad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor := \frac{1}{x} \pmod 1, \quad \text{for } x \in (0, 1). \tag{2.1}$$

Each irrational number $x \in (0, 1)$ has a unique simple continued fraction expansion as follows

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} := [a_1(x), a_2(x), a_3(x), \ldots],$$

where $a_1(x) = \lfloor 1/x \rfloor$, $a_n(x) = \lfloor a_1(T^{n-1}(x)) \rfloor$ for $n \geq 2$ and the positive integers $a_n(x)$ are the $n$th partial quotients of $x$.

The map in Eq. (2.1) is commonly known as the the Gauss map and has the convenient property that $T([a_1, a_2, a_3, \dots]) = [a_2, a_3, a_4, \dots]$.

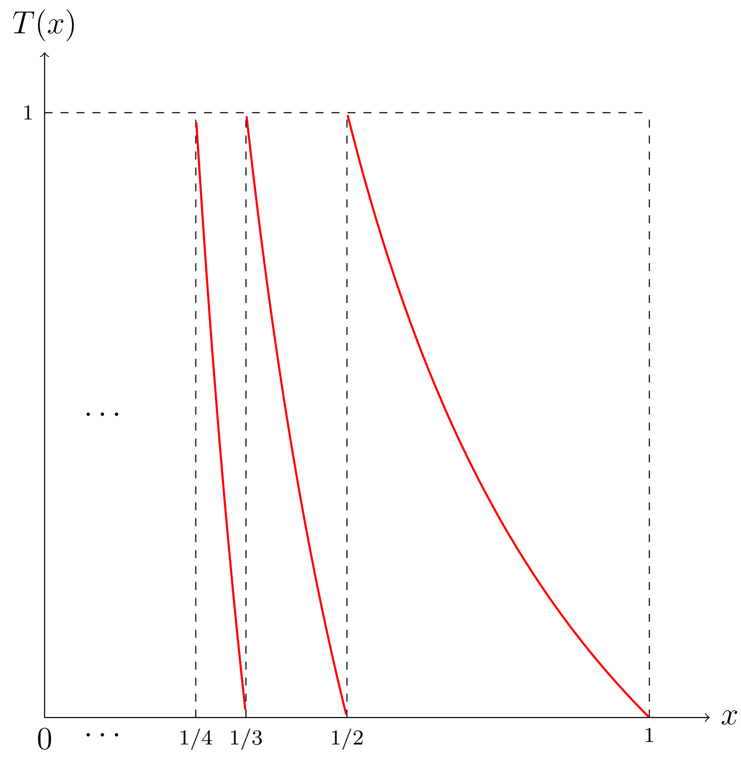
Henceforth in this thesis, except where it is explicitly nominated as something else (such as in Section 2.4), the transformation $T$ is to be taken as the Gauss map.

The Gauss measure, $\mu$ is defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}, \quad A \in \mathcal{B}_{\mathbb{I}};$$

where $\mathcal{B}_I$ denotes the $\sigma$-algebra of Borel subsets of $\mathbb{I}$.

Gauss measure is a probability measure over $\mathbb{I}$. Gauss measure is important for us since it is preserved by the Gauss map $T$, that is, $\mu(T^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{B}_I$. The Gauss transformation $T$ is ergodic with respect to the Gauss measure.




*The Gauss map*

**Remark 2.2.1** Gauss stated in 1812 that, in current notation,

$$\lim_{n\to\infty} \lambda\left(T^{-n}([0,x))\right) = \mu([0,x]), \quad x \in \mathbb{I},$$

where, as usual, $\lambda$ denotes Lebesgue measure. Gauss asked for an estimate of the convergence rate in the above limiting relation, and this has actually been the first problem of the metrical theory of continued fractions. Ramifications of this problem, which was given a first solution only in 1928 (Gauss' proof has never been found), still pervade the current developments. We refer the interested reader to chapter 2 of [31] which is devoted to a thorough treatment of Gauss' problem. ∎

## 2.3 Properties of Continued Fractions

We will need the following result in the proof of Lemma 3.1.1.

**Theorem 2.3.1 (Lagrange)** *The convergents of $x$ are optimal rational approximations of $x$ in the sense that*

$$\min_{q<q_n(x), p\in\mathbb{N}} |qx-p| = |q_{n-1}(x)x - p_{n-1}(x)|.$$ ∎

The theory of continued fractions plays a significant role in the metric theory of one dimensional Diophantine approximation. We refer the reader to some standard texts [31, 34, 38] for a detailed description of these notions.

### 2.3.1 Cylinders

For any integer vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, define a *basic cylinder* $I_n$ of order $n$ as follows:

$$I_n(a_1, \ldots, a_n) := \{x \in [0,1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

In simple terms, the basic cylinder of order $n$ consists of all real numbers in $[0,1)$ whose continued fraction expansions begin with $(a_1, \ldots, a_n)$.

In [34, page 57], Khinchin describes the positions of cylinders $I_{n+1}$ of order $n+1$ inside the $n$th order cylinder $I_n$. We write it as a lemma, to refer to it later.

**Lemma 2.3.2 ([34], page 57)** Let $I_n = I_n(a_1, \ldots, a_n)$ be a basic cylinder of order $n$, which is partitioned into sub-cylinders $\{I_{n+1}(a_1, \ldots, a_n, a_{n+1}) : a_{n+1} \in \mathbb{N}\}$. When $n$ is odd, these sub-cylinders are positioned from left to right, as $a_{n+1}$ increases from 1 to $\infty$; when $n$ is even, they are positioned from right to left. ∎

An explanatory diagram of this phenomenon may be found in [40, Figure 1. Distribution of cylinders.]

In the following lemma, we collect some basic properties of continued fractions, due to Khinchin [34] and Wu [56], that we will be referred to in our later proofs. They are also found in the seminal paper of Wang and Wu [55].

**Lemma 2.3.3** For any positive integers $a_1, \ldots, a_n$, let $p_n = p_n(a_1, \ldots, a_n)$ and $q_n = q_n(a_1, \ldots, a_n)$ be defined recursively by Eq. (1.7). Then:

(P$_1$)

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n, \text{ for all } n \geq 1.$$

(P$_2$)

$$I_n(a_1, a_2, \ldots, a_n) = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } n \text{ is even;} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Thus, its length is given by

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \tag{2.3}$$

(P$_3$) For any $n \geq 1$, $q_n \geq 2^{(n-1)/2}$ and for any $1 \leq k \leq n$

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, \ldots, a_n)}{q_n(a_1, \ldots, a_{k-1}, a_{k+1} \ldots, a_n)} \leq a_k + 1.$$

(P$_4$)

$$\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \ldots, a_1].$$

(P$_5$) For any $n \geq 1$ and $k \geq 1$ , we have (by induction on $n$) that

$$q_{n+k}(a_1, \ldots, a_n, a_{n+1} \ldots, a_{n+k}) \geq q_n(a_1, \ldots, a_n) q_k(a_{n+1}, \ldots, a_{n+k}), \tag{2.4}$$

$$q_{n+k}(a_1, \ldots, a_n, a_{n+1} \ldots, a_{n+k}) \leq 2q_n(a_1, \ldots, a_n) q_k(a_{n+1}, \ldots, a_{n+k}). \tag{2.5}$$

(P$_6$)

$$\big|q_{n-1}(x)x - p_{n-1}(x)\big| = \frac{1}{q_n(x) + T^n(x)q_{n-1}(x)}. \tag{2.6}$$

(P$_7$)

$$\begin{aligned} \frac{1}{3a_{n+1}(x)q_n^2(x)} < \left|x - \frac{p_n(x)}{q_n(x)}\right| &\overset{\text{by (2.6)}}{=} \frac{1}{q_n(x)(q_{n+1}(x) + T^{n+1}(x)q_n(x))} \\ &< \frac{1}{a_{n+1}q_n^2(x)}, \end{aligned}$$

and the derivative of $T^n$ is given by

$$(T^n)'(x) = \frac{(-1)^n}{(xq_{n-1} - p_{n-1})^2}. \tag{2.7}$$

Further,

$$q_n^2(x) \leq \prod_{k=0}^{n-1} |T'^k(x))| \leq 4q_n^2(x).$$

■

The next theorem connects one-dimensional Diophantine approximation with continued fractions.

**Theorem 2.3.4 (Legendre)** *Let $\frac{p}{q}$ be a rational number. Then*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2} \quad \Longrightarrow \quad \frac{p}{q} = \frac{p_n(x)}{q_n(x)}, \quad \text{for some } n \geq 1.$$

According to Legendre's theorem if an irrational $x$ is well approximated by a rational $\frac{p}{q}$, then this rational must be a convergent of $x$. Thus in order to find good rational approximates to an irrational number we only need to focus on its convergents. Note that, from ($P_3$) of Lemma 2.3.3, a real number $x$ is well approximated by its convergent $\frac{p_n}{q_n}$ if its $(n+1)$th partial quotient $(a_{n+1})$ is sufficiently large.

We will use the next result, due to Łuczak [41], in the proof of Theorem 1.7.6.

**Lemma 2.3.5 ([41], Łuczak)** For any $a, b > 1$, the sets

$$\left\{x \in [0,1) : a_n(x) \geq a^{b^n}, \text{ for infinitely many } n \in \mathbb{N}\right\}$$

and

$$\left\{x \in [0,1) : a_n(x) \geq a^{b^n}, \text{ for all sufficiently large } n \in \mathbb{N}\right\}$$

are of the same Hausdorff dimension $\dfrac{1}{1+b}$.

## 2.4 Pressure Function and Hausdorff Dimension

Pressure functions play an important role in finding the Hausdorff dimension of sets connected with infinite systems, such as those generated by continued fractions. We shall see their use in chapters 5 and 6. In this section we will define the concept and establish the properties we need, in the continued fraction setting.

The general idea of the pressure function (often simply called pressure), in particular topological pressure, is comprehensively explained by Walters [53, pages 207-210]. From this and other references, we shall produce a function, from which we can deduce a lower bound for the Hausdorff dimension of our set of interest.

The early work by Moran [45], in calculating the Hausdorff dimension in the context of a family of linear maps, is seen in [49, Theorem 2.2.1]. In the non-linear case, however, the corresponding generalisation of Moran, involves the pressure function.

Let $T : X \to X$ be a continuous transformation of a compact metric space $(X, d)$. Let $C(X, \mathbb{R})$ denote the Banach algebra of real-valued continuous functions of $X$ equipped with the supremum norm. The topological pressure of $T$ will be a map $\mathsf{P}(T, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$.

**Definition 2.4.1 ([49], page 31)** Given any continuous function $f : X \to \mathbb{R}$ we define its *pressure* $\mathsf{P}(f)$ (with respect to $T$) as

$$\mathsf{P}(f) := \limsup_{n\to\infty} \frac{1}{n} \log \underbrace{\left( \sum_{\substack{T^n x = x \\ x \in X}} e^{f(x)+f(Tx)+\cdots+f\left(T^{n-1}x\right)} \right)}_{\text{Sum over periodic points}}$$ ■

As is seen in [49], the actual limit exists and so the "lim sup" can be replaced by a "lim". In practice, we shall mainly be interested in a family of functions $f_t(x) = -t \log |T'(x)|$, $x \in X$ and $0 \le t \le d$, so that the above function reduces to

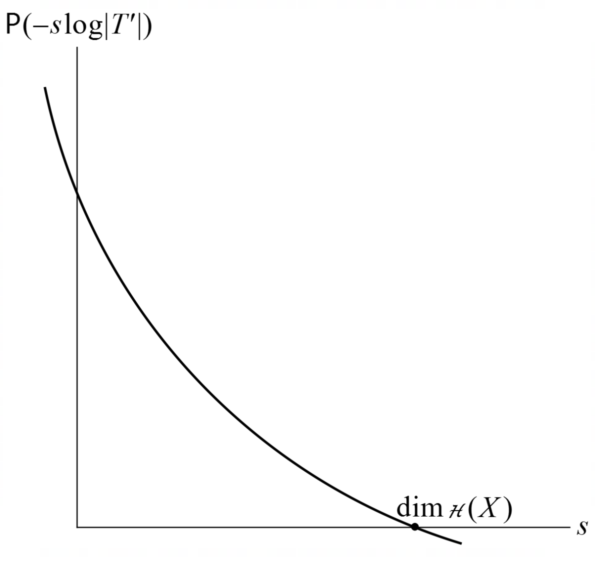
$$[0,d] \to \mathbb{R}, \qquad t \mapsto \mathsf{P}(f_t) = \limsup_{n\to\infty} \tfrac{1}{n} \log \left( \sum_{\substack{T^n x = x \\ x \in X}} \frac{1}{\left|(T^n)'(x)\right|^t} \right).$$

The following standard result is essentially due to Bowen and Ruelle. This theorem was proven originally for Julia sets with the acting map $T$ being the rational function for which $X$ is the Julia set. When the Julia set is a quasi-circle, it is due to Bowen. For the case of hyperbolic Julia sets, it is due to Ruelle. From [49, Theorem 2.3.2], we have:

**Theorem 2.4.2 (Bowen-Ruelle)** *Let $T : X \to X$ be a $C^{1+\alpha}$ conformal expanding map, for some $\alpha > 0$. There is a unique solution $0 \le s \le d$ to*

$$\mathsf{P}(-s \log |T'|) = 0,$$

*which occurs precisely at $s = \dim_{\mathcal{H}} X$.* ■




*A plot of pressure gives* $\dim_{\mathcal{H}} X$

The pressure function is strictly monotone decreasing. See [49, page 32].

### 2.4.1 The Continued Fraction Setting

In this subsection, we apply the previous more general comments about pressure functions to the system of continued fractions. The program that we follow is led by the work of Li, Wang, Wu and Xu in [40, Section 2.2]. A similar exposition to this subsection is found in the recent paper by Huang, Wu and Xu, see [26, Section 2.2]. For more thorough results on pressure function in infinite, conformal, iterated, function systems, we refer to the work of Mauldin and Urbański in [42, 43, 44].

In [42], after defining the limit set (see [42, Eq. (2.4)]) Mauldin and Urbański prove an analogue of the Moran-Bowen formula, identifying its Hausdorff dimension as the zero of the pressure function $\mathsf{P}(t)$. In [43, page 4998] they presented a form of pressure function in conformal iterated function systems with applications to the geometry of continued fractions.

From these papers, a pressure function with a continuous potential can be approximated by the pressure function restricted to the subsystems in continued fractions.

Consider a subset $\mathcal{A}$ of $\mathbb{N}$ and define

$$Y_{\mathcal{A}} = \{x \in [0,1) : \text{for all } n \geq 1, a_n(x) \in \mathcal{A}\}.$$

Then $(Y_{\mathcal{A}}, T)$ is a subsystem of $([0,1), T)$ where $T$ is the Gauss map as defined in Eq. (2.1). Given any real function $\varphi : [0,1) \to \mathbb{R}$, the pressure function restricted to the system $(Y_{\mathcal{A}}, T)$ is defined as

$$\mathsf{P}_{\mathcal{A}}(T, \varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\cdots,a_n \in \mathcal{A}} \sup_{x \in Y_{\mathcal{A}}} e^{S_n\varphi([a_1,\cdots,a_n+x])}, \tag{2.8}$$

where $S_n\varphi(x)$ denotes the ergodic sum $\varphi(x) + \cdots + \varphi(T^{n-1}x)$. Denote $\mathsf{P}_{\mathbb{N}}(T,\varphi)$ by $\mathsf{P}(T,\varphi)$ for $\mathcal{A} = \mathbb{N}$. Also note that if $\varphi$ satisfy the continuity property than we can remove the supremum from Eq. (2.8).

For each $n \geq 1$ we represent the $n$th variation of $\varphi$ by

$$\mathrm{Var}_n(\varphi) := \sup\Big\{|\varphi(x) - \varphi(y)| : I_n(x) = I_n(y)\Big\}.$$

The existence of the limit in the definition of the pressure function in Eq. (2.8) is due to the following result from [40].

**Proposition 2.4.3 ([40], Proposition 2.4)** *Let $\varphi : [0,1) \to \mathbb{R}$ be a real function with $\mathrm{Var}_1(\varphi) < \infty$ and $\mathrm{Var}_n(\varphi) \to 0$ as $n \to \infty$. Then the limit defining $\mathsf{P}_{\mathcal{A}}(T,\phi)$ exists and the value of $\mathsf{P}_{\mathcal{A}}(T,\phi)$ remains the same even without taking supremum over $x \in Y_{\mathcal{A}}$ in Eq. (2.8).* ■

The next result, which Li, Wang, Wu and Xu prove in [40], which in turn is from the work of Hanus, Mauldin and Urbański in [24], shows that in the system of continued fractions, the pressure function has a continuity property, when the system $([0,1),T)$ is approximated by its subsystems $(Y_{\mathcal{A}},T)$.

**Proposition 2.4.4 ([40], Proposition 2.5)** *Let* $\varphi:[0,1)\to\mathbb{R}$ *be a real function with* $\mathrm{Var}_1(\varphi)<\infty$ *and* $\mathrm{Var}_n(\varphi)\to 0$ *as* $n\to\infty$. *We have*

$$\mathsf{P}_{\mathbb{N}}(T,\varphi)=\sup\{\mathsf{P}_{\mathcal{A}}(T,\varphi):\mathcal{A}\text{ is a finite subset of }\mathbb{N}\}.$$ ■

We refer the reader to [40, Proposition 2.5] for an elementary proof.

From now onwards we consider the specific potential

$$\varphi_1(x)=-s(s\log B+\log|T'(x)|)$$

where $1<B<\infty$, $s\geq 0$ and $T'$ is the derivative of the Gauss map $T$. By applying Proposition 2.4.4 to $\varphi_1$, it is clear that $\varphi_1$ satisfies the variation condition.

By using Eq. (2.7) of Lemma 2.3.3, it is easy to check that

$$S_n(-s(s\log B+\log|T'(x)|))=-ns^2\log B-s\log q_n^2.$$

Therefore, the pressure function Eq. (2.8) with potential $\varphi_1$ becomes

$$\begin{aligned}\mathsf{P}_{\mathcal{A}}(T,s(s\log B+\log|T'(x)|))&=\lim_{n\to\infty}\frac{1}{n}\log\sum_{a_1,\dots,a_n\in\mathcal{A}}e^{S_n(-s(s\log B+\log|T'(x)|))}\\&=\lim_{n\to\infty}\frac{1}{n}\log\sum_{a_1,\dots,a_n\in\mathcal{A}}\left(\frac{1}{B^{ns}q_n^2}\right)^s.\end{aligned}$$

For any $n\geq 1$ and $s\geq 0$, let

$$g_n(s)=\sum_{a_1,\dots,a_n\in\mathcal{A}}\frac{1}{(B^{ns}q_n^2)^s}.$$

Define

$$\begin{aligned}&t_{n,B}(\mathcal{A}):=\inf\{s\geq 0:g_n(s)\leq 1\},\\&t_B(\mathcal{A}):=\inf\{s\geq 0:\mathsf{P}_{\mathcal{A}}(T,-s(s\log B+\log|T'|))\leq 0\},\\&t_B(\mathbb{N}):=\inf\{s\geq 0:\mathsf{P}(T,-s(s\log B+\log|T'|))\leq 0\}.\end{aligned}$$

If we take $\mathcal{A}$ to be a finite subset of $\mathbb{N}$, then it is easy to check that both $g_n(s)$ and $\mathsf{P}_{\mathcal{A}}(T,-s(s\log B+\log|T'|))$ are monotonically decreasing and continuous with respect to $s$ (for details see [55]). Therefore, $t_{n,B}(\mathcal{A})$ and $t_B(\mathcal{A})$ are respectively the unique solutions to $g_n(s)=1$ and $\mathsf{P}_{\mathcal{A}}(T,-s(s\log B+\log|T'|))=0$.

For any $M\in\mathbb{N}$, take $\mathcal{A}_M=\{1,2,\dots,M\}$. For simplicity, write $t_{n,B}(M)$ for $t_{n,B}(\mathcal{A}_M)$, $t_B(M)$ for $t_B(\mathcal{A}_M)$, $t_{n,B}$ for $t_{n,B}(\mathbb{N})$ and $t_B$ for $t_B(\mathbb{N})$.

From Proposition 2.4.4 and the definition of $t_{n,B}(M)$ we have the following result.

**Corollary 2.4.5 ([40], Corollary 2.6)** *For any integer $M \in \mathbb{N}$,*

$$\lim_{n\to\infty} t_{n,B}(M) = t_B(M), \quad \lim_{M\to\infty} t_B(M) = t_B.$$

*Since the function of $B$ belongs to $(1,\infty)$, therefore the dimensional number $t_B$ is continuous with respect to $B$ and*

$$\lim_{B\to 1} t_B = 1, \quad \lim_{B\to\infty} t_B = 1/2. \tag{2.9}$$

**Proof:** This can be proved by following similar steps as for $s_B$ in [55]. ∎

Also note that from Eq. (2.9) and definition of $t_{n,B}(M)$, we have $0 \leq t_B(M) \leq 1$. For further discussion on the dimensional number, see [55].

## 2.5 The Mass Distribution Principle and its Generalisation

Calculating Hausdorff measure or Hausdorff dimension normally splits into two parts: the upper bound and the lower bound. Within the frameworks of independent variable theory, obtaining the upper bound is often easier to establish relying on estimating the natural cover of the set under consideration. To calculate the lower bound, for both Hausdorff measure or Hausdorff dimension is often challenging. A general and classical method for obtaining the lower bound is the following mass distribution principle. But before we state it, we introduce some basic definitions.

**Definition 2.5.1 ([20], §1.3)** Let $\mu$ be a measure on the Hausdorff locally compact topological space $X$. The *support* of $\mu$ is the smallest closed set $F$ such that $\mu(X \setminus F) = 0$.

We think of the support of a measure as the set on which the measure is concentrated. We say that $\mu$ is a measure *on* a set $F$, if $F$ contains the support of $\mu$.

**Definition 2.5.2 ([20], §1.3)** A mass distribution on a set $F$, is a measure with support contained in $F$, such that $0 < \mu(F) < \infty$.

We think of $\mu(F)$ as the mass of the set $F$. Intuitively: we take a finite mass and spread it in some way across a set $F$ to get a mass distribution on $F$.

We will be employing the following mass distribution principle to calculate the lower Hausdorff dimension bound for both Theorems 1.6.2 and 1.7.6. We refer the reader to [20, §4.1].

**Proposition 2.5.3 (Mass Distribution Principle)** *Let $\mu$ be a mass distribution on a set $F$ and suppose that for some $s > 0$, there are numbers $c > 0$ and $\rho > 0$ such that*

$$\mu(U) \leq c\,|U|^s \tag{2.10}$$

*for all sets $U$ with $|U| \leq \rho$. Then*

$$\mathcal{H}^s\,(F) \geq \frac{\mu(F)}{c} > 0.$$

*Hence, in particular,* $\dim_{\mathcal{H}}(F) \geq s$. ■

**Proof:** If $\{U_i\}$ is any cover of $F$, so $F \subset \bigcup_i U_i$, then

$$0 < \mu(F) \leq \mu\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu\,(U_i) \leq c \sum_{i=1}^{\infty} |U_i|^s\,,$$

using properties of a measure and (2.10). Rewrite this last inequality as

$$\sum_{i=1}^{\infty} |U_i|^s \geq \frac{\mu(F)}{c}$$

for all $\rho$-covers of $F$. Thus, from the definition of Hausdorff measure, it follows that, by taking the infimum over all $\rho$-covers such that $\rho$ is small enough, we have that $\mathcal{H}^s\,(F) \geq \frac{\mu(F)}{c} > 0$. Hence, in particular, $\dim_{\mathcal{H}} F \geq s$. ■

### 2.5.1 A Generalised Hausdorff Measure Criterion

To calculate the lower bound of $\mathcal{H}^f$-measure for a general dimension functions $f$, the following variation of the mass distribution principle stated above is often used, see [5, page 975].

**Lemma 2.5.4 (Mass Distribution Principle for $f$-measure)** Let $\mu$ be a probability measure supported on a subset $F$ of $X$. Suppose there are positive constants $c > 0$ and $\rho > 0$ such that for some dimension function $f$

$$\mu(U) \leq cf(|U|)$$

for all sets $U$ with $|U| \leq \rho$. Then $\mathcal{H}^f(F) \geq \mu(F)/c$. ■

Specifically, the mass distribution principle replaces the consideration of all coverings by the construction of a particular measure $\mu$. Given a sequence of sets $(B_i)_i$, if we want to prove that $\mathcal{H}^f(\limsup_{i\to\infty} B_i)$ is infinite (and in particular strictly positive), one possible strategy is to deploy the mass distribution principle in two steps.

1. Construct a suitable Cantor type subset $\mathcal{K} \subset F = \limsup_{i\to\infty} B_i$ and a probability measure $\mu$ supported on $\mathcal{K}$.

2. Show that for any fixed $c > 0$, $\mu$ satisfies the condition that for any measurable set $U$ of sufficiently small diameter, $\mu(U) \leq cf(|U|)$.

If this can be done, then by the mass distribution principle, it follows that

$$\mathcal{H}^f(F) \geq \mathcal{H}^f(\mathcal{K}) \geq c^{-1}.$$

Then since $c$ is arbitrary, it follows that $\mathcal{H}^f(F) = \infty$.

The main intricate and substantive part of this entire process is the construction of a suitable Cantor type subset of $F$ which supports a probability measure. In 2019, Hussain and Simmons [30] introduced a generalised principle to determine the $f$-dimensional Hausdorff measure of limsup sets which relieves one of the need to utilise the Cantor type construction from this process.

To state the result of Hussain and Simmons, we introduce some notation. Let $X$ be a metric space. For $\delta > 0$, a measure $\mu$ is *Ahlfors $\delta$-regular* if and only if there exist positive constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that the inequality

$$c_1 r^\delta \leq \mu(B(x,r)) \leq c_2 r^\delta$$

holds for every ball $B := B(x,r)$ in $X$ of radius $r \leq r_0$ centred at $x \in \mathrm{Supp}(\mu)$, where $\mathrm{Supp}(\mu)$ denotes the topological support of $\mu$. The space $X$ is called Ahlfors $\delta$-regular if there is an Ahlfors $\delta$-regular measure whose support is equal to $X$. If $X$ is Ahlfors $\delta$-regular, then so is the $\mathcal{H}^\delta$ measure restricted to $X$, written as, $\mathcal{H}^\delta \upharpoonright X$.

**Theorem 2.5.5 ([30], Theorem 1)** *Fix $\delta > 0$, let $(B_i)_i$ be a sequence of open sets in an Ahlfors $\delta$-regular metric space $X$, and let $f$ be a dimension function such that*

$$\begin{aligned} r \mapsto & r^{-\delta} f(r) \textit{ is decreasing, and} \\ & r^{-\delta} f(r) \to \infty \textit{ as } r \to 0. \end{aligned} \tag{2.11}$$

*Fix $C > 0$, and suppose that the following hypothesis holds:*

*(*) For every ball $B_0 \subset X$ and for every $N \in \mathbb{N}$, there exists a probability measure $\mu = \mu(B_0, N)$ with $\mathrm{Supp}(\mu) \subset \bigcup_{i \geq N} B_i \cap B_0$, such that for every ball $B = B(x,\rho) \subset X$, we have*

$$\mu(B) \ll \max\left(\left(\frac{\rho}{\operatorname{diam} B_0}\right)^\delta, \frac{f(\rho)}{C}\right). \tag{2.12}$$

*Then for every ball $B_0$,*

$$\mathcal{H}^f\left(B_0 \cap \limsup_{i\to\infty} B_i\right) \quad C.$$

*In particular, if the hypothesis (*) holds for all $C$, then*

$$\mathcal{H}^f\left(B_0 \cap \limsup_{i\to\infty} B_i\right) = \infty.$$

■

The condition (2.11) is a natural condition which implies that $\mathcal{H}^f(B) = \infty$. The hypothesis (*) is the main ingredient of this theorem and gives a systematic way of constructing the probability measure on the limsup set.

We will use Hussain and Simmons' criterion to prove the divergence case of Theorem 1.5.9.

Chapter 3

# Lebesgue Measure Theory for Uniform Approximation

The aim of this chapter is to reproduce, for the ease of reader and to fully describe the metrical theory for the set $D(\psi)$, the Lebesgue measure criterion of Kleinbock and Wadleigh [36].

Recall from Definition 1.4.1 that the set of $\psi$-Dirichlet improvable numbers $D(\psi)$ is the set of all real numbers $x$ such that the system

$$|qx-p| < \psi(t) \quad \text{and} \quad |q| < t$$

has a nontrivial integer solution for all large enough $t$; where $\psi : [t_0,\infty) \to \mathbb{R}_+$ is a non-increasing function with $t_0 \geq 1$ fixed and $t\psi(t) < 1$ for all $t \geq t_0$.

Clearly $D(\psi)$ is contained in $W(\psi)$ whenever $\psi$ is non-increasing. If $\psi$ decreases faster than $\psi_1 = 1/t$, then we also know that $D(\psi)$ and $W(\psi)$ differ. This follows from Theorem 1.3.1 and that Davenport and Schmidt [13] proved that for any $\epsilon > 0$,

$$\lambda\Big(D\big((1-\epsilon)\psi_1\big)^c\Big) = 1.$$

So it is natural to ask how small are the corresponding sets, that is, what is the size of the complement $D(\psi)^c$, in the sense of measure or dimension, for functions $\psi$ which decrease faster than $(1-\epsilon)\psi_1$ for any $\epsilon > 0$?

Around the time of Davenport and Schmidt's work on improving Dirichlet's theorem, there were contributions made by Diviš in the papers [15, 16] and some made very recently by Haas [23]. Then recently, Kleinbock and Wadleigh [36] produced their main result.

**Theorem 1.5.1 ([36], Theorem 1.8)** *Let $\psi : [t_0,\infty) \to \mathbb{R}_+$ be non-increasing, and suppose the function $t \mapsto t\psi(t)$ is non-decreasing and $t\psi(t) < 1$ for all $t > t_0$. Then*

$$\lambda\left(D(\psi)^c \cap [0,1]\right) = \begin{cases} 0, & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} < \infty; \\ 1, & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} = \infty. \end{cases} \tag{1.11}$$

∎

## 3.1 Dirichlet Improvability via Continued Fractions

The results of Davenport and Schmidt [13] and Kleinbock and Wadleigh [36], rely crucially on the observation that $\psi$-Dirichlet improvability is equivalent to the following condition on the growth rate of partial quotients.

**Lemma 3.1.1 ([36], Lemma 2.1)** Let $\psi : \mathbb{N} \to \mathbb{R}_+$ be non-increasing and suppose $t\psi(t) < 1$ for all $t$ sufficiently large. Then

$$\begin{aligned} x \in D(\psi) &\iff |q_{n-1}x - p_{n-1}| \; < \; \psi(q_n) \text{ for all } n \gg 1 \\ &\iff [a_{n+1}, a_{n+2}, \dots] \cdot [a_n, a_{n-1}, \dots, a_1] \; < \; \frac{1}{\Psi(q_n)} \text{ for all } n \gg 1. \end{aligned}$$

**Proof:** For the first equivalence, if $x$ is in $D(\psi)$, then for all $n$ sufficiently large, there exist $p, q \in \mathbb{N}$ with $q < q_n$ such that

$$|qx - p| \; < \; \psi(q_n).$$

Then by Lagrange's theorem 2.3.1,

$$|q_{n-1}x - p_{n-1}| \leq |qx - p| \; < \; \psi(q_n).$$

Conversely, assume that

$$|q_{n-1}x - p_{n-1}| \; < \; \psi(q_n), \text{ for all } n \gg 1.$$

Then for any large $t$ there exists $n$ such that $q_{n-1} < t \leq q_n$. Hence, by the monotonicity of $\psi$, one has

$$|q_{n-1}x - p_{n-1}| \; < \; \psi(q_n) \leq \psi(t).$$

The second equivalence follows from Eq. (1.10) and from ($P_4$) and ($P_6$) in Lemma 2.3.3 via a simple computation. ∎

Lemma 3.1.1 is one step towards rephrasing the $\psi$-Dirichlet property of $x$ in terms of the growth of the continued fraction entries $a_n(x)$. For a fixed $x = [a_1, a_2, \ldots]$, consider the sequences of the form

$$\theta_{n+1} \; = \; [a_{n+1}(x), a_{n+2}(x), \dots] \quad \text{and} \quad \phi_n \; = \; [a_n(x), a_{n-1}(x), \dots, a_1(x)].$$

Cassels derived the following formula, see [11, eq. (16) pp. 7].

**Lemma 3.1.2 ([11])**

$$(1 + \theta_{n+1}\phi_n)^{-1} = q_{n+1}|q_n x - p_n|. \tag{3.1}$$

This is the second step for passing from Lemma 3.1.1 to the continued fractions statement of Lemma 1.6.1.

**Remark 3.1.3** We present a precis of Cassels' proof. Cassels uses $\theta$, we use $x$ for the real number being approximated. Cassels also indexes his $a_n$ from a different index and builds his version [11, Equation (8) and (9) of page 3] of our recursion Eq. (1.7) accordingly. Kleinbock and Wadleigh also note Cassels' index difference and make adjustments accordingly. They mention this in a footnote to [36, Lemma 2.1].

Cassels' approach to continued fractions also differs in that he does not present them as a natural consequence of the Euclidean algorithm, rather he uses a number of inequalities flowing from Dirichlet's theorem to prove the existence of a continued fraction expansion for each real number. Hence the proof we present here serves to modernise his results. ∎

**Proof: Part 1:** Firstly, we explain Cassels' meaning of $\theta$ and $\phi$.

Using the recurrence relation Eq. (1.7), we get

$$q_{n+1}x - p_{n+1} = (a_{n+1}q_n + q_{n-1})\,x - (a_{n+1}p_n + p_{n-1})$$

so

$$(q_{n-1}x - p_{n-1}) = a_{n+1}\,(q_n x - p_n) - (q_{n+1}x - p_{n+1})\,.$$

From Olds [47, Theorem 3.3] we deduce that in the equation above, two of three bracketed terms have the same sign, so that when we take absolute values, we have

$$|q_{n-1}x - p_{n-1}| = a_{n+1}\,|q_n x - p_n| + |q_{n+1}x - p_{n+1}|\,. \tag{3.2}$$

Write

$$\theta_0 = 1, \quad \theta_n = \frac{|q_n x - p_n|}{|q_{n-1}x - p_{n-1}|} \quad (n \geq 1) \tag{3.3}$$

so that

$$\theta_1 = x, \quad 0 \leq \theta_n < 1 \quad (n \geq 1)\,.$$

Eq. (3.2) becomes

$$\theta_n^{-1} = a_n + \theta_{n+1}. \tag{3.4}$$

Then

$$\theta_n = [a_n(x), a_{n+1}(x), \ldots]$$

follows from Eq. (3.4), which can also be read from the long form of the continued fraction expansion:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots\, a_{n-1}(x) + \cfrac{1}{a_n(x) + \ddots}}}}$$

Similarly write

$$\phi_n = \frac{q_n}{q_{n+1}} \quad (n \geq 0) \tag{3.5}$$

so that

$$0 \leq \phi_n \leq 1.$$

Then

$$q_{n+1} = a_{n+1} q_n + q_{n-1}$$

becomes

$$\phi_n^{-1} = a_n + \phi_{n-1}.$$

Hence as with $\theta_n$, we have

$$\phi_n = [a_n(x), a_{n-1}(x), \ldots, a_1(x)].$$

**Part 2:** Secondly, we produce an identity that in Part 3, we will use with terms of the identity replaced by $\theta$ and $\phi$, to arrive at our final result.

Adding and subtracting $q_n q_{n+1} x$ we have the identity

$$q_{n+1} p_n - q_n p_{n+1} = q_n \left( q_{n+1} x - p_{n+1} \right) - q_{n+1} \left( q_n x - p_n \right). \tag{3.6}$$

But from $(\mathrm{P}_1)$ of Lemma 2.3.3

$$q_{n+1} p_n - q_n p_{n+1} = (-1)^{n+1} \tag{3.7}$$

and again Olds [47, Theorem 3.3] tells us that

$$\left( q_n x - p_n \right) \left( q_{n+1} x - p_{n+1} \right) \leq 0. \tag{3.8}$$

Apply Eq. (3.7) and inequality (3.8) to Eq. (3.6) to deduce that [11, Lemma 2, Corollary 3]

$$q_n \left\| q_{n+1} x \right\| + q_{n+1} \left\| q_n x \right\| = 1. \tag{3.9}$$

**Part 3:** On the first line in the calculations below, we start with Eq. (3.9). On the second line we use Eq. (3.3) (with its index increased by 1, as explained in Remark 3.1.3), and Eq. (3.5). We get

$$\begin{aligned}
1 &= q_n \left\| q_{n+1} x \right\| + q_{n+1} \left\| q_n x \right\| \\
&= q_n \left\| q_n x \right\| \theta_{n+1} + q_n \phi_n^{-1} \left\| q_n x \right\| \\
&= \left( \theta_{n+1} + \phi_n^{-1} \right) q_n \left\| q_n x \right\| \\
\text{Hence} \quad q_n \left\| q_n x \right\| &= \frac{1}{\theta_{n+1} + \phi_n^{-1}} \\
\Rightarrow \frac{q_n}{\phi_n} \left\| q_n x \right\| &= \frac{1}{\phi_n} \left( \frac{1}{\theta_{n+1} + \phi_n^{-1}} \right) \\
\Rightarrow q_{n+1} \left\| q_n x \right\| &= \left( 1 + \theta_{n+1} \phi_n \right)^{-1}, \text{ by Eq. (3.5).}
\end{aligned}$$

■

By combining the relation (3.1) with the first $\psi$-Dirichlet property of $x$ in Lemma 3.1.1, Kleinbock and Wadleigh proved the following important $\psi$-Dirichlet improvability criterion, which rephrases the $\psi$-Dirichlet improvability of $x$ in terms of the growth of the product of consecutive partial quotients. This leads to the criterion that a number is $\psi$-Dirichlet improvable, if and only if the partial quotients of $x$ do not grow too quickly.

**Lemma 3.1.4 ([36], Lemma 2.2)** Let $x \in [0,1) \setminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t\psi(t) < 1$ for all $t \geq t_0$ and $\Psi(t)$ as in Eq. (1.10). Then

(i) $x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \leq \Psi(q_n)/4$ for all sufficiently large $n$.

(ii) $x \in D(\psi)^c$ if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many $n$.

**Proof:** Let $x \in [0,1) \setminus \mathbb{Q}$. From Lemma 3.1.1 and Cassels' formula (3.1), we have

$$\begin{aligned} x \in D(\psi) &\iff (1+\theta_{n+1}\phi_n)^{-1} < q_n\psi(q_n) \text{ for sufficiently large } n \\ x \in D(\psi) &\iff \Big(1 + [a_{n+1}, a_{n+2}, \ldots] \cdot [a_n, a_{n-1}, \ldots, a_1]\Big)^{-1} < q_n\psi(q_n) \\ &\qquad \text{for sufficiently large } n. \end{aligned}$$

Since $\left(a_{n+1} + \frac{1}{a_{n+2}}\right)\left(a_n + \frac{1}{a_{n-1}}\right) \leq 4a_na_{n+1}$, we have

$$\begin{aligned} \left(1 + \frac{1}{a_na_{n+1}}\right)^{-1} &< (1+\theta_{n+1}\phi_n)^{-1} \\ &< \left(1 + \frac{1}{\left(a_{n+1} + \frac{1}{a_{n+2}}\right)} \frac{1}{\left(a_n + \frac{1}{a_{n-1}}\right)}\right)^{-1} \leq \left(\frac{1}{1 + \frac{1}{4a_{n+1}a_n}}\right). \end{aligned}$$

Hence, $x \in D(\psi)$ if

$$\left(1 + \frac{1}{4a_{n+1}a_n}\right)^{-1} < q_n\psi(q_n) \quad \text{for all sufficiently large } n.$$

Hence $x \in D(\psi)$ if

$$a_na_{n+1} < \frac{1}{4}\left(\frac{q_n\psi(q_n)}{1 - q_n\psi(q_n)}\right) := \frac{1}{4}\Psi(q_n) \quad \text{for all sufficiently large } n.$$

The proof for (ii) follows similarly. ∎

### 3.1.1 A Dynamical Borel-Cantelli Lemma

The Gauss measure $\mu$ defined by $d\mu = \frac{1}{\log 2}\frac{dx}{x+1}$, is equivalent to the Lebesgue measure $\lambda$ and the Gauss transformation $T$ is ergodic with respect to the Gauss measure. Gauss measure is a probability measure over $\mathbb{I}$. Kleinbock and Wadleigh combine two results of Philipp [48, theorems 2.3 and 3.2] related to the divergence case of the Borel-Cantelli lemma, and the following strong mixing property of the Gauss map $T$ with respect to the Gauss measure, to establish a quite general dynamical Borel-Cantelli Lemma in [36, Lemma 3.5].

**Theorem 3.1.5 ([48], Theorem 3.2)** *There exist constants $c_0 > 0$ and $0 < \gamma < 1$ with the following property. Fix $\mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ and write*

$$I_k := \{x \in [0,1) : a_1(x) = r_1, a_2(x) = r_2, ..., a_k(x) = r_k\}.$$

*Let $F \subset [0,1]$ be any measurable set. Then for all $n \geq 0$,*

$$\left|\mu(I_k \cap T^{-n-k}F) - \mu(I_k)\mu(F)\right| \leq c_0\mu(I_k)\mu(F)\gamma^{\sqrt{n}}. \tag{3.10}$$

As Philipp [48] observed, the estimate (3.10) admits passing to unions, in the sense described in the next corollary.

**Corollary 3.1.6 ([36], Corollary 3.4)** *Let $c_0$ and $\gamma$ be as in Theorem 3.1.5. Let $F \subset [0,1]$ be any measurable set. Fix $k \in \mathbb{N}$, and let $\mathfrak{R} \subset \mathbb{N}^k$. Then Eq. (3.10) holds for all $n \geq 0$ when $I_k$ is replaced with $\cup_{\mathbf{r}\in\mathfrak{R}} I_k$*

**Proof:** We have

$$\begin{aligned}
&\left|\mu\left(\cup_{\mathbf{r}\in\mathfrak{R}} I_k \cap T^{-n-k}F\right) - \mu\left(\cup_{\mathbf{r}\in\mathfrak{R}} I_k\right)\mu\left(F\right)\right| \\
&= \left|\sum_{\mathbf{r}\in\mathfrak{R}} \left(\mu(I_k \cap T^{-n-k}F) - \mu(I_k)\mu(F)\right)\right| \\
&\leq \sum_{\mathbf{r}\in\mathfrak{R}} c_0\mu(I_k)\mu(F)\gamma^{\sqrt{n}} = c_0\mu\left(\cup_{\mathbf{r}\in\mathfrak{R}} I_k\right)\mu(F)\gamma^{\sqrt{n}}.
\end{aligned}$$

∎

We now combine the above statements to establish a quite general dynamical Borel-Cantelli lemma:

**Lemma 3.1.7 ([36], Lemma 3.5)** Fix $k \in \mathbb{N}$. Suppose $A_n$ $(n \in \mathbb{N})$ is a sequence of sets such that each $A_n$ is a union of sets of the form $I_k$, $\mathbf{r} \in \mathbb{N}^k$ where ($I_k$ as defined in Theorem 3.1.5). If $\sum_n \mu(A_n) = \infty$ (resp.$< \infty$) then for almost every (resp. almost no) $x \in [0,1]$ one has $T^n(x) \in A_n$ for infinitely many $n$.

**Proof:** The convergence case follows from the Borel-Cantelli lemma and the fact that $\mu$ is $T$-invariant.

Suppose $\sum_n \mu(A_n) = \infty$. For $m \geq n+k$ write

$$\begin{aligned}
\mu(T^{-n}A_n \cap T^{-m}A_m) &= \mu(A_n \cap T^{-(m-n)}A_m) \\
&\leq \mu(A_n)\mu(A_m) + c_0\mu(A_n)\mu(A_m)\gamma^{\sqrt{m-n-k}} \\
&\leq \mu(A_n)\mu(A_m) + \mu(A_m)c_0\gamma^{\sqrt{m-n-k}} \\
&= \mu(T^{-n}A_n)\mu(T^{-m}A_m) + \mu(T^{-m}A_m)c_0\gamma^{\sqrt{m-n-k}}
\end{aligned}$$

for $c_0$, $\gamma$ as in Theorem 3.1.5. The sets $T^{-n}A_n$ therefore satisfy the condition of [36, Theorem 3.1] (in light of [36, Remark 3.2]). By that theorem, $\sum_n \mu(T^{-n}A_n) = \infty$ guarantees that almost all $x$ lie in $T^{-n}A_n$ for infinitely many $n$. ∎

This lemma can now be applied to the set

$$\begin{aligned}
\mathcal{E}_2(\Phi) &:= \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\} \qquad (3.11) \\
&= \limsup_{n\to\infty} A_n := \limsup_{n\to\infty}\{x \in (0,1) : a_1(x)a_2(x) \geq \Phi(n)\}
\end{aligned}$$

with $n \geq 1$, to obtain the following theorem.

**Theorem 1.7.4 ([36], Theorem 3.6)** *Let $\Phi : \mathbb{N} \to [1,\infty)$ be an arbitrary function such that $\lim_{n\to\infty}\Phi(n) = \infty$. Then*

$$\lambda\big(\mathcal{E}_2(\Phi)\big) = \begin{cases} 0, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{\log\Phi(n)}{\Phi(n)} < \infty; \\ 1, & \text{if} \quad \sum\limits_{n=1}^{\infty} \frac{\log\Phi(n)}{\Phi(n)} = \infty. \end{cases}$$

To prove Theorem 1.7.4, in view of Lemma 3.1.7 it suffices to show that

$$\lambda(A_n) \asymp \mu(A_n) \asymp \frac{\log\Phi(n)}{\Phi(n)}.$$

We prove this in the following lemma.

**Lemma 3.1.8** The Gauss measure of $A_n$ satisfies

$$\frac{1}{\log 2}\cdot(1+o(1))\cdot\frac{\log\Phi(n)}{\Phi(n)} \leq \mu(A_n) \leq \frac{1}{\log 2}\cdot\frac{\log\Phi(n)+O(1)}{\Phi(n)}$$

where '$o$' and '$O$' are little-$o$ and big-$O$ notations respectively.

**Proof:** For any $n \geq 1$,

$$\begin{aligned}
A_n &= \bigcup_{1\leq a\leq\Phi(n)}\left[\frac{1}{a+1/\lfloor\frac{\Phi(n)}{a}\rfloor}, \frac{1}{a}\right) \bigcup \left(\bigcup_{a>\Phi(n)}\left[\frac{1}{a+1},\frac{1}{a}\right)\right) \\
&\subseteq \bigcup_{a\leq\Phi(n)}\left[\frac{1}{a+\frac{a}{\Phi(n)}},\frac{1}{a}\right) \bigcup \left(0,\frac{1}{\Phi(n)}\right)
\end{aligned}$$

and

$$\begin{aligned}\mu\left(\left[\frac{1}{a+\frac{a}{\Phi(n)}},\frac{1}{a}\right)\right) &= \frac{1}{\log 2}\cdot\int_{\frac{1}{a+\frac{a}{\Phi(n)}}}^{\frac{1}{a}}\frac{1}{1+x}dx\\ &= \frac{1}{\log 2}\cdot\log\left(1+\frac{1}{a(\Phi(n)+1)+\Phi(n)}\right)\\ &\le \frac{1}{\log 2}\cdot\frac{1}{a(\Phi(n)+1)+\Phi(n)}.\end{aligned}$$

Note that the Lebesgue measure and the Gauss measure $\mu$ are equivalent. Then there exists an absolute constant $c$ such that $\mu((0,\frac{1}{\Phi(n)}))\le\frac{c}{\Phi(n)}$. So

$$\begin{aligned}\mu(A_n) &\le \frac{1}{\log 2}\cdot\sum_{a=1}^{\lfloor\Phi(n)\rfloor}\frac{1}{a(\Phi(n)+1)+\Phi(n)}+\frac{c}{\Phi(n)}\\ &\le \frac{1}{\log 2}\cdot\left(\frac{1}{2\Phi(n)+1}+\int_1^{\Phi(n)}\frac{dx}{(\Phi(n)+1)x+\Phi(n)}\right)+\frac{c}{\Phi(n)}\\ &\le \frac{1}{\log 2}\cdot\frac{\log\Phi(n)+O(1)}{\Phi(n)}.\end{aligned}$$

On the other hand,

$$A_n\supset\bigcup_{a\le\Phi(n)}\left(\frac{1}{a+\frac{a}{a+\Phi(n)}},\frac{1}{a}\right).$$

Then we have

$$\begin{aligned}\mu(A_n) &\ge \frac{1}{\log 2}\cdot\sum_{a=1}^{\lfloor\Phi(n)\rfloor}\left(\log\left(1+\frac{1}{a}\right)-\log\left(1+\frac{1}{a+\frac{a}{a+\Phi(n)}}\right)\right)\\ &= \frac{1}{\log 2}\cdot\sum_{a=1}^{\lfloor\Phi(n)\rfloor}\log\left(1+\frac{\frac{1}{a+\Phi(n)+1}}{a+\frac{a+\Phi(n)}{a+\Phi(n)+1}}\right)\\ &\ge \frac{1}{\log 2}\cdot\sum_{a=1}^{\lfloor\Phi(n)\rfloor}\log\left(1+\frac{1}{(a+1)(a+\Phi(n)+1)}\right).\end{aligned}$$

Thus

$$\begin{aligned}\mu(A_n) &\ge \frac{1}{\log 2}\cdot\sum_{a=1}^{\lfloor\Phi(n)\rfloor}(1+o(1))\frac{1}{(a+1)(a+\Phi(n)+1)}\\ &\ge \frac{1}{\log 2}\cdot(1+o(1))\cdot\frac{1}{\Phi(n)}\sum_{a=1}^{\lfloor\Phi(n)\rfloor}\left(\frac{1}{a+1}-\frac{1}{a+\Phi(n)+1}\right)\\ &\ge \frac{1}{\log 2}\cdot(1+o(1))\cdot\frac{1}{\Phi(n)}\left(\int_1^{\lfloor\Phi(n)\rfloor}\frac{dx}{x+1}-\int_1^{\lfloor\Phi(n)\rfloor}\frac{dx}{x+\Phi(n)+1}\right)\\ &\ge \frac{1}{\log 2}\cdot(1+o(1))\cdot\frac{\log\Phi(n)}{\Phi(n)},\end{aligned}$$

which completes the proof. ∎

Comparing Theorem 1.7.4 with Lemma 1.6.1 one can see that in order to finish proving Theorem 1.5.1, one would need to replace $\Phi(n)$ with $\Phi(q_n)$ in Eq. (3.11). This can be achieved using known facts about the growth of $q_n(x)$ for almost all $x$.

**Corollary 3.1.9 ([36], Corollary 3.7)** *Let $\Phi : \mathbb{N} \to [1, \infty]$ be a non-decreasing function with $\lim_{n\to\infty} \Phi(n) = \infty$. If*

$$\sum_n \frac{\log \Phi(n)}{n\Phi(n)} < \infty \quad (\textit{resp.} = \infty),$$

*then almost every (resp. almost no) $x \in [0,1] \setminus \mathbb{Q}$ has $a_{n+1}(x)\, a_n(x) \leq \Phi(q_n(x))$ for sufficiently large $n$.* ■

We follow the proof of this corollary from [26, Corollary 1.6] using $m = 2$.

**Proof:** Define

$$G(\Phi) = \{x \in [0,1) : a_n(x)\, a_{n+1}(x) \geq \Phi(q_n(x)) \quad \text{for i.m. } n \in \mathbb{N}\}.$$

Choose a number $b$ such that $1 < b \leq 2^{\frac{1}{4}}$. Recall a result due to Khinchin[1] about the growth rate of $q_n$: There exists a positive number $C > 1$ such that for almost all $x$,

$$q_n(x) \leq C^n \quad \text{for all large enough } n.$$

Note that for any $x$, $q_n(x) \geq 2^{(n-1)/2} \geq b^n$ for all $n \geq 2$. (We used property ($P_3$) of Lemma 2.3.3 in the first inequality).

Assume that

$$\sum_{n=1}^{\infty} \frac{\log \Phi(n)}{n\Phi(n)} = \infty,$$

by using Cauchy's condensation principle, this is equivalent to saying that

$$\sum_{n=1}^{\infty} \frac{\log \Phi(C^n)}{\Phi(n)} = \infty.$$

Then applying Theorem 1.7.4, we get that the set

$$\{x \in [0,1) : a_n(x)\, a_{n+1}(x) \geq \Phi(C^n) \quad \text{for i.m. } n \in \mathbb{N}\}$$

is full, and hence so is the set

$$\begin{aligned} &\{x \in [0,1) : a_n(x)\, a_{n+1}(x) \geq \Phi(C^n) \quad \text{for infinitely many } n \in \mathbb{N}\} \\ &\cap \{x \in [0,1) : q_n(x) \leq C^n \quad \text{for sufficiently large } n \in \mathbb{N}\}. \end{aligned}$$

Since $\Phi$ is non-decreasing, the intersection is clearly a subset of $G(\Phi)$.

[1] One may also use Lévy's result here which states that $\frac{\log(q_n)}{n} \to \frac{\pi^2}{12 \log 2}$ almost surely.

Assume that

$$\sum_{n=1}^{\infty} \frac{\log \Phi(n)}{n\Phi(n)} < \infty,$$

which is equivalent to (by Cauchy condensation) saying that

$$\sum_{n=1}^{\infty} \frac{\log \Phi(b^n)}{\Phi(b^n)} < \infty.$$

Then applying Theorem 1.7.4, we get that the set

$$\{x \in [0,1) : a_n(x)\, a_{n+1}(x) \geq \Phi(b^n) \quad \text{for i.m. } n \in \mathbb{N}\}$$

is null. Since $\Phi(b^n) \leq \Phi(q_n(x))$ for all $x$ and $n \geq 2$, clearly

$$G(\Phi) \subset \{x \in [0,1) : a_n(x)\, a_{n+1}(x) \geq \Phi(b^n) \quad \text{for i.m. } n \in \mathbb{N}\}.$$

So, $G(\Phi)$ is null. ∎

Finally, in [36, Proof of Theorem 1.8], Kleinbock and Wadleigh complete the work of characterising $\psi$ such that $D(\psi)$ has zero/full measure.

The interested reader is referred to both [36] and the recent work of [26]. Together they include the latest developments in the Lebesgue measure theory for uniform approximation.

CHAPTER 4

# Hausdorff Measure Theory for Uniform Approximation

In this chapter, for any dimension function $f$, we present a complete $f$-dimensional Hausdorff measure characterisation of the set $D(\psi)^c$ of Dirichlet non-improvable numbers. Recall that,

**Definition 1.5.3 (ESL dimension function)** A dimension function $f$ is *essentially sub-linear* (ESL), if

$$\text{there exists } B > 1 \text{ such that } \limsup_{x \to 0} \frac{f(Bx)}{f(x)} < B. \tag{1.12}$$

We prove the following two theorems depending upon whether the dimension function is ESL or NESL (see Definition 1.5.8). The first theorem was proved by Hussain, Kleinbock, Wadleigh and Wang [29] and is reproduced in Section 4.3 for completeness.

**Theorem 1.5.4 ([29], Theorem 1.6)** *Let $\psi$ be a non-increasing, positive function with $t\psi(t) < 1$ for all sufficiently large $t$. Let $f$ be an ESL dimension function. Then*

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0, & \text{if } \quad \sum\limits_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty, & \text{if } \quad \sum\limits_t tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

The second theorem is new and it is a joint work with Hussain and Simmons [10] and is proved in Section 4.4.

**Theorem 1.5.9** *Let $\psi$ be a non-increasing, positive function with $t\psi(t) < 1$ for all sufficiently large $t$. Let $f$ be an NESL dimension function. Then*

$$\mathcal{H}^f\left(D(\psi)^c\right) = \begin{cases} 0, & \text{if } \quad \sum\limits_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty, & \text{if } \quad \sum\limits_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

## 4.1 Comparing ESL with NESL

Before commencing the proofs, we compare ESL dimension functions with NESL dimension functions. To classify the dimension function examples as ESL or NESL, we shall calculate the $\limsup_{x\to 0} \frac{f(Bx)}{f(x)}$ in each case.

**Example 4.1.1** The ESL condition is clearly satisfied for the dimension functions $f(x) = x^s$ when $0 \leq s < 1$. Less obvious, are the following ESL examples:

1. $f(x) = \frac{x^{2/3}}{\log\left(\frac{1}{x}\right)}$ is ESL as
$$\limsup_{x\to 0} \frac{(Bx)^{2/3}\left(\log\left(\frac{1}{Bx}\right)\right)^{-1}}{x^{2/3}\left(\log\left(\frac{1}{x}\right)\right)^{-1}} = B^{2/3}.$$

2. $f(x) = x^{2/3}\log\left(\log\left(\frac{1}{x}\right)\right)$ is ESL as
$$\limsup_{x\to 0} \frac{(Bx)^{2/3}\log\left(\log\left(\frac{1}{Bx}\right)\right)}{x^{2/3}\log\left(\log\left(\frac{1}{x}\right)\right)} = B^{2/3}.$$

3. $f(x) = x^{2/t}\log^{\epsilon}\left(x^{-1/t}\right)$ is ESL, when $t \geq 2$ and $\epsilon > 0$, as
$$\limsup_{x\to 0} \frac{(Bx)^{2/t}\log^{\epsilon}\left((Bx)^{-1/t}\right)}{x^{2/t}\log^{\epsilon}\left(x^{-1/t}\right)} = B^{2/t}$$

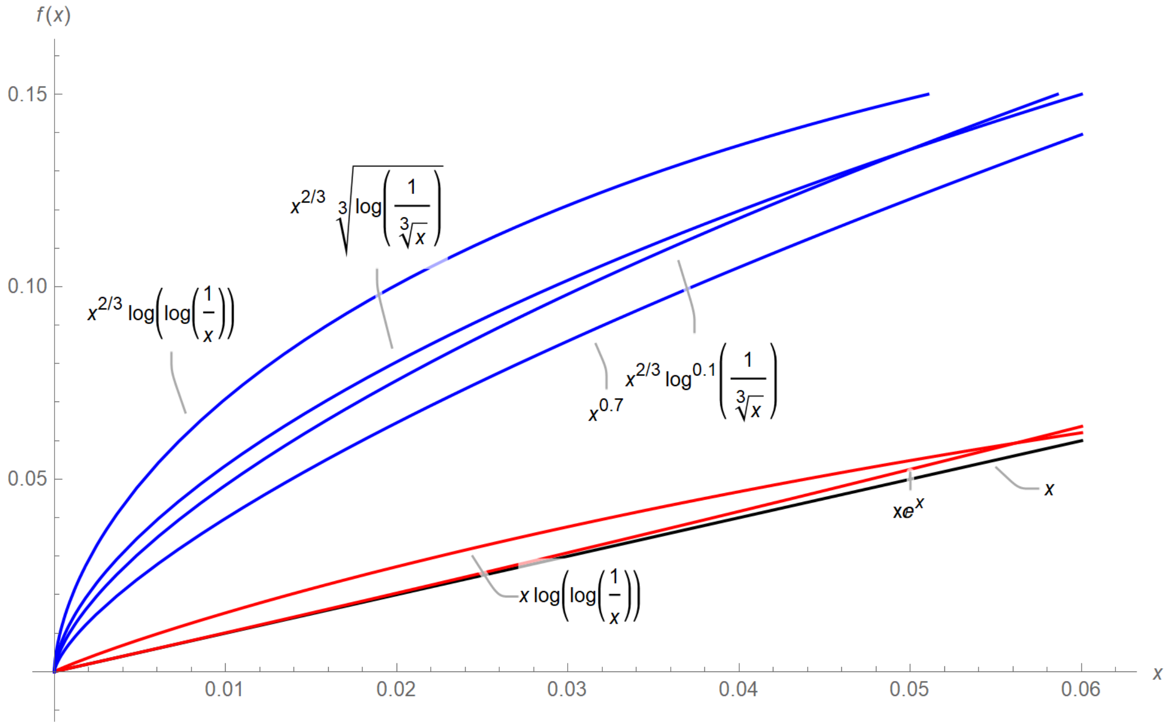
Overleaf we use $t = 3$ and $\epsilon = 1/3$, giving $f(x) = x^{2/3}\log\left(\frac{1}{x^{1/3}}\right)^{1/3}$ ∎

**Example 4.1.2** NESL examples are the dimension function $f(x) = x$, and

1. $f(x) = xe^x$ is NESL as
$$\limsup_{x\to 0} \frac{(Bx)e^{Bx}}{xe^x} = B.$$

2. $f(x) = x^t\log(1/x)$, for $t \geq 1$, is NESL as
$$\limsup_{x\to 0} \frac{(Bx)^t\log\left(\frac{1}{Bx}\right)}{x^t\log\left(\frac{1}{x}\right)} = B^t.$$

3. $f(x) = x\log\left(\log\left(\frac{1}{x}\right)\right)$ is NESL as
$$\limsup_{x\to 0} \frac{Bx\log\left(\log\left(\frac{1}{Bx}\right)\right)}{x\log\left(\log\left(\frac{1}{x}\right)\right)} = B.$$ ∎




*ESL versus NESL: close to $x = 0$*

The picture shows that, close to $x = 0$, only the NESL dimension functions are nearing the identity function $f(x) = x$.

## 4.2 Some Set Inclusions

As indicated in chapter 1, we rewrite the set theoretic descriptions of $\Psi$-approximable and $\Psi$-Dirichlet non-improvable numbers in terms of continued fraction expressions. This allows us to recognise some inclusions and then examine the set theoretic differences. Recall from (1.16) that we have the inclusions

$$\mathcal{K}(3\Psi) \subset G_1(\Psi) \subset G(\Psi) \subset D(\psi)^c \subset G(\Psi/4), \tag{4.1}$$

where

$$G(\Psi) := \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \ > \ \Psi\big(q_n(x)\big) \ \text{ for infinitely many } n \in \mathbb{N}\Big\},$$
$$G_1(\Psi) := \Big\{x \in [0,1) : a_{n+1}(x) \ > \ \Psi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N}\Big\},$$
$$\mathcal{K}(\Psi) := \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\},$$

and $\Psi : [1,\infty) \to \mathbb{R}_+$ is a non-decreasing function.

Except for the first inclusion, all other inclusions follow from Lemma 1.6.1. We prove that $\mathcal{K}(3\Psi) \subset G_1(\Psi)$ for completeness.

**Lemma 4.2.1** $\mathcal{K}(3\Psi) \subset G_1(\Psi)$.

**Proof:** If there are infinitely many $(p, q)$ with

$$|x - p/q| < \frac{1}{3\Psi(q)q^2} < \frac{1}{2q^2},$$

then, by Legendre's Theorem 2.3.4,

$$\frac{p}{q} = \frac{p_n(x)}{q_n(x)} \quad \text{for some } n \geq 1.$$

(We can assume that $\Psi(q) > 2/3$, which is required in the second inequality in this proof, see also remark 4.3.2).

Since $p_n, q_n$ are coprime, we must have $q_n \leq q$. So, by the monotonicity of $\Psi$,

$$\left|x - \frac{p_n}{q_n}\right| = \left|x - \frac{p}{q}\right| < \frac{1}{3\Psi(q)q^2} \leq \frac{1}{3\Psi(q_n)q_n^2}.$$

On the other hand, in view of $(P_5)$,

$$\left|x - \frac{p_n}{q_n}\right| \geq \frac{1}{3a_{n+1}q_n^2}.$$

This implies $a_{n+1} > \Psi(q_n)$, for infinitely many $n$, proving the lemma. ∎

In view of the above inclusions, Theorem 1.5.4 can be restated for $G(\Psi)$.

**Theorem 4.2.2 ([29], Theorem 1.8)** *Let $\Psi : [t_0, \infty] \to \mathbb{R}_+$ be a non-decreasing function and let $f$ be an ESL dimension function. Then*

$$\mathcal{H}^f\big(G(\Psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

In view of the inclusions (4.1), Theorem 4.2.2 readily implies Theorem 1.5.4.

## 4.3 Hausdorff Measure: ESL Dimension Functions

In this section we prove Theorem 4.2.2 in two parts: the convergence and divergence cases.

### 4.3.1 Divergence Case

Jarník Theorem 1.3.21 is an elegant zero-infinity law for the Hausdorff measure of the set $W(\psi)$. For the divergence case of Theorem 4.2.2, we need the most modern version, in terms of $\mathcal{K}(\Psi)$.

#### 4.3.1.1 Jarník's theorem for $\mathcal{K}(\Psi)$

We look to the seminal paper of Beresnevich and Velani [5], with a slight improvement as noticed in [29].

**Theorem 4.3.1 ([29], Theorem 2.6 )** *Let $\Psi$ be a non-increasing function, and let $f$ be a dimension function satisfying the following properties:*

$$\lim_{x\to 0}\frac{f(x)}{x}=\infty, \tag{4.2}$$

*and*

$$\exists C\geq 1 \text{ such that } \frac{f(x_2)}{x_2}\leq C\frac{f(x_1)}{x_1} \text{ whenever } x_1<x_2\ll 1. \tag{4.3}$$

*Then*

$$\mathcal{H}^f\big(\mathcal{K}(\Psi)\big)=\begin{cases}0 & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right)<\infty;\\ \infty & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right)=\infty.\end{cases} \tag{4.4}$$

**Proof (Sketch):** The difference between the result of Beresnevich and Velani [5, Theorem 2], and the result of [29, Theorem 2.6], is that the original formulation of Jarník ([3, §1.1]) assumes condition (4.3) with $C=1$, that is, the monotonicity of the function $x\mapsto\frac{f(x)}{x}$. However, this condition can be replaced by the weaker "quasi-monotonicity" condition of (4.3). The modern proof of Jarník's theorem, due to Beresnevich and Velani [5], is given by a combination of Khinchin's classical theorem [34] and the mass transference principle [5, Theorem 2]. However, in proving the latter theorem the monotonicity assumption on the function $x\mapsto\frac{f(x)}{x}$ is used only in the last step of the proof, that is, in the last inequality in formula (29) in [5]. The proof still works if (4.3) is used instead. ■

Condition (4.3) is only required for the divergence case and the convergence case is free from any assumptions on the dimension and the approximating functions.

**Remark 4.3.2** In proving the divergence case of Theorem 4.2.2, we can assume that $\Psi(t)\geq 1$ for all $t\gg 1$. Otherwise, $\Psi(t)<1$ for all large $t$ since we have assumed $\Psi$ to be non-decreasing. Then it is obvious that $G_1(\Psi)$, and thus $G(\Psi)$, contains all irrational numbers in $[0,1]$, and that the sum in Theorem 4.2.2 diverges.

Applying Theorem 4.2.2 and Theorem 4.3.1 to the inclusions (4.1), we have

$$\mathcal{H}^f\big(G(\Psi)\big)\geq\mathcal{H}^f\big(G_1(\Psi)\big)\geq\mathcal{H}^f\big(\mathcal{K}(3\Psi)\big)=\infty$$

whenever one can show that the dimension function $f$ satisfies Eq. (4.2) and condition (4.3), and that for the sum dichotomy Eq. (4.4), we have the case that

$$\sum_t tf\left(\frac{1}{3t^2\Psi(t)}\right)=\infty. \tag{4.5}$$

This is done via the following lemma.

**Lemma 4.3.3 ([29], Lemma 3.1 )** Let $f$ be an essentially sub-linear dimension function. Then both Eq. (4.2) and condition (4.3) hold.

**Proof:** Condition (1.12) implies that there exist $\epsilon, \delta > 0$ and $B > 1$ such that

$$0 < x < \delta \quad \Longrightarrow \quad \frac{f(Bx)}{f(x)} < B - \epsilon. \tag{4.6}$$

Therefore for some $0 < x_0 < \delta$ and all $n \geq 1$ one has

$$f(x_0/B^n) > f(x_0)/(B-\epsilon)^n \quad \Longleftrightarrow \quad \frac{f(x_0/B^n)}{x_0/B^n} > \left(\frac{B}{B-\epsilon}\right)^n \frac{f(x_0)}{x_0}.$$

This shows that $f(x)/x \to \infty$ as $x \to 0$. So we have shown that Eq. (4.2) holds.

As for condition (4.3), let $x_1 < x_2 < \delta$. Assume

$$B^{-k} \leq x_2 < B^{-k+1}, \; B^{-\ell} \leq x_1 < B^{-\ell+1}, \text{ with } k \leq \ell.$$

Then

$$\frac{f(x_2)}{f(x_1)} \cdot \frac{x_1}{x_2} \leq \frac{f(B^{-k+1})}{f(B^{-\ell})} \cdot \frac{B^{-\ell+1}}{B^{-k}} \leq (B-\epsilon)^{\ell-k+1} \cdot B^{-\ell+k+1} \leq B^2.$$

Therefore,

$$\frac{f(x_2)}{x_2} \leq B^2 \cdot \frac{f(x_1)}{x_1},$$

and (4.3) follows. ∎

Finally, notice that (4.5) is equivalent to (4.4) as $f$ is increasing and, by (4.6), has the doubling property. This settles the divergence case of the ESL Theorem.

### 4.3.2 Convergence Case

The set $G(\Psi)$ can be written in terms of the following basic cylinders:

$$G(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} J_n(a_1,\ldots,a_n),$$

where

$$J_n(a_1,\ldots,a_n) := \bigcup_{a_{n+1} > \frac{\Psi(q_n)}{a_n}} I_{n+1}(a_1,\ldots,a_n,a_{n+1}).$$

Using ($P_2$) in Lemma 2.3.3 and the recursive relation (1.7), the diameter of $J_n(a_1, \ldots, a_n)$ can be bounded as follows:

$$\begin{aligned}|J_n(a_1, \ldots, a_n)| &= \sum_{a_{n+1} > \frac{\Psi(q_n)}{a_n}} \left| \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} - \frac{(a_{n+1}+1)p_n + p_{n-1}}{(a_{n+1}+1)q_n + q_{n-1}} \right| \\ &\leq \left| \frac{\frac{\Psi(q_n)}{a_n}p_n + p_{n-1}}{\frac{\Psi(q_n)}{a_n}q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{\left(\frac{\Psi(q_n)}{a_n}q_n + q_{n-1}\right) q_n} \\ &\leq \frac{1}{\Psi(q_n)q_{n-1}q_n}.\end{aligned}$$

Therefore,

$$\mathcal{H}^f\big(G(\Psi)\big) \leq \liminf_{N\to\infty} \sum_{n\geq N} \sum_{a_1,\ldots a_n} f\left(\frac{1}{\Psi(q_n)q_{n-1}q_n}\right).$$

So, the remaining task is to estimate the summation over

$$\mathcal{A}_N := \{(a_1, \ldots, a_n) : n \geq N\}.$$

We first partition $\mathcal{A}_N$ by introducing a function $g_N : \mathcal{A}_N \to \mathbb{N}^2$, defined as

$$g_N\big((a_1, \ldots, a_n)\big) = \big(q_{n-1}(a_1, \ldots, a_n), q_n(a_1, \ldots, a_n)\big),$$

where $q_n$ is defined by the recurrence relation (1.7). The following observations can readily be verified.

1. The function $g_N$ is two-to-one. This is because a *rational* number has *two* continued fraction representations. More precisely, continued fraction expansions of rational numbers are not allowed to terminate in 1; if $p/q$ has continued fraction expansion $[b_1, \ldots, b_k]$, one must have $b_k \geq 2$. However it is also true that

   $$p/q = [b_1, \ldots, b_k - 1, 1].$$

   Now fix a positive integer vector $(p, q)$ such that $p/q$ has expansion $[b_1, \ldots, b_k]$, and assume that $g_N\big((a_1, \ldots, a_n)\big) = (p, q)$. Then by ($P_4$) in Lemma 2.3.3,

   $$\frac{p}{q} = \frac{q_{n-1}(a_1, \ldots, a_n)}{q_n(a_1, \ldots, a_n)} = [a_n, a_{n-1}, \ldots, a_1],$$

   This gives a continued fraction representation of $p/q$. So,

   $$(a_n, \ldots, a_1) = (b_1, \ldots, b_k) \text{ or } (b_1, \ldots, b_k - 1, 1).$$

   On the other hand, it is straightforward to check that

   $$g_N\big((b_k, \ldots, b_1)\big) = g_N\big((1, b_k - 1, \ldots, b_1)\big) = (p, q).$$

2. The range of $g_N$ is a subset of

$$\mathcal{C}_N := \Big\{(p,q) \in \mathbb{N}^2 : gcd(p,q) = 1,\ 1 \le p \le q,\ q \ge 2^{(N-1)/2}\Big\}.$$

3. The following is a partition of $\mathcal{A}_N$:

$$\mathcal{A}_N = \bigcup_{(p,q)\in\mathcal{C}_N} g_N^{-1}(p,q).$$

As a result, for a dimension function $f$, we have

$$\begin{aligned}\mathcal{H}^f\big(G(\Psi)\big) \le& \liminf_{N\to\infty} \sum_{(p,q)\in\mathcal{C}_N} \sum_{g_N^{-1}(p,q)} f\left(\frac{1}{q_{n-1}q_n\Psi(q_n)}\right)\\ \le& 2\liminf_{N\to\infty} \sum_{(p,q)\in\mathcal{C}_N} f\left(\frac{1}{pq\Psi(q)}\right)\\ \le& 2\liminf_{N\to\infty} \sum_{q\ge 2^{(N-1)/2}} \sum_{1\le p\le q} f\left(\frac{1}{pq\Psi(q)}\right).\end{aligned}$$

It can be seen that if

$$\sum_{q=1}^{\infty} \sum_{1\le p\le q} f\left(\frac{1}{pq\Psi(q)}\right) < \infty,$$

then it readily follows from Proposition 1.3.18 that $\mathcal{H}^f\big(G(\Psi)\big) = 0$.

To complete the proof of the convergence case, it remains to show that

$$\sum_{q=1}^{\infty} \sum_{1\le p\le q} f\Big(\frac{1}{pq\Psi(q)}\Big) \quad \text{and} \quad \sum_{q=1}^{\infty} qf\Big(\frac{1}{q^2\Psi(q)}\Big)$$

have the same convergence and divergence property. It is straightforward to establish that

$$\sum_{q=1}^{\infty} qf\Big(\frac{1}{q^2\Psi(q)}\Big) = \infty \Longrightarrow \sum_{q=1}^{\infty} \sum_{1\le p\le q} f\Big(\frac{1}{pq\Psi(q)}\Big) = \infty,$$

since by the increasing property of $f$,

$$\sum_{1\le p\le q} f\Big(\frac{1}{pq\Psi(q)}\Big) \ge \sum_{1\le p\le q} f\Big(\frac{1}{q^2\Psi(q)}\Big) = qf\Big(\frac{1}{q^2\Psi(q)}\Big).$$

So, to finish the proof of the convergence case, it remains to prove that

$$\sum_{q=1}^{\infty} qf\Big(\frac{1}{q^2\Psi(q)}\Big) < \infty \Longrightarrow \sum_{q=1}^{\infty} \sum_{1\le p\le q} f\Big(\frac{1}{pq\Psi(q)}\Big) < \infty. \tag{4.7}$$

Clearly this implication is not true when $f(x) = x$. This is also not true for $f(x) = x\log(1/x)$, see Remark 4.3.5 below. Therefore, a natural question is to classify dimension functions $f$ for which the assertion (4.7) holds. It turns out that this assertion is satisfied for essentially sub-linear dimension functions.

**Proposition 4.3.4** *Let $f$ be an essentially sub-linear dimension function. Then the assertion in (4.7) is true.*

**Proof:** Fix $b < B$ such that

$$\frac{f(Bx)}{f(x)} < b \text{ when } x < x_0. \tag{4.8}$$

Let

$$q_0 > \max\{x_0^{-1}, B\},$$

which is designed so that the inequality (4.8) can be utilised later.

Now for each $q \geq q_0$ we estimate the inner summation in the series

$$\sum_{q=1}^{\infty} \sum_{1\leq p\leq q} f\Big(\frac{1}{pq\Psi(q)}\Big).$$

Let $t$ be the integer such that $B^{t-1} \leq q < B^t$. Then

$$\begin{aligned}\sum_{1\leq p\leq q} f\Big(\frac{1}{pq\Psi(q)}\Big) &= \sum_{k=1}^{t} \sum_{B^{-k}q<p\leq B^{-k+1}q} f\Big(\frac{1}{pq\Psi(q)}\Big)\\ &\leq \sum_{k=1}^{t} B^{-k+1} q f\Big(\frac{B^k}{q^2\Psi(q)}\Big) = B\sum_{k=1}^{t} C_k,\end{aligned}$$

where $C_k := B^{-k} q f\Big(\frac{B^k}{q^2\Psi(q)}\Big)$. Notice that for any $k < t$,

$$\frac{C_{k+1}}{C_k} = \frac{B^{-k-1} q f\Big(\frac{B^{k+1}}{q^2\Psi(q)}\Big)}{B^{-k} q f\Big(\frac{B^k}{q^2\Psi(q)}\Big)} := \frac{f(Bx)}{Bf(x)} < \frac{b}{B},$$

since

$$x := \frac{B^k}{q^2\Psi(q)} \leq \frac{1}{q\Psi(q)} < x_0.$$

Thus for any $1 \leq k \leq t$,

$$C_k \leq \Big(\frac{b}{B}\Big)^{k-1} C_1.$$

As a result,

$$\sum_{k=1}^{t} C_k \leq \sum_{k=1}^{t} \Big(\frac{b}{B}\Big)^{k-1} C_1 \leq cC_1 = \frac{c}{B} q f\Big(\frac{B}{q^2\Psi(q)}\Big) \leq \frac{cb}{B} \cdot q f\Big(\frac{1}{q^2\Psi(q)}\Big).$$

In summary, we have proved

$$\sum_{1\leq p\leq q} f\Big(\frac{1}{pq\Psi(q)}\Big) \leq cb \cdot q f\Big(\frac{1}{q^2\Psi(q)}\Big).$$ ■

So, the desired assertion follows, and the proof of the ESL Theorem is thus completed.

**Remark 4.3.5** As stated earlier, the functions $f(x) = x$ and $f(x) = x\log(1/x)$ are not essentially sub-linear. For these particular examples, the argument of the ESL convergence case only leads to a weaker/incomplete characterization of $\mathcal{H}^f\big(D(\psi)^c\big)$. For clarity we refer the reader to examples 5.1 and 5.2 in [29].

## 4.4 Hausdorff Measure: NESL Dimension Functions

In this section we prove Theorem 1.5.9 which splits into two parts: the convergence case and the divergence case.

### 4.4.1 Convergence Case

We are given that the series

$$\sum_t t \log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) \tag{4.9}$$

converges. We can assume that $\Psi(t) \geq 1$ for all $t \gg 1$. Otherwise, $\Psi(t) < 1$ for all large $t$ since we have assumed $\Psi$ to be non-decreasing. Then it is obvious that the set

$$G_1(\Psi) = \{x \in [0,1) : a_{n+1}(x) \ > \ \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\}$$

and thus

$$G(\Psi) = \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\Big\},$$

contains all irrational numbers in $[0,1]$, and that the sum in (4.9) diverges. Since $\Psi$ is non-decreasing and from (1.7), it follows that $q_n \geq a_n q_{n-1}$. We notice some obvious inclusions,

$$\begin{aligned}
G(\Psi) =& \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\Big\} \\
\subseteq& \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(a_n q_{n-1}) \ \text{ for i.m. } n \in \mathbb{N}\Big\} \\
\subseteq& \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\ldots,a_n,a_{n+1}). \\
=& \mathcal{A}_1(\Psi) \cup \mathcal{A}_2(\Psi),
\end{aligned}$$

where

$$\mathcal{A}_1(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_n \leq \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\ldots,a_n,a_{n+1}).$$

$$\mathcal{A}_2(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_n > \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\ldots,a_n,a_{n+1}).$$

#### 4.4.1.1 Covering for $\mathcal{A}_1(\Psi)$

To estimate the Hausdorff measure of the set $\mathcal{A}_1(\Psi)$, we proceed as follows. Let

$$J_n(a_1,\dots,a_n) := \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\dots,a_n,a_{n+1}).$$

Using ($P_2$) of Lemma 2.3.3 and the recursive relation (1.7), the diameter of $J_n(a_1,\dots,a_n)$ can be bounded as follows:

$$\begin{aligned}
|J_n(a_1,\dots,a_n)| &= \sum_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} \left| \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} - \frac{(a_{n+1}+1)p_n + p_{n-1}}{(a_{n+1}+1)q_n + q_{n-1}} \right| \\
&\le \left| \frac{\frac{\Psi(a_n q_{n-1})}{a_n} p_n + p_{n-1}}{\frac{\Psi(q_n)}{a_n} q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{\left(\frac{\Psi(a_n q_{n-1})}{a_n} q_n + q_{n-1}\right) q_n} \\
&\le \frac{1}{\Psi(a_n q_{n-1}) a_n q_{n-1}^2}.
\end{aligned}$$

Let $Q > 1$ and $Q < q_{n-1} \le 2Q$. Then we can bound the diameter of $J_n(a_1,\dots,a_n)$ as

$$|J_n(a_1,\dots,a_n)| \ll \frac{1}{\Psi(a_n Q) a_n Q^2}.$$

Hence, the cost of the cover, when $a_{n+1} > a_n$, is

$$\sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right).$$

In the case $a_{n+1} = a_n$ the cost of the cover is given by

$$f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Since $Q > 1$, it follows from Eq. (2.3) that for each window $[Q, 2Q]$, there are at most $Q^2$ cylinders $I_n$ of length comparable (up to a constant) to $Q^{-2}$. Multiplying the cost of the cover given above by $Q^2$ which are the number of such intervals, and then summing over all the windows $Q = 2^k$, we have

$$\sum_{Q=2^k;k\ge1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) + \sum_{Q=2^k;k\ge1} Q^2 f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Cauchy's condensation test tells us whether a series converges or not. Applying Cauchy's condensation test to the second term gives the total cost as

$$\sum_{Q=2^k;k\ge1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) + \sum_{q} qf\left(\frac{1}{q^2\Psi(q)}\right). \tag{4.10}$$

The second term in Eq. (4.10) is clearly smaller than that of Eq. (4.9), so we can ignore it. For the first term in Eq. (4.10), applying Cauchy's condensation test in reverse gives

$$\sum_{k\geq 1, j\geq 1} \sum_{Q=2^k, A=2^j, A<\Psi(Q)} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right).$$

Now apply the change of variables $R = QA^{1/2}$. Notice that in the case where the dimension function $f$ is close to an identity function $f(x) = x$, $\Psi$ grows subpolynomially and for convenience we can assume it is logarithmic. This is possible, since $A$ is small in comparison to $Q$; we have

$$Q < R < QA < Q^2,$$

so

$$\log(Q) \approx \log(R) \approx \log(QA),$$

where the symbol $\approx$ means approximately equal. This means that if $\Psi$ is like a logarithm function, then the same will be true if we replace log by $\Psi$. Since $f$ is close to the identity function, it rules out the case that $\Psi(q) > q^x$ for some $x$, since the series (4.9) converges for all such $\Psi$. So for $\Psi$ close to the boundary of convergence/divergence, we have $\Psi(q) \ll q^x$ for all $x$. Thus we have

$$\sum_{k\geq 1, j\geq 1} \sum_{R=2^k, A=2^j, A<\Psi(R)} R^2 f\left(\frac{1}{R^2\Psi(R)}\right).$$

Evaluating the summation with respect to $A$ and applying Cauchy's condensation test gives

$$\sum_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right).$$

So if Eq. (4.9) converges, then by the Hausdorff-Cantelli lemma

$$\mathcal{H}^f(G(\Psi)) = 0.$$

#### 4.4.1.2 Covering for $\mathcal{A}_2(\Psi)$

Next we notice that the set $\mathcal{A}_2(\Psi)$ is a subset of the Jarník set, that is, it is contained in the set $G_1(\Psi)$. The continued fraction expansion for any $x \in G_1(\Psi)$ should have unbounded partial quotients. Therefore if $a_n \geq \Psi(q_{n-1})$ then it follows that $a_{n+k} \geq \Psi(q_{n-1})$, for some $k \geq 1$, for infinitely many $n$. This implies that for any dimension function $f$

$$\mathcal{H}^f(\mathcal{A}_2(\Psi)) \leq \mathcal{H}^f(G_1(\Psi)) \leq \mathcal{H}^f(\mathcal{K}(\Psi)) = 0 \iff \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty.$$

However notice that the series above is smaller than the one we are claiming in our theorem, that is,

$$\sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) \leq \sum_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right).$$

Hence combining both the coverings for $\mathcal{A}_1(\Psi)$ and $\mathcal{A}_2(\Psi)$, we conclude that

$$\mathcal{H}^f(G(\Psi)) = 0 \iff \sum_t t\log\left(\Psi(t)\right) f\left(\frac{1}{t^2\Psi(t)}\right) < \infty.$$

**Remark 4.4.1** Let $f(x) = x\log(1/x)$, then we have

$$\begin{aligned}
\sum_{Q=2^k,\ k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{Q^2 a\Psi(Qa)}\right) &= \sum_{Q=2^k,\ k\geq 1} \sum_{a=1}^{\Psi(Q)} \frac{\log\left(Q^2 a\Psi(Qa)\right)}{a\Psi(Qa)} \\
&\leq \sum_{Q=2^k,\ k\geq 1} \sum_{a=1}^{\Psi(Q)} \frac{\log\left(Q^2 a\Psi(Q)\right)}{a\Psi(Q)} \qquad (\dagger) \\
&= \sum_{Q=2^k,\ k\geq 1} \sum_{a=1}^{\Psi(Q)} \left(\frac{\log\left(Q^2\Psi(Q)\right)}{a\Psi(Q)} + \frac{\log a}{a\Psi(Q)}\right) \\
&\leq \sum_{Q=2^k,\ k\geq 1} \frac{\log\left(Q^2\Psi(Q)\right)\cdot\log\Psi(Q)}{\Psi(Q)} + \frac{\log^2\Psi(Q)}{\Psi(Q)}
\end{aligned}$$

where the inequality $(\dagger)$ follows from the fact the $x^{-1}\log(x)$ is monotonic.

Now split the last inequality in two parts depending upon the cases $\Psi(Q) \geq Q$ or otherwise.

(i) When $\Psi(Q) \geq Q$. In this case, $Q^2 \leq \Psi^2(Q)$. Hence we have

$$\begin{aligned}
\sum_{Q=2^k,\ k\geq 1} \frac{\log\left(Q^2\Psi(Q)\right)\cdot\log\Psi(Q)}{\Psi(Q)} + \frac{\log^2\Psi(Q)}{\Psi(Q)} &\leq 4 \sum_{Q=2^k,\ k\geq 1, \Psi(Q)\geq Q} \frac{\log^2\Psi(Q)}{\Psi(Q)} \\
&\leq 4 \sum_{Q=2^k,\ k\geq 1} \frac{\log^2(Q)}{(Q)} = C < \infty
\end{aligned}$$

(ii) When $\Psi(Q) < Q$.

$$\sum_{Q=2^k,\ k\geq 1} \frac{\log\left(Q^2\Psi(Q)\right)\cdot\log\Psi(Q)}{\Psi(Q)} + \frac{\log^2\Psi(Q)}{\Psi(Q)} \leq 4 \sum_{Q=2^k,\ k\geq 1, \Psi(Q)<Q} \frac{\log Q\cdot\log\Psi(Q)}{\Psi(Q)}$$

Hence the dominating series for convergence is $\sum_{q=1}^{\infty}\frac{\log q\log\Psi(q)}{q\Psi(q)}$. Note that from [29] the governing series is given by $\sum_{q=1}^{\infty}\frac{\log q\log\log q}{q\Psi(q)}$, and

$$\sum_{q=1}^{\infty} \frac{\log q \cdot \log \log q}{q\Psi(q)} < \infty \Longrightarrow \sum_{q=1}^{\infty} \frac{\log q \cdot \log \Psi(q)}{q\Psi(q)} < \infty$$

and

$$\sum_{q=1}^{\infty} \frac{\log q \cdot \log \Psi(q)}{q\Psi(q)} = \infty \Longrightarrow \sum_{q=1}^{\infty} \frac{\log q \cdot \log \log q}{q\Psi(q)} = \infty.$$

This is because for the first series in this sum, the summation over the terms $\Psi(q) \geq \log^3(q)$ converges. So, the dominating terms are $\Psi(q) \leq \log^3(q)$. While, the terms in the second series is larger than that in the first series along these terms. ∎

### 4.4.2 Divergence Case

For the divergence case, we appeal to a recent criterion introduced by Hussain and Simmons [30]. With this criterion, see Section 2.5.1, now we are in a position to prove the divergence case. We will prove, in particular, the following generalised form, from which the divergence case of Theorem 1.5.9 readily follows.

A collection of maps $u = (u_a)_{a \in E}$ is a Gauss iterated function system (GIFS) on $\mathbb{R}$, if:

- $E$ is a countable (finite or infinite) index set, which is referred to as an alphabet;
- $X \subseteq \mathbb{R}$ is a nonempty compact set which is equal to the closure of its interior;
- for all $a \in E, u_a(X) \subset X$.

**Theorem 4.4.2** *Let $(u_a)_{a\in E}$ be the Gauss iterated function system. For each finite word $\omega \in E^*$ and $a \leq \Psi(Q_\omega)$ let*

$$S_{\omega a} = u_{\omega a}([0, a/\Psi(Q_\omega a)]).$$

*Let $f$ be a dimension function such that $\sum_{\omega,a} f(\operatorname{diam} S_{\omega a})$ diverges. Then*

$$\mathcal{H}^f \left( \limsup_{\omega,a} S_{\omega a} \right) = \infty.$$

∎

To apply Theorem 4.4.2 to derive the Hausdorff measure of the set $G(\Psi)$, notice that the set $S_{\omega a}$ corresponds to the set

$$\bigcup_{a_n \leq \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

Hence

$$\limsup_{\omega,a} S_{\omega a} \subseteq G(\Psi).$$

As in the convergence case for the set $\mathcal{A}_1(\Psi)$, let $Q > 1$ and $Q < q_{n-1} \leq 2Q$. Then we have

$$\begin{aligned}\infty &= \sum_{\omega,a} f(\operatorname{diam} S_{\omega a})\\ &\asymp \sum_{Q=2^k;k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right)\\ &\asymp \sum_t t\log(\Psi(t))\, f\left(\frac{1}{t^2\Psi(t)}\right).\end{aligned}$$

**Proof (Theorem 4.4.2):** Fix $B_0 \subset [0,1]$ and $N \in \mathbb{N}$, and we will construct the measure $\mu = \mu(B_0, N)$ such that the hypothesis (*) in Theorem 2.5.5 holds.

For each $k, \ell \in \mathbb{N}$ let

$$A_{k,\ell} = \{(\omega, a) : 2^k \leq Q_\omega < 2^{k+1},\ \ 2^\ell \leq a < 2^{\ell+1}\}.$$

Then for all $(\omega, a) \in A_{k,\ell}$, we have $\operatorname{diam}(S_{\omega a}) \asymp \rho_{k,\ell}$. Thus

$$\sum_{k,\ell} \#(A_{k,\ell}) f(\rho_{k,\ell}) = \infty.$$

Let

$$A'_{k,\ell} = \{(\omega, a) \in A_{k,\ell} : S_{\omega a} \subset B_0\}.$$

**Claim 4.4.3**

$$\#(A'_{k,\ell}) \gtrsim \#(A_{k,\ell})|B_0|$$

for all $k, \ell \geq N_0$, for some sufficiently large $N_0$. ■

**Proof:** Consider the set

$$\left\{\tau b : Q_\tau < 2^k \leq Q_{\tau b}\right\}.$$

Fix $\tau$ such that $Q_\tau < 2^k$. Since $Q_{\tau b} = bQ_\tau + Q_{\tau'}$, where $\tau'$ is $\tau$ minus its last symbol, it follows that there are at most $2^k/Q_\tau$ values of $b$ such that $2^k \leq Q_{\tau b} < 2^{k+1}$. On the other hand, the set

$$\bigcup_{b:2^k\leq Q_{\tau b}} u_{\tau b}([0,1]) \tag{4.11}$$

has measure comparable to $(2^k/Q_\tau)^{-1}Q_\tau^{-2} = 2^{-k}/Q_\tau$.

Since the sets (4.11) form a disjoint cover of $[0,1]$, it follows that

$$
\begin{aligned}
\#(A'_{k,\ell}) &\geq 2^\ell \#\{\omega : 2^k \leq Q_\omega < 2^{k+1},\ S_\omega \cap B_0 \neq \emptyset\} \\
&\geq 2^\ell \sum_{\substack{\tau \\ Q_\tau < 2^k \\ S_\tau \cap B_0 \neq \emptyset}} \left| \bigcup_{b:2^k \leq Q_{\tau b}} u_{\tau b}([0,1]) \right| \\
&\asymp 2^\ell |B_0| \# \left\{\tau : Q_\tau < 2^k,\ S_\tau \cap B_0 \neq \emptyset\right\} \\
&\asymp \#(A_{k,\ell})|B_0|.
\end{aligned}
$$

■

It follows that

$$\sum_{k,\ell \geq N_0} \#(A'_{k,\ell}) f(\rho_{k,\ell}) = \infty.$$

Fix $N_1$ such that

$$\Omega = \sum_{N_0 \leq k,\ell \leq N_1} \#(A'_{k,\ell}) f(\rho_{k,\ell}) \geq C$$

and define the measure $\mu$ as follows:

$$\mu = \frac{1}{\Omega} \sum_{N_0 \leq k,\ell \leq N_1} \sum_{(\omega,a) \in A'_{k,\ell}} f(\rho_{k,\ell}) \lambda_{S_{\omega a}},$$

where $\lambda_A$ is normalized Lebesgue measure on a set $A$.

Let $B = u_\tau([1/b_1, 1/b_2])$ for some $\tau, b_1, b_2$ (possibly $b_2 = 1$). Next we estimate $\mu(B)$ and show that it satisfies (2.12). Let

$$A''_{k,\ell} = \{(\omega, a) \in A_{k,\ell} : S_{\omega a} \subset B\}.$$

Then clearly

$$\#(A''_{k,\ell}) \ll \#(A_{k,\ell})|B| \asymp \#(A'_{k,\ell}) \frac{|B|}{|B_0|}$$

and thus

$$
\begin{aligned}
\frac{1}{\Omega} \sum_{N_0 \leq k,\ell \leq N_1} \sum_{(\omega,a) \in A''_{k,\ell}} f(\rho_{k,\ell}) \lambda_{S_{\omega a}}(B) &\ll \frac{1}{\Omega} \sum_{N_0 \leq k,\ell \leq N_1} \#(A'_{k,\ell}) \frac{|B|}{|B_0|} f(\rho_{k,\ell}) \\
&= \frac{|B|}{|B_0|}.
\end{aligned}
$$

Now for all $(\omega, a)$ such that $S_{\omega a} \cap B \neq \emptyset$, we have either $S_{\omega a} \subset B$ or $B \subset S_{\omega a}$. If the latter case never holds, then we are done. Otherwise, we have

$$\mu(B) = \frac{1}{\Omega} f(\rho_{k,\ell}) \lambda_{S_{\omega a}}(B),$$

where $(\omega, a) \in A'_{k,\ell}$ is chosen so that $B \subset S_{\omega a}$. Since $\Omega \geq C$ and $\rho_{k,\ell} \asymp |S_{\omega a}|$, we have

$$\mu(B) \ll \frac{f(\operatorname{diam} B)}{C}$$

in this case.

Now let $B_1$ be an arbitrary ball, and let $\omega$ be the longest word such that $B \subset S_\omega$. If there exist distinct $n_1, n_2 \in \mathbb{N}$ such that $u_\omega(1/n_i) \in B_1$, then let $b_1 = \lfloor 1/\max(B_1) \rfloor$ and $b_2 = \lceil 1/\min(B_1) \rceil$ and let $B$ be as above. Then $\operatorname{diam}(B_1) \asymp \operatorname{diam}(B)$ and $B_1 \subset B$, so it follows from the previous paragraph that

$$\mu(B_1) \ll \frac{f(\operatorname{diam} B_1)}{C}.$$

On the other hand, if there do not exist such distinct $n_1, n_2$, then by the maximality of $\omega$ there exists one such $n \in \mathbb{N}$ such that $u_\omega(1/n) \in B_1$. Let $n_1 = n$ and $n_2 = n + 1$, and let $b_i$ be maximal such that

$$B_1 \subset u_{n_1}([0, 1/b_1]) \cup u_{n_2}([0, 1/b_2]).$$

The argument of the previous paragraph shows that for all $i = 1, 2$,

$$\mu(u_{\omega n_i}([0, 1/b_i])) \ll \frac{f(\operatorname{diam} u_{\omega n_i}([0, 1/b_i]))}{C}$$

and on the other hand, the maximality of $b_i$ gives

$$\operatorname{diam}(u_{\omega n_i}([0, 1/b_i])) \asymp \operatorname{diam}(B_1).$$

It follows that

$$\mu(B_1) \ll \frac{f(\operatorname{diam} B_1)}{C}.$$

Hence the proof of Theorem 4.4.2 is complete. ■

# Chapter 5

# Dirichlet Non Improvability versus Well Approximability

In this chapter we prove Theorem 1.6.2, that is, we derive the Hausdorff dimension of the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$, for $C > 0$. Recall, that for a non-decreasing function $\Psi : [1, \infty) \to \mathbb{R}_+$, we have

$$G(\Psi) := \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \ > \ \Psi\big(q_n(x)\big) \ \text{ for infinitely many } n \in \mathbb{N}\Big\},$$
$$\mathcal{K}(\Psi) := \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{q^2\Psi(q)} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\},$$

and

$$\mathcal{K}(3\Psi) \subset G(\Psi).$$

We show that the set difference $G(\Psi) \setminus \mathcal{K}(C\Psi)$ is uncountable, indicating a large disparity between the size of the set of $\Psi$-Dirichlet non-improvable numbers $G(\Psi)$ and the classical set of $1/q^2\Psi(q)$-approximable numbers $\mathcal{K}(\Psi)$.

**Theorem 1.6.2** *Let $\Psi : [1, \infty) \to \mathbb{R}_+$ be a non-decreasing function and $C > 0$. Then*

$$\dim_{\mathcal{H}} \Big(G(\Psi) \setminus \mathcal{K}(C\Psi)\Big) = \frac{2}{\tau + 2}, \quad \text{where} \quad \tau = \liminf_{q \to \infty} \frac{\log \Psi(q)}{\log q}. \qquad \blacksquare$$

## 5.1 Upper Bound

For ease of calculations, we choose $C = 1$ throughout the remainder of the chapter.

The result for the $s$-dimensional Hausdorff measure for $G(\Psi)$, which was proved in Theorem 4.2.2, is all that we need in proving the upper bound for the Hausdorff dimension of the set $G(\Psi) \setminus \mathcal{K}(\Psi)$.

As a consequence of Theorem 4.2.2, the Hausdorff dimension of the set $G(\Psi)$ is given by

$$\dim_{\mathcal{H}} G(\Psi) = \frac{2}{2+\tau}, \text{ where } \tau = \liminf_{t \to \infty} \frac{\log \Psi(t)}{\log t}.$$

Since

$$G(\Psi) \setminus \mathcal{K}(\Psi) \subseteq G(\Psi),$$

therefore,

$$\dim_{\mathcal{H}}\Big(G(\Psi)\setminus\mathcal{K}(\Psi)\Big) \leq \frac{2}{\tau+2}. \tag{5.1}$$

Thus the proof of Theorem 1.6.2 follows from establishing the complementary lower bound.

## 5.2 Lower Bound

For convenience, we denote $E := G(\Psi)\setminus\mathcal{K}(\Psi)$ which can be written as

$$E = \left\{ x \in [0,1) : \begin{array}{l} a_{n+1}(x)a_n(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Psi(q_n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

To illustrate the main ideas and to simplify the presentation, we first prove the result for a specific choice of the approximating function $\Psi(q_n) := q_n^\tau$ for any $\tau > 0$. Proving the result for the general approximating function $\Psi(q_n)$ instead of $q_n^\tau$ will require slight modification to the arguments presented below but essentially the process is the same. We will briefly sketch this process in the last section.

The set $E$ can now be written as

$$E = \left\{ x \in [0,1) : \begin{array}{l} a_{n+1}(x)a_n(x) \geq q_n^\tau \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < q_n^\tau \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

We aim to show that

$$\dim_{\mathcal{H}} E \geq \frac{2}{\tau+2}.$$

Fix a large integer $L$, and define $S = S(L, M)$ to be the solution to the equation

$$\sum_{\substack{1\leq a_i\leq M \\ 1\leq i\leq L}} \left(\frac{1}{q_L^{2+\tau}(a_1,\cdots,a_L)}\right)^S = 1. \tag{5.2}$$

**Remark 5.2.1** It follows from the definition of the pressure function (Section 2.4), that as $L, M \to \infty$, then $S \to \frac{2}{2+\tau}$. The process of proving this follows as in [55, Lemma 2.6], and we refer the reader to that paper for full details. ■

For more thorough results on pressure function in infinite conformal iterated function systems we refer to [43].

So, it remains to show that

$$\dim_{\mathcal{H}} E \geq S.$$

The main strategy in obtaining the lower bound is to use the mass distribution principle Proposition 2.5.3. To employ it, we systematically divide the process into the following subsections.

### 5.2.1 Cantor Subset Construction

The main idea is to construct a Cantor subset within $E$ supporting a probability measure so that the hypothesis of the mass distribution principle is satisfied.

Consider a sequence of integers $\{n_0, n_1, n_2, \dots\}$ such that $n_0 = 0$ and $n_k \ll n_{k+1}$ to mean that the sequence $(n_k)_{k\in\mathbb{N}}$ is rapidly increasing. Define the subset $\mathcal{E}_M$ of $E$ as follows

$$\mathcal{E}_M = \left\{ x \in [0,1) : \begin{array}{l} \frac{1}{4} q_{n_k-1}^{\tau} \le a_{n_k}(x) \le \frac{1}{2} q_{n_k-1}^{\tau} \text{ and } a_{n_k-1}(x) = 4 \\ \text{and } 1 \le a_j(x) \le M, \text{ for all } j \ne n_k - 1, n_k \end{array} \right\}. \tag{5.3}$$

For any $n \ge 1$, define strings $(a_1, \dots, a_n)$ by

$$D_n = \left\{ (a_1, \dots, a_n) \in \mathbb{N}^n : \begin{array}{l} \frac{1}{4} q_{n_k-1}^{\tau} \le a_{n_k}(x) \le \frac{1}{2} q_{n_k-1}^{\tau} \text{ and } a_{n_k-1}(x) = 4 \\ \text{and } 1 \le a_j(x) \le M, \text{ for all } 1 \le j \ne n_k - 1, n_k \le n \end{array} \right\}.$$

**Definition 5.2.2** For any $n \ge 1$ and $(a_1, \dots, a_n) \in D_n$, we call $I_n(a_1, \dots, a_n)$ a *basic interval of order $n$* and

$$J_n := J_n(a_1, \dots, a_n) := \bigcup_{a_{n+1}} I_{n+1}(a_1, \dots, a_n, a_{n+1}) \tag{5.4}$$

a *fundamental interval of order $n$*, where the union in Eq. (5.4) is taken over all $a_{n+1}$ such that $(a_1, \dots, a_n, a_{n+1}) \in D_{n+1}$. ∎

The construction of $\mathcal{E}_M$ naturally splits into three distinct cases for the fundamental intervals $J_n$. These cases match the constraints on the partial quotients evidenced in Eq. (5.3). Accordingly, the following table (commencing from $k = 1$), summarises our Cantor subset construction, such that for $(a_1, \dots, a_n, a_{n+1}) \in D_{n+1}$:

| | | |
|---|---|---|
| Case I. | $n_k \le n \le n_{k+1} - 3,$ | $J_n = \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}(a_1, \dots, a_n, a_{n+1});$ |
| Case II. | $n = n_{k+1} - 2,$ | $J_n = I_{n+1}(a_1, \dots, a_n, 4);$ |
| Case III. | $n = n_{k+1} - 1,$ | $J_n = \bigcup_{\frac{1}{4} q_n^{\tau} \le a_{n+1}(x) \le \frac{1}{2} q_n^{\tau}} I_{n+1}(a_1, \dots, a_n, a_{n+1}).$ |

*Table 1*

It is now clear that

$$\mathcal{E}_M = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \dots, a_n) \in D_n} J_n(a_1, \dots, a_n).$$

### 5.2.2 Lengths of Fundamental Intervals

In this subsection, we calculate the lengths of the fundamental intervals for the three cases in Table 1.

**Case I.** When $n_k \le n \le n_{k+1} - 3$ for any $k \ge 1$ , since

$$J_n(a_1, \ldots, a_n) = \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

Therefore,

$$|J_n(a_1, \ldots, a_n)| = \frac{M}{(q_n + q_{n-1})((M+1)q_n + q_{n-1})}$$

and

$$\frac{1}{6q_n^2} \le |J_n(a_1, \ldots, a_n)| \le \frac{1}{q_n^2}.$$

In particular for $n = n_k$, since $\frac{1}{4}q_{n-1}^\tau \le a_n(x) \le \frac{1}{2}q_{n-1}^\tau$, we have

$$|J_n(a_1, \ldots, a_n)| \le \frac{1}{q_n^2} = \frac{1}{(a_n q_{n-1} + q_{n-2})^2} \le \frac{1}{(a_n q_{n-1})^2} = \frac{1}{\frac{1}{16}q_{n-1}^{2+2\tau}},$$

and

$$|J_n(a_1, \ldots, a_n)| \ge \frac{1}{6q_n^2} = \frac{1}{6(a_n q_{n-1} + q_{n-2})^2} \ge \frac{1}{\frac{3}{2}q_{n-1}^{2+2\tau}}.$$

Therefore, for $n = n_k$ we have

$$\frac{1}{\frac{3}{2}q_{n-1}^{2+2\tau}} \le |J_n(a_1, \ldots, a_n)| \le \frac{1}{\frac{1}{16}q_{n-1}^{2+2\tau}}.$$

**Case II.** When $n = n_{k+1} - 2$, we have

$$J_n = I_n(a_1, \ldots, a_n, 4).$$

Therefore,

$$|J_n(a_1, \ldots, a_n)| = \frac{1}{(4q_n + q_{n-1})(5q_n + q_{n-1})}$$

and

$$\frac{1}{60q_n^2} \le |J_n(a_1, \ldots, a_n)| \le \frac{1}{16q_n^2}.$$

**Case III.** When $n = n_{k+1} - 1$, since

$$J_n = \bigcup_{\frac{1}{4}q_n^\tau \le a_{n+1}(x) \le \frac{1}{2}q_n^\tau} I_{n+1}\left(a_1, \ldots, a_n, a_{n+1}\right).$$

Therefore

$$|J_n(a_1, \ldots, a_n)| = \frac{\frac{1}{4}q_n^\tau + 1}{(\frac{1}{4}q_n^{\tau+1} + q_{n-1})(\frac{1}{2}q_n^{\tau+1} + q_n + q_{n-1})}$$

and

$$\frac{1}{\frac{3}{2}q_n^{2+\tau}} \le |J_n(a_1, \ldots, a_n)| \le \frac{1}{\frac{1}{4}q_n^{2+\tau}}.$$

### 5.2.3 Gap Estimation

In this section we estimate the gap between $J_n(a_1, \ldots, a_n)$ and its adjoint fundamental interval of the same order $n$. These gaps are helpful for estimating the measure on general balls.

Let $J_{n-1}(a_1, \ldots, a_{n-1})$ be the mother fundamental interval of $J_n(a_1, \ldots, a_n)$. With no loss of generality, assume that $n$ is even, since if $n$ is odd we can carry out the estimation in almost the same way. Let the left and the right gap between $J_n(a_1, \ldots, a_n)$ and its adjoint fundamental interval at each side be represented by $g_n^{\ell}(a_1, \ldots, a_n)$ and $g_n^r(a_1, \ldots, a_n)$ respectively.

Denote by $g_n(a_1, \ldots, a_n)$ the minimum distance between $J_n(a_1, \ldots, a_n)$ and its adjacent interval of the same order $n$, that is,

$$g_n(a_1, \ldots, a_n) = \min\{g_n^{\ell}(a_1, \ldots, a_n), g_n^r(a_1, \ldots, a_n)\}.$$

Since $n$ is even, the right adjoint fundamental interval to $J_n$, which is contained in $J_{n-1}$, is

$$J_n' = J_n(a_1, \ldots, a_{n-1}, a_n + 1) \text{ (if it exists)}$$

and the left adjoint fundamental interval to $J_n$, which is contained in $J_{n-1}$, is

$$J_n'' = J_n(a_1, \ldots, a_{n-1}, a_n - 1) \text{ (if it exists).}$$

The gap estimation work proceeds according to the range of $n$ defined for $\mathcal{E}_M$. The three cases are described in Table 1.

**Gap I.** For the case $n_k \le n \le n_{k+1} - 3$, we have

$$\begin{aligned} J_n &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \ldots, a_n, a_{n+1}\right), \\ J_n' &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \ldots, a_n, a_{n+1} + 1\right), \\ J_n'' &= \bigcup_{1 \le a_{n+1}(x) \le M} I_{n+1}\left(a_1, \ldots, a_n, a_{n+1} - 1\right). \end{aligned}$$

Then by Lemma 2.3.2, for the right gap

$$g_n^r(a_1, \ldots, a_n) \ge \frac{1}{(q_n + q_{n-1})((M+1)(q_n + q_{n-1}) + q_{n-1})}$$

and for the left gap

$$g_n^l(a_1, \ldots, a_n) \ge \frac{1}{q_n((M+1)q_n + q_{n-1})}.$$

So

$$g_n(a_1, \ldots, a_n) = \frac{1}{(q_n + q_{n-1})((M+1)(q_n + q_{n-1}) + q_{n-1})}.$$

Also, by comparing $g_n(a_1,\dots,a_n)$ with $J_n(a_1,\dots,a_n)$, we notice that

$$g_n(a_1,\dots,a_n) \geq \frac{1}{2M}|J_n(a_1,\dots,a_n)|.$$

**Gap II.** For the case $n = n_{k+1}-2$, we have

$$\begin{aligned} J_n &= I_{n+1}(a_1,\dots,a_n,4) \subset I_n(a_1,\dots,a_n),\\ J_n' &= I_{n+1}(a_1,\dots,a_n+1,4) \subset I_n(a_1,\dots,a_n+1),\\ J_n'' &= I_{n+1}(a_1,\dots,a_n-1,4) \subset I_n(a_1,\dots,a_n-1). \end{aligned}$$

Since $J_n$ lies in the middle of $I_n(a_1,\dots,a_n)$ and $J_n'$ lies on the right to $I_n(a_1,\dots,a_n)$ therefore the right gap is larger than the distance between the right endpoint of $J_n$ and that of $I_n$. Also, as $J_n''$ lies on the left to $I_n(a_1,\dots,a_n)$ therefore the left gap is larger than the distance between the left endpoint of $J_n$ and that of $I_n$.

Hence, for the right gap

$$g_n^r(a_1,\dots,a_n) \geq \frac{p_n+p_{n-1}}{q_n+q_{n-1}} - \frac{4p_n+p_{n-1}}{4q_n+q_{n-1}} = \frac{3}{(q_n+q_{n-1})(4q_n+q_{n-1})}.$$

and for the left gap

$$g_n^l(a_1,\dots,a_n) \geq \frac{5p_n+p_{n-1}}{5q_n+q_{n-1}} - \frac{p_n}{q_n} = \frac{1}{(5q_n+q_{n-1})q_n}.$$

Therefore,

$$g_n(a_1,\dots,a_n) \geq \frac{1}{(5q_n+q_{n-1})(q_n+q_{n-1})}.$$

Also, by comparing $g_n(a_1,\dots,a_n)$ with $J_n(a_1,\dots,a_n)$, we notice that

$$g_n(a_1,\dots,a_n) \geq \frac{4}{3}|J_n(a_1,\dots,a_n)|.$$

**Gap III.** For the case $n = n_{k+1}-1$, we have

$$\begin{aligned} J_n &= \bigcup_{\frac14 q_n^\tau \leq a_{n+1}(x) \leq \frac12 q_n^\tau} I_{n+1}(a_1,\dots,a_n,a_{n+1}),\\ J_n' &= \bigcup_{\frac14 q_n^\tau \leq a_{n+1}(x) \leq \frac12 q_n^\tau} I_{n+1}(a_1,\dots,a_n+1,a_{n+1}),\\ J_n'' &= \bigcup_{\frac14 q_n^\tau \leq a_{n+1}(x) \leq \frac12 q_n^\tau} I_{n+1}(a_1,\dots,a_n-1,a_{n+1}). \end{aligned}$$

In this case also the gap position geometry is the same as the case when $n = n_{k+1}-2$.

Hence, for the right gap

$$g_n^r(a_1,\dots,a_n) \geq \frac{(\frac14 q_n^\tau - 1)}{(\frac14 q_n^\tau q_n + q_{n-1})(q_n+q_{n-1})}$$

and for the left gap

$$g_n^l(a_1,\ldots,a_n) \geq \frac{1}{((\frac{1}{2}q_n^\tau+1)q_n+q_{n-1})q_n}.$$

Therefore,

$$g_n(a_1,\ldots,a_n) \geq \frac{1}{((\frac{1}{2}q_n^\tau+1)q_n+q_{n-1})(q_n+q_{n-1})}.$$

Also, by comparing $g_n(a_1,\ldots,a_n)$ with $J_n(a_1,\ldots,a_n)$, we notice that

$$g_n(a_1,\ldots,a_n) \geq \frac{1}{3}|J_n(a_1,\ldots,a_n)|.$$

### 5.2.4 Mass Distribution on $\mathcal{E}_M$

We define a measure $\mu$ supported on $\mathcal{E}_M$. For this we start by defining the measure on the fundamental intervals of order $n_k-2$, $n_k-1$ and $n_k$. The measure on other fundamental intervals can be obtained by using the consistency of a measure. Because the sparse set $\{n_k\}_{k\geq 1}$ is of our choosing, we may let $m_{k+1}L = n_{k+1}-2-n_k$ for any $k\geq 0$. In other words, the $m_{k+1}$ may also be regarded as an increasing sequence. This simplifies calculations without loss of generality.

Note that the sum in Eq. (5.2) induces a measure $\mu$ on a *basic* interval of order $L$

$$\mu(I_L(a_1,\ldots,a_L)) = \left(\frac{1}{q_L^{2+\tau}}\right)^S,$$

for each $1\leq a_1,\ldots,a_L\leq M$.

**Step I.** Let $1\leq i\leq m_1$. We first define a positive measure for the *fundamental* intervals $J_{iL}(a_1,\ldots,a_{iL})$

$$\mu\left(J_{iL}(a_1,\ldots,a_{iL})\right) = \prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}(a_{tL+1},\ldots,a_{(t+1)L})}\right)^S$$

and then we distribute this measure uniformly over its next offspring.

**Step II.** For $J_{n_1-1}$ and $J_{n_1-2}$, define a measure

$$\begin{aligned}\mu\left(J_{n_1-1}(a_1,\ldots,a_{n_1-1})\right) &= \mu\left(J_{n_1-2}(a_1,\ldots,a_{n_1-2}\right)\\ &= \prod_{t=0}^{m_1-1}\left(\frac{1}{q_L^{2+\tau}(a_{tL+1},\ldots,a_{(t+1)L})}\right)^S.\end{aligned}$$

**Step III.** For $J_{n_1}$, define a measure

$$\mu\left(J_{n_1}(a_1,\ldots,a_{n_1})\right) = \frac{1}{\frac{1}{4}q_{n_1-1}^\tau}\mu\left(J_{n_1-1}(a_1,\ldots,a_{n_1-1}\right).$$

In other words, the measure of $J_{n_1-1}$ is uniformly distributed on its next offspring $J_{n_1}$.

**Measure of other levels.** The measure of fundamental intervals for other levels can be defined inductively.

To define the measure on general fundamental interval $J_{n_{k+1}-2}$ and $J_{n_{k+1}-1}$, we assume that $\mu\left(J_{n_k}\right)$ has been defined. Then define

$$\begin{aligned}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right) &= \mu\left(J_{n_{k+1}-2}(a_1,\ldots,a_{n_{k+1}-2})\right)\\ &= \mu\left(J_{n_k}(a_1,\ldots,a_{n_k})\right)\cdot\\ &\prod_{t=0}^{m_{k+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S.\end{aligned}$$

Next, we equally distribute the measure of the fundamental interval $J_{n_{k+1}-1}$ among its next offspring which is a fundamental interval of order $n_{k+1}$, that is,

$$\mu\left(J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})\right) = \frac{1}{\frac{1}{4}q^{\tau}_{n_{k+1}-1}}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right).$$

The measure of other fundamental intervals, of level less than $n_{k+1}-2$ , is given by using the consistency of the measure. Therefore, for $n = n_k + iL$ where $1 \le i \le m_{k+1}$, we define

$$\begin{aligned}\mu\left(J_{n_k+iL}(a_1,\ldots,a_{n_k+iL})\right) &= \mu\left(J_{n_k}(a_1,\ldots,a_{n_k})\right)\\ &\cdot\prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S.\end{aligned}$$

### 5.2.5 Hölder Exponent of the Measure $\mu$

For the lower bound, we aim to apply the mass distribution principle to the Cantor subset $\mathcal{E}_M$, which requires the measure of a *general ball*. Thus far we have only calculated $\mu\left(J_n(a_1,\ldots,a_n)\right)$. We show that there is a Hölder condition between $\mu\left(J_n(a_1,\ldots,a_n)\right)$ and $|J_n(a_1,\ldots,a_n)|$ and another Hölder condition between $\mu(B(x,r))$ and $r$. The derived inequalities continue the program of establishing our lower bound. For brevity in some calculations, we write $J_n$ as short for $J_n(a_1,\ldots,a_n)$. Note that in the statement of the mass distribution principle, Proposition 2.5.3, the measure of a general set is compared to its diameter. However, it can simply be tailored to compare the measure of a ball to its radius.

#### 5.2.5.1 The Hölder Exponent of the Measure $\mu$ on Fundamental Intervals

First, we estimate the Hölder exponent of $\mu\left(J_n(a_1,\ldots,a_n)\right)$ in relation to $|J_n|$.

**Step I.** When $n = iL$ for some $1 \leq i < m_1$

$$\begin{aligned}
\mu\left(J_{iL}\left(a_1,\ldots,a_{iL}\right)\right) &= \prod_{t=0}^{i-1}\left(\frac{1}{q_L^{2+\tau}\left(a_{tL+1},\ldots,a_{(t+1)L}\right)}\right)^S \\
&\overset{(2.5)}{\leq} 2^{(2+\tau)(i-1)}\left(\frac{1}{q_{iL}^{2+\tau}\left(a_1,\ldots,a_{iL}\right)}\right)^S \\
&\overset{(2.4)}{\leq} \left(\frac{1}{q_{iL}^{2+\tau}\left(a_1,\ldots,a_{iL}\right)}\right)^{S-2/L} \\
&\ll \left|J_{iL}\left(a_1,\ldots,a_{iL}\right)\right|^{S-2/L}.
\end{aligned} \tag{5.5}$$

**Step II(a).** When $n = m_1L = n_1 - 2$

$$\begin{aligned}
\mu\left(J_{n_1-2}\left(a_1,\ldots,a_{n_1-2}\right)\right) &= \prod_{t=0}^{m_1-1}\left(\frac{1}{q_L^{2+\tau}\left(a_{tL+1},\ldots,a_{(t+1)L}\right)}\right)^S \\
&\overset{(5.5)}{\leq} 2^{(2+\tau)(m_1-1)}\left(\frac{1}{q_{m_1L}^{2+\tau}(a_1,\ldots,a_{m_1L})}\right)^S \\
&\leq 2^{(2+\tau)(m_1-1)}\left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1,\ldots,a_{n_1-2})}\right)^S \\
&\leq \left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1,\ldots,a_{n_1-2})}\right)^{S-\frac{2}{L}} \\
&\ll \left|J_{n_1-2}\left(a_1,\ldots,a_{n_1-2}\right)\right|^{S-2/L}.
\end{aligned} \tag{5.6}$$

**Step II(b).** When $n = n_1 - 1 = m_1L + 1$

$$\begin{aligned}
\mu\left(J_{n_1-1}\left(a_1,\ldots,a_{n_1-1}\right)\right) &= \mu\left(J_{n_1-2}(a_1,\ldots,a_{n_1-2})\right) \\
&\overset{(5.6)}{\leq} \left(\frac{1}{q_{n_1-2}^{2+\tau}(a_1,\ldots,a_{n_1-2})}\right)^{S-\frac{2}{L}} \\
&\asymp \left(\frac{1}{q_{n_1-1}^{2+\tau}(a_1,\ldots,a_{n_1-1})}\right)^{S-\frac{2}{L}} \\
&\leq c\left|J_{n_1-1}\left(a_1,\ldots,a_{n_1-1}\right)\right|^{S-\frac{2}{L}},
\end{aligned} \tag{5.7}$$

where $c = \frac{3}{2}$ and inequality (5.7) is obtained from the relation

$$q_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-2},4) \asymp q_{n_{k+1}-2}(a_1,\ldots,a_{n_{k+1}-2})$$

defined for any $k$.

**Step III.** For $n = n_1$ using the inequality (5.7), we have

$$\begin{aligned}
\mu\left(J_{n_1}(a_1,\ldots,a_{n_1})\right) &= \frac{1}{\frac{1}{4}q^{\tau}_{n_1-1}}\mu\left(J_{n_1-1}(a_1,\ldots,a_{n_1-1}\right) \\
&\leq \frac{1}{\frac{1}{4}q^{\tau}_{n_1-1}}\ c\ \left(\frac{1}{q^{2+\tau}_{n_1-1}(a_1,\ldots,a_{n_1-1})}\right)^{S-\frac{2}{L}} \\
&\leq \frac{1}{\frac{1}{4}}\ c\ \left(\frac{1}{q^{2+2\tau}_{n_1-1}(a_1,\ldots,a_{n_1-1})}\right)^{S-\frac{2}{L}} \\
&\ll |J_{n_1}(a_1,\ldots,a_{n_1})|^{S-\frac{2}{L}}.
\end{aligned}$$

Next we find the Hölder exponent for the general fundamental interval $J_{n_{k+1}-1}$. Determining the Hölder exponent for intervals of other levels can be carried out in the same way.

Let $n = n_{n_{k+1}-1}$. Recall that,

$$\begin{aligned}
&\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right) \\
&= \mu\left(J_{n_{k+1}-2}(a_1,\ldots,a_{n_{k+1}-2})\right) \\
&= \left[\prod_{j=0}^{k-1}\left(\frac{1}{\frac{1}{4}q^{\tau}_{n_{j+1}-1}}\prod_{t=0}^{m_{j+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_j+tL+1},\ldots,a_{n_j+(t+1)L})}\right)^S\right)\right] \\
&\qquad \cdot \prod_{t=0}^{m_{k+1}-1}\left(\frac{1}{q_L^{2+\tau}(a_{n_k+tL+1},\ldots,a_{n_k+(t+1)L})}\right)^S.
\end{aligned}$$

By arguments similar to **Step I** and **Step II**, we obtain

$$\begin{aligned}
\mu\left(J_{n_{k+1}-1}\right) \leq \prod_{j=0}^{k-1}&\left(\frac{1}{\frac{1}{4}q^{\tau}_{n_{j+1}-1}}\left(\frac{1}{q^{2+\tau}_{m_{j+1}L}(a_{n_j+1},\ldots,a_{n_j+(m_{j+1})L})}\right)^{S-\frac{2}{L}}\right) \\
&\cdot\left(\frac{1}{q^{2+\tau}_{m_{k+1}L}(a_{n_k+1},\ldots,a_{n_k+(m_{k+1})L})}\right)^{S-\frac{2}{L}} \\
&\leq 2^{2k}\cdot\left(\frac{1}{q^{2+\tau}_{n_{k+1}-2}}\right)^{S-\frac{6}{L}} \leq \left(\frac{1}{q^{2+\tau}_{n_{k+1}-2}}\right)^{S-\frac{10}{L}} \\
&\asymp \left(\frac{1}{q^{2+\tau}_{n_{k+1}-1}}\right)^{S-\frac{10}{L}} \\
&\leq c_3\ |J_{n_{k+1}-1}|^{S-\frac{10}{L}},
\end{aligned}$$

where $c_3 = \frac{3}{2}$. Here for the third inequality, we use

$$q^{2(2+\tau)}_{n_{k+1}-2} \geq q^2_{n_{k+1}-2} \geq 2^{n_{k+1}-3} \geq 2^{L(m_1+\ldots+m_{k+1})} \geq 2^{L(k+1)} \geq 2^{Lk} = 2^{2k\cdot\frac{L}{2}}.$$

Consequently,

$$\begin{aligned}\mu\left(J_{n_{k+1}}\left(a_1,\ldots,a_{n_{k+1}}\right)\right) &= \frac{1}{\frac{1}{4}q^{\tau}_{n_{k+1}-1}}\mu\left(J_{n_{k+1}-1}(a_1,\ldots,a_{n_{k+1}-1})\right)\\ &\le \frac{1}{\frac{1}{4}}\left(\frac{1}{q^{2+2\tau}_{n_{k+1}-1}}\right)^{S-\frac{10}{L}}\\ &\ll \left|J_{n_{k+1}}\left(a_1,\ldots,a_{n_{k+1}}\right)\right|^{S-\frac{10}{L}}.\end{aligned}$$

In summary, we have shown that for any $n \ge 1$ and $(a_1,\ldots,a_n)$,

$$\mu\left(J_n\left(a_1,\ldots,a_n\right)\right) \ll \left|J_n\left(a_1,\ldots,a_n\right)\right|^{S-\frac{10}{L}}.$$

#### 5.2.5.2 Hölder Exponent for a General Ball

Assume that $x \in \mathcal{E}_M$ and $B(x,r)$ is a ball centred at $x$ with radius $r$ small enough. For each $n \ge 1$, let $J_n = J_n(a_1,\ldots,a_n)$ contain $x$ and

$$g_{n+1}(a_1,\ldots,a_{n+1}) \le r < g_n(a_1,\ldots,a_n).$$

Clearly, by the definition of $g_n$ we see that

$$B(x,r)\cap\mathcal{E}_M \subset J_n(a_1,\ldots,a_n).$$

The calculations for the Hölder exponent for a general ball proceed according to the three cases described in Table 1.

**Case I.** When $n = n_{k+1}-1$.

(i) $r \le |I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|$. In this case the ball $B(x,r)$ can intersect at most four basic intervals of order $n_{k+1}$, which are

$$\begin{aligned}&I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}-1), \quad I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}),\\ &I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}+1), \quad I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}+2).\end{aligned}$$

Thus we have

$$\begin{aligned}\mu(B(x,r)) &\le 4\mu(J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}}))\\ &\le 4c_0|J_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|^{S-\frac{10}{L}}\\ &\le 8c_0Mg_{n_{k+1}}^{S-\frac{10}{L}}\\ &\le 8c_0Mr^{S-\frac{10}{L}}.\end{aligned}$$

(ii) $r > |I_{n_{k+1}}(a_1,\ldots,a_{n_{k+1}})|$. In this case, since

$$|I_{n_k}(a_1,\ldots,a_{n_k})| = \frac{1}{q_{n_{k+1}}(q_{n_{k+1}}+q_{n_{k+1}-1})} \ge \frac{1}{2q^{2+2\tau}_{n_{k+1}-1}},$$

the number of fundamental intervals of order $n_{k+1}$ contained in $J_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-1})$ that the ball $B(x,r)$ intersects is at most

$$4rq_{n_{k+1}-1}^{2+2\tau} + 2 \leq 8rq_{n_{k+1}-1}^{2+2\tau}.$$

Thus we have

$$\begin{aligned}
\mu(B(x,r)) &\leq \min\left\{\mu(J_{n_{k+1}-1}), 8rq_{n_{k+1}-1}^{2\tau} q_{n_{k+1}-1}^{2}\mu(J_{n_{k+1}})\right\} \\
&\leq \mu(J_{n_{k+1}-1}) \min\left\{1, 8rq_{n_{k+1}-1}^{2\tau}\ q_{n_{k+1}-1}^{2}\frac{1}{q_{n_{k+1}-1}^{\tau}}\right\} \\
&\leq c|J_{n_{k+1}-1}|^{S-\frac{10}{L}} \min\left\{1, 8rq_{n_{k+1}-1}^{\tau} q_{n_{k+1}-1}^{2}\right\} \\
&\leq c\left(\frac{1}{q_{n_{k+1}-1}^{2+\tau}}\right)^{S-\frac{10}{L}} \min\left\{1, 8rq_{n_{k+1}-1}^{\tau} q_{n_{k+1}-1}^{2}\right\} \\
&\leq c\left(\frac{1}{q_{n_{k+1}-1}^{2+\tau}}\right)^{S-\frac{10}{L}} (8r\ q_{n_{k+1}-1}^{\tau} q_{n_{k+1}-1}^{2})^{S-\frac{10}{L}} \\
&\leq Cr^{S-\frac{10}{L}}, \text{ where } C = c8^{S-\frac{10}{L}}.
\end{aligned}$$

Here we use $\min\{a,b\} \leq a^{1-s}b^s$ for any $a, b > 0$ and $0 \leq s \leq 1$.
**Case II.** When $n = n_{k+1} - 2$. For $r > |I_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-1})|$.

In this case, since

$$|I_{n_{k+1}-1}(a_1, \ldots, a_{n_{k+1}-1})| \geq \frac{1}{128\ q_{n_{k+1}-2}^{2}},$$

the number of fundamental intervals of order $n_{k+1} - 1$ contained in $J_{n_{k+1}-2}$ that the ball $B(x,r)$ intersects, is at most

$$2(128)rq_{n_{k+1}-2}^{2} + 2 \leq 256rq_{n_{k+1}-2}^{2}.$$

Thus

$$\begin{aligned}
\mu(B(x,r)) &\leq \min\left\{\mu(J_{n_{k+1}-2}), 256rq_{n_{k+1}-2}^{2}\mu(J_{n_{k+1}-1})\right\} \\
&\asymp \min\left\{\mu(J_{n_{k+1}-2}), c_1rq_{n_{k+1}-2}^{2}\mu(J_{n_{k+1}-2})\right\} \\
&= \mu(J_{n_{k+1}-2}) \min\left\{1, 256rq_{n_{k+1}-1}^{2}\right\} \\
&\leq c\left(\frac{1}{q_{n_{k+1}-1}^{2+\tau}}\right)^{S-\frac{10}{L}} \min\left\{1, 256rq_{n_{k+1}-1}^{2}\right\} \\
&\leq c\left(\frac{1}{q_{n_{k+1}+1}^{2}}\right)^{S-\frac{10}{L}} \min\left\{1, 256rq_{n_{k+1}+1}^{2}\right\} \\
&\leq Cr^{S-\frac{10}{L}}, \text{ where } C = c256^{S-\frac{10}{L}}.
\end{aligned}$$

**Case III.** When $n_k \leq n \leq n_{k+1} - 3$. In such a range for $n$, we know that $1 \leq a_n \leq M$ and $|J_n| \asymp 1/q_n^2$. So,

$$\begin{aligned}
\mu(B(x,r)) \leq \mu(J_n) &\leq c|J_n|^{S-\frac{10}{L}} \\
&\leq c\left(\frac{1}{q_n^2}\right)^{S-\frac{10}{L}} \\
&\leq c4M^2\left(\frac{1}{q_{n+1}^2}\right)^{S-\frac{10}{L}} \\
&\ll c4M^2|J_{n+1}|^{S-\frac{10}{L}} \\
&\leq c8M^3 g_{n+1}^{S-\frac{10}{L}} \\
&\leq 8cM^3 r^{S-\frac{10}{L}}.
\end{aligned}$$

### 5.2.6 Conclusion of the Lower Bound Calculations

Finally, by combining all of the above cases with the mass distribution principle Proposition 2.5.3, we have proved that

$$\dim_{\mathcal{H}}\mathcal{E}_M \geq S - 10/L.$$

Let $L, M \to \infty$. Because $\mathcal{E}_M$ is a subset of $E$ and by Remark 5.2.1 we conclude that

$$\dim_{\mathcal{H}} E \geq \dim_{\mathcal{H}}\mathcal{E}_M \geq \lim_{L,M\to\infty}(S - 10/L) = \frac{2}{\tau+2}. \tag{5.8}$$

This completes the calculations for the lower bound.

Now Eq. (5.1) together with Eq. (5.8) allows us to conclude that

$$\dim_{\mathcal{H}}\Big(G(\Psi) \setminus \mathcal{K}(\Psi)\Big) = \frac{2}{\tau+2}.$$

This completes the proof of Theorem 1.6.2.

## 5.3 The General Case

The case for the general approximating function $\Psi$ follows almost exactly the same line of investigations as for the particular case $\Psi(q_n) = q_n^\tau$ for any $\tau > 0$; as discussed above. There are some added subtleties which we will outline and then direct the reader to mimic the proof for the particular approximating function, $q_n^\tau$, earlier.

Consider a rapidly increasing sequence $\{Q_n\}_{n\geq 1}$ of positive integers. For a fixed $\epsilon > 0$, let $\delta \geq 3\epsilon$. Without loss of generality, we can assume that the approximating function $\Psi$ is defined as

$$Q_n^{\tau-\epsilon} \leq \Psi(Q_n) \leq Q_n^{\tau+\epsilon} \quad \text{for all } n \geq 1,$$

where
$$\tau=\liminf_{n\to\infty}\frac{\log\Psi(Q_n)}{\log(Q_n)}.$$
Let
$$A_M=\{x\in[0,1):1\le a_n(x)\le M,\ \text{for all } n\ge 1\}.$$
For all $x\in A_M$, there exists a large $n_1\in\mathbb{N}$ such that
$$q_{n_1-2}\le Q_1^{1-\delta}\Longrightarrow q_{n_1-2}\le Q_1^{1-\delta}\le 2Mq_{n_1-2}.$$
Let
$$a_{n_1-1}(x)=\frac{1}{4}Q_1^{\delta}\ \text{ and }\ \frac{1}{2}\,q_{n_1-1}^{\tau-\epsilon}\le a_{n_1}(x)\le q_{n_1-1}^{\tau-\epsilon}.$$
Then the basic intervals of order $n_1-2,n_1-1$ and $n_1$ can be defined as,
$$I_{n_1-2}\left(a_1,\dots,a_{n_1-2}\right):x\in A_M,$$
$$I_{n_1-1}\left(a_1,\dots,a_{n_1-2},\frac{1}{4}Q_1^{\delta}\right):x\in A_M,$$
$$I_{n_1}\left(a_1,\dots,a_{n_1-2},\frac{1}{4}Q_1^{\delta},a_{n_1}\right):x\in A_M\ \text{ and }\ \frac{1}{2}\,q_{n_1-1}^{\tau-\epsilon}\le a_{n_1}(x)\le q_{n_1-1}^{\tau-\epsilon}.$$
Now fix the basic interval $I_{n_1}(a_1,\cdots,a_{n_1})$, that is, choose it to be an element in the first level of the Cantor set. Consider the set of points:
$$\{[a_1,\cdots,a_{n_1},b_1,b_2,\cdots],1\le b_i\le M\ \text{for all } i\ge 1\}.$$
Then do the same as for the definition of $n_1$. That is for each $x$, find $n_2$ such that $q_{n_2-2}$ is almost $Q_2$.

Continuing in this way define $n_k$ recursively as follows. Collect the $n_k\in\mathbb{N}$ satisfying
$$q_{n_k-2}\le Q_k^{1-\delta}\le 2Mq_{n_k-2}.$$
Define the subset $\mathcal{E}_M^*$ of $G(\Psi)\setminus\mathcal{K}(\Psi)$ as
$$\mathcal{E}_M^*=\left\{x\in[0,1):\begin{array}{l}\frac{1}{2}q_{n_k-1}^{\tau-\epsilon}\le a_{n_k}(x)\le q_{n_k-1}^{\tau-\epsilon}\text{ and } a_{n_k-1}(x)=\frac{1}{4}Q_k^{\delta}\\ \text{and } 1\le a_j(x)\le M,\text{ for all } j\ne n_k-1,n_k\end{array}\right\}.$$
For any $n\ge 1$, define strings $(a_1,\dots,a_n)$ by
$$D_n^*=\left\{(a_1,\dots,a_n)\in\mathbb{N}^n:\begin{array}{l}\frac{1}{2}q_{n_k-1}^{\tau-\epsilon}\le a_{n_k}(x)\le q_{n_k-1}^{\tau-\epsilon}\text{ and } a_{n_k-1}(x)=\frac{1}{4}Q_k^{\delta}\text{ and}\\ 1\le a_j(x)\le M,\text{ for all } 1\le j\ne n_k-1,n_k\le n\end{array}\right\}.$$
For any $n\ge 1$ and $(a_1,\dots,a_n)\in D_n^*$, define
$$J_n\left(a_1,\dots,a_n\right):=\bigcup_{a_{n+1}}I_{n+1}(a_1,\dots,a_n,a_{n+1})\tag{5.9}$$

to be the fundamental interval of order $n$, where the union in Eq. (5.9) is taken over all $a_{n+1}$ such that $(a_1, \ldots, a_n, a_{n+1}) \in D^*_{n+1}$. Then

$$\mathcal{E}^*_M = \bigcap_{n=1}^{\infty} \bigcup_{(a_1,\ldots,a_n)\in D^*_n} J_n\left(a_1, \ldots, a_n\right).$$

As can be seen, the Cantor type structure of the set $\mathcal{E}^*_M$, for the general approximating function $\Psi(Q_n)$, includes similar steps as for particular function, $\Psi(q_n) = q_n^{\tau}$, from the earlier sections. Also, the process of finding the dimension for this set follows similar steps and calculations as we have done for finding the dimension of the Cantor set $\mathcal{E}_M$. However, the calculations involve lengthy expressions and complicated constants. In order to avoid unnecessary intricacy, we will not produce these expressions.

**Remark 5.3.1** Finally, it is worth pointing out that the set $E$ may be generalised to the following form. Let $m \geq 1$, $\Psi : [1, \infty) \to \mathbb{R}_+$ be a non-decreasing function. Define

$$E^* = \left\{ x \in [0,1) : \begin{array}{c} \displaystyle\prod_{i=1}^{m} a_{n+i}(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ \displaystyle\prod_{i=1}^{m-1} a_{n+i}(x) < \Psi(q_n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

The Hausdorff dimension of this set is

$$\dim_{\mathcal{H}} E^* = \frac{2}{\tau + 2}, \quad \text{where} \quad \tau = \liminf_{q \to \infty} \frac{\log \Psi(q)}{\log q}$$

The calculations become very intricate and therefore we leave the details aside in this thesis. ■

Chapter 6

# Hausdorff Dimension of an Exceptional Set

In this chapter we prove Theorem 1.7.6. Recall that, for an arbitrary positive function $\Phi : \mathbb{N} \to (1, \infty)$ such that $\lim_{n\to\infty} \Phi(n) = \infty$, we have

$$\mathcal{E}_1(\Phi) := \{x \in [0,1) : a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$
$$\mathcal{E}_2(\Phi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

and

$$\begin{aligned}\mathcal{F}(\Phi) :=&\ \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi)\\ =&\left\{x \in [0,1) : \begin{array}{l} a_{n+1}(x)a_n(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array}\right\}.\end{aligned}$$

We calculate the Hausdorff dimension of the set $\mathcal{F}(\Phi)$.

**Theorem 1.7.6** *Let $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim\limits_{n\to\infty} \Phi(n) = \infty$. Suppose*

$$\log B = \liminf_{n\to\infty} \frac{\log \Phi(n)}{n} \text{ and } \log b = \liminf_{n\to\infty} \frac{\log\log \Phi(n)}{n}.$$

*Then*

$$\dim_{\mathcal{H}}\mathcal{F}(\Phi) = \begin{cases} t_B := \inf\{s \geq 0 : \mathsf{P}\left(T, -s^2 \log B - s\log(|T'|\right) \leq 0\}, & \text{if } 1 < B < \infty; \\ \frac{1}{1+b}, & \text{if } B = \infty. \end{cases}$$

■

Note that if we take $B = 1$ then from the definition of $\mathcal{F}(\Phi)$ we have $a_{n+1}(x) < 1$ which contradicts $a_{n+1}(x) \geq 1$. Therefore, $B > 1$.

The proof of Theorem 1.7.6 consists of two cases:

(i) $1 < B < \infty$,

(ii) $B = \infty$.

## 6.1 Case (i): $1 < B < \infty$

**Proof:** By the choice of $B$ in the statement of Theorem 1.7.6 one notes that

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) = \dim_{\mathcal{H}} \mathcal{F}(\Phi : n \to B^n) \quad \text{when } 1 < B < \infty.$$

The reason that the function $\Phi : n \to B^n$ yields the Hausdorff dimension of the set $\mathcal{F}(\Phi)$, follows verbatim from [55, Theorem 4.2]. Therefore, we can simply take the approximating function $\Phi(n) := B^n$ and rewrite the set $\mathcal{F}(\Phi)$ as

$$\begin{aligned} \mathcal{F}(\Phi) :=\ & \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi) \\ =\ & \left\{ x \in [0,1) : \begin{array}{l} a_{n+1}(x)a_n(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}. \end{aligned}$$

The aim is to show $\dim_{\mathcal{H}} \mathcal{F}(B) = t_B$. The details of the proof of Theorem 1.7.6 is divided into two further subsections. That is finding the upper bound

$$\dim_{\mathcal{H}} \mathcal{F}(B) \leq t_B,$$

and the lower bound

$$\dim_{\mathcal{H}} \mathcal{F}(B) \geq t_B$$

separately. Taken together, this will conclude our proof for **Case (i)**.

### 6.1.1 Upper Bound for $\mathcal{F}(B)$

For the upper bound of $\dim_{\mathcal{H}} \mathcal{F}(B)$, we consider two sets:

$$\mathcal{F}_1(B) = \left\{ x \in [0,1) : a_n(x) \geq B^n \ \text{ for infinitely many } n \in \mathbb{N} \right\}$$

and

$$\mathcal{F}_2(B) = \left\{ x \in [0,1) : \begin{array}{c} 1 \leq a_n(x) \leq B^n,\ a_{n+1}(x) \geq B^n / a_n(x) \\ \text{for infinitely many } n \in \mathbb{N}, \text{ and} \\ a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\} \tag{6.1}$$

From the definition of Hausdorff dimension it follows that

$$\dim_{\mathcal{H}} \mathcal{F}(B) \leq \max\{\dim_{\mathcal{H}} \mathcal{F}_1(B), \dim_{\mathcal{H}} \mathcal{F}_2(B)\}.$$

The Hausdorff dimension of $\mathcal{F}_1(B)$ follows from Theorem 1.7.2. So it remains to obtain the upper bound for the Hausdorff dimension of $\mathcal{F}_2(B)$. Recall that the pressure function $P(T)$ is monotonic with respect to the potential which implies then $s_B \leq t_B$. So, once we can show $\dim_{\mathcal{H}} \mathcal{F}_2(B) \leq t_B$, the upper bound for the $\dim_{\mathcal{H}} \mathcal{F}(B)$ follows.

Fix $\epsilon > 0$ and let $s = t_B + 2\epsilon$. We will show that $\dim_{\mathcal{H}}\mathcal{F}_2(B) \le s$.

By the definition of $t_B$, one has for any large $n$,

$$\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{ns}q_{n-1}^2}\right)^s \le \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{n(t_B+\epsilon)}q_{n-1}^2}\right)^{t_B+\epsilon} \cdot B^{-n\epsilon^2} \le B^{-n\epsilon^2}. \quad (6.2)$$

From Eq. (6.1)

$$\begin{aligned}
\mathcal{F}_2(B) \;&\subset\; \left\{x \in [0,1) : \begin{array}{l} 1 \le a_n(x) \le B^n, \text{ and} \\ (B^n/a_n(x)) \le a_{n+1}(x) < B^n \text{ for i. m. } n \in \mathbb{N} \end{array}\right\} \\
&= \bigcap_{N=1}^{\infty}\bigcup_{n\ge N} \left\{x \in [0,1) : \begin{array}{l} 1 \le a_n(x) \le B^n, \text{ and} \\ (B^n/a_n(x)) \le a_{n+1}(x) < B^n \end{array}\right\} \\
&= \bigcap_{N=1}^{\infty}\bigcup_{n\ge N} \mathcal{F}_I \cup \mathcal{F}_{II} \qquad (6.3)
\end{aligned}$$

where

$$\mathcal{F}_I = \{x \in [0,1) : 1 \le a_n(x) < \alpha^n, (\ B^n/a_n(x)) \le a_{n+1}(x) < B^n\}$$
$$\mathcal{F}_{II} = \{x \in [0,1) : \alpha^n \le a_n(x) \le B^n, (B^n/a_n(x)) \le a_{n+1}(x) < B^n\}$$

and $\alpha^n > 1$. Here we have assumed that $\alpha > 1$ and therefore $\alpha^n > 1$ for any $n \in \mathbb{N}$.

Next we will separately find suitable coverings for set $\mathcal{F}_I$ and $\mathcal{F}_{II}$ whereas the union of the coverings for both these sets will serve as an appropriate covering for $\mathcal{F}_2(B)$.

The set $\mathcal{F}_I$ can be covered by collections of fundamental intervals $J_n$ of order $n$:

$$\begin{aligned}
\mathcal{F}_I &\subset \{x \in [0,1) : 1 \le a_n(x) \le \alpha^n, (B^n/a_n(x)) \le a_{n+1}(x)\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left\{x \in [0,1) : \begin{array}{l} a_k(x) = a_k,\ 1 \le k \le n-1, \\ 1 \le a_n(x) \le \alpha^n, \text{ and} \\ (B^n/a_n(x)) \le a_{n+1}(x) \end{array}\right\} \\
&= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \bigcup_{1\le a_n<\alpha^n} \bigcup_{a_{n+1}\ge B^n/a_n} I_{n+1}(a_1,\cdots,a_{n+1}) \\
&= \bigcup_{\substack{a_1,\cdots,a_{n-1}\in\mathbb{N}, \\ 1\le a_n\le\alpha^n}} J_n(a_1,\cdots,a_n).
\end{aligned}$$

Note that since

$$J_n(a_1,\cdots,a_n) = \bigcup_{a_{n+1}\ge B^n/a_n} I_{n+1}(a_1,\cdots,a_{n+1}),$$

therefore we have

$$|J_n(a_1,\cdots,a_n)| \asymp \frac{1}{B^n a_n q_{n-1}^2}.$$

Cover the set $\mathcal{F}_{II}$ by the collection of fundamental intervals $J'_{n-1}$ of order $n-1$:

$$\begin{aligned}\mathcal{F}_{II} &\subset \left\{x \in [0,1) : a_n(x) \ge \alpha^n\right\} \\ &= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left\{x \in [0,1) : a_k(x) = a_k, 1 \le k \le n-1, a_n(x) \ge \alpha^n\right\} \\ &= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \bigcup_{a_n \ge \alpha^n} I_n(a_1,\cdots,a_n) \\ &= \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} J'_{n-1}(a_1,\cdots,a_{n-1}).\end{aligned}$$

Since

$$J'_{n-1}(a_1,\cdots,a_{n-1}) = \bigcup_{a_n \ge \alpha^n} I_n(a_1,\cdots,a_n),$$

therefore we have

$$|J'_{n-1}(a_1,\cdots,a_{n-1})| \asymp \frac{1}{\alpha^n q_{n-1}^2}.$$

Now we consider the $s$-volume of the cover of $\mathcal{F}_I \bigcup \mathcal{F}_{II}$:

$$\begin{aligned}&\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \sum_{1\le a_n \le \alpha^n} \left(\frac{1}{B^n a_n q_{n-1}^2}\right)^s + \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s \\ \asymp &\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \alpha^{n(1-s)} \left(\frac{1}{B^n q_{n-1}^2}\right)^s + \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s \quad \text{(integrating on } a_n) \\ = &\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left[\left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s + \left(\frac{1}{\alpha^n q_{n-1}^2}\right)^s\right] \quad (\text{by } \alpha = B^s) \\ \asymp &\sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{ns} q_{n-1}^2}\right)^s.\end{aligned}$$

Therefore, from (6.3), we obtain

$$\mathcal{F}_2(B) \subset \bigcap_{N=1}^{\infty} \bigcup_{n\ge N} \left\{ \bigcup_{\substack{a_1,\cdots,a_{n-1}\in\mathbb{N} \\ 1\le a_n \le \alpha^n}} J_n(a_1,\cdots,a_n) \bigcup \bigcup_{a_1,\cdots,a_{n-1}\in\mathbb{N}} J'_{n-1}(a_1,\cdots,a_{n-1}) \right\}. \tag{6.4}$$

Thus from (6.4) and (6.2), we obtain the $s$-dimensional Hausdorff measure of $\mathcal{F}_2(B)$ as

$$\mathcal{H}^s(\mathcal{F}_2(B)) \le \liminf_{N\to\infty} \sum_{n\ge N}^{\infty} \sum_{a_1,\cdots,a_{n-1}\in\mathbb{N}} \left(\frac{1}{B^{ns} q_{n-1}^2}\right)^s \le \liminf_{N\to\infty} \sum_{n\ge N}^{\infty} \frac{1}{B^{n\epsilon^2}} = 0.$$

This gives $\dim_{\mathcal{H}} \mathcal{F}_2(B) \le s = t_B + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\dim_{\mathcal{H}} \mathcal{F}_2(B) \le t_B$. Consequently,

$$\dim_{\mathcal{H}} \mathcal{F}(B) \le t_B. \tag{6.5}$$

### 6.1.2 Lower Bound for $\mathcal{F}(B)$

In this subsection we will determine the lower bound for $\dim_{\mathcal{H}}\mathcal{F}(B)$. As in the previous chapter, we will use the mass distribution principle to obtain the lower bound of the Hausdorff dimension. Recall that the main ingredient is to construct a suitable Cantor subset within $\mathcal{F}(B)$ supporting a probability measure and satisfying the hypothesis of the mass distribution principle.

We proceed as follows: to prove $\dim_{\mathcal{H}}\mathcal{F}(B) \geq t_B$ it is sufficient to show that $\dim_{\mathcal{H}}\mathcal{F}(B) \geq t_{L,B}(M)$ for all large enough $M$ and $L$ (Corollary 2.4.5). For this we will construct a subset $\mathcal{F}_M(B) \subset \mathcal{F}(B)$ and use the lower bound for Hausdorff dimension of $\mathcal{F}_M(B)$ to approximate that of $\mathcal{F}(B)$.

Fix $s < t_{L,B}(M)$. Let $\alpha = B^s$. Choose a rapidly increasing sequence of integers $\{n_k\}_{k\geq 1}$ and, for convenience, we let $n_0 = 0$.

Define the subset $\mathcal{F}_M(B)$ of $\mathcal{F}(B)$ as follows

$$\mathcal{F}_M(B) = \left\{ x \in [0,1) : \begin{array}{l} \dfrac{B^{n_k}}{2\alpha^{n_k}} \leq a_{n_k+1}(x) \leq \dfrac{B^{n_k}}{\alpha^{n_k}}, a_{n_k}(x) = 2\alpha^{n_k} \text{ for all } k \geq 1 \\ \text{and } 1 \leq a_j(x) \leq M, \text{ for all } j \neq n_k, n_k+1 \end{array} \right\}. \tag{6.6}$$

#### 6.1.2.1 Structure of $\mathcal{F}_M(B)$

For any $n \geq 1$, define the set of strings

$$D_n = \left\{ (a_1, \ldots, a_n) \in \mathbb{N}^n : \begin{array}{l} \dfrac{B^{n_k}}{2\alpha^{n_k}} \leq a_{n_k+1}(x) \leq \dfrac{B^{n_k}}{\alpha^{n_k}}, a_{n_k}(x) = 2\alpha^{n_k} \\ \text{and } 1 \leq a_j(x) \leq M, \ j \neq n_k, n_k+1 \end{array} \right\}.$$

Recall Definition 5.2.2 that for any $n \geq 1$ and $(a_1, \ldots, a_n) \in D_n$, we call $I_n(a_1, \ldots, a_n)$ a *basic interval of order* $n$ and

$$J_n := J_n(a_1, \ldots, a_n) := \bigcup_{a_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}) \tag{6.7}$$

a *fundamental interval of order* $n$, where the union in Eq. (6.7) is taken over all $a_{n+1}$ such that $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$.

Note that in Eq. (6.6) according to the limitations on the partial quotients we have three distinct cases for $J_n$. Accordingly, the following table (commencing from $k = 1$), summarises our Cantor subset construction, such that for $(a_1, \ldots, a_n, a_{n+1}) \in D_{n+1}$:

| | | |
|---|---|---|
| I. | $n_{k-1}+1 \le n \le n_k - 2,$ | $J_n = \bigcup_{1 \le a_{n+1} \le M} I_{n+1}(a_1, \ldots, a_n, a_{n+1});$ (6.8) |
| II. | $n = n_k - 1,$ | $J_n = \bigcup_{a_{n+1} = 2\alpha^n} I_{n+1}(a_1, \ldots, a_n, a_{n+1});$ (6.9) |
| III. | $n = n_k,$ | $J_n = \bigcup_{\frac{B^n}{2\alpha^n} \le a_{n+1} \le \frac{B^n}{\alpha^n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$ (6.10) |

*Table 2*

Then,

$$\mathcal{F}_M(B) = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \ldots, a_n) \in D_n} J_n\left(a_1, \ldots, a_n\right).$$

#### 6.1.2.2 Lengths of Fundamental Intervals

In this subsubsection, we will estimate the lengths of the fundamental intervals in the three cases defined above in the structure of $\mathcal{F}_M(B)$, as shown in Table 2.

**I.** If $n_{k-1}+1 \le n \le n_k - 2$ then from Eq. (6.8) and using Eq. (2.3) we have

$$\begin{aligned} |J_n(a_1, \ldots, a_n)| &= \sum_{1 \le a_{n+1} \le M} \left|I_{n+1}\left(a_1, \ldots, a_n, a_{n+1}\right)\right| \\ &= \sum_{1 \le a_{n+1} \le M} \frac{1}{q_{n+1}\left(q_{n+1}+q_n\right)} \\ &= \sum_{a_{n+1}=1}^{M} \frac{1}{q_n} \left(\frac{1}{q_{n+1}} - \frac{1}{q_{n+1}+q_n}\right) \\ &= \frac{1}{q_n} \sum_{a_{n+1}=1}^{M} \left(\frac{1}{a_{n+1}q_n + q_{n-1}} - \frac{1}{\left(a_{n+1}+1\right)q_n + q_{n-1}}\right) \\ &= \frac{1}{q_n}\left(\frac{1}{q_n + q_{n-1}} - \frac{1}{\left(M+1\right)q_n + q_{n-1}}\right) \\ &= \frac{M}{\left(\left(M+1\right)q_n + q_{n-1}\right)\left(q_n + q_{n-1}\right)}. \end{aligned} \tag{6.11}$$

Also, from (6.11) we have

$$\frac{1}{6q_n^2} \le |J_n(a_1, \cdots, a_n)| \le \frac{1}{q_n^2}. \tag{6.12}$$

In particular for $n = n_k + 1$,

$$\frac{1}{24B^{2n}q_{n-2}^2} \le |J_n(a_1, \cdots, a_n)| \le \frac{1}{4B^{2n}q_{n-2}^2}. \tag{6.13}$$

**II.** If $n = n_k - 1$ then from Eq. (6.9) and following the same steps as for case **I** we have

$$|J_n(a_1, \ldots, a_n)| = \frac{1}{(2\alpha^n q_n + q_{n-1})((2\alpha^n + 1)q_n + q_{n-1})}$$

and

$$\frac{1}{12\alpha^{n+1}q_n^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{2\alpha^{n+1}q_n^2}. \tag{6.14}$$

**III.** If $n = n_k$ then from Eq. (6.10) and following the similar steps as for case **I** we obtain

$$|J_n(a_1, \ldots, a_n)| = \frac{\frac{B^n}{2\alpha^n} + 1}{(\frac{B^n}{2\alpha^n}q_n + q_{n-1})((\frac{B^n}{\alpha^n} + 1)q_n + q_{n-1})}$$

and

$$\frac{\alpha^n}{6B^n q_n^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{2\alpha^n}{B^n q_n^2}.$$

Further,

$$\frac{1}{32\alpha^n B^n q_{n-1}^2} \leq |J_n(a_1, \cdots, a_n)| \leq \frac{1}{2\alpha^n B^n q_{n-1}^2}. \tag{6.15}$$

#### 6.1.2.3 Supporting Measure on $\mathcal{F}_M(B)$

To construct a suitable measure supported on $\mathcal{F}_M(B)$ first recall that $t_{L,B}(M)$ is the solution to

$$\sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{B^{Ls}q_L^2}\right)^s = 1.$$

For $\alpha = B^s$ this sum becomes

$$\sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{\alpha^L q_L^2}\right)^s = 1.$$

Let $m_k L = n_k - n_{k-1} - 1$ for any $k \geq 1$. Note that $m_1 L = n_1 - 1$ since we have assumed $n_0 = 0$ and define

$$w = \sum_{a_1,\ldots,a_L \in \mathcal{A}_M} \left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+t+1}, \cdots, a_{n_{k-1}+(t+1)L})}\right)^s$$

where $0 \leq t \leq m_k - 1$.

**Step I.** Let $1 \leq m \leq m_1$. We first define a positive measure for the *fundamental* interval $J_{mL}(a_1, \ldots, a_{mL})$ as

$$\mu(J_{mL}(a_1, \ldots, a_{mL})) = \prod_{t=0}^{m-1} \frac{1}{w} \left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})}\right)^s,$$

and then we distribute this measure uniformly over its next offspring.

**Step II.** When $n = m_1L = n_1 - 1$ then define a measure

$$\mu(J_{m_1L}(a_1, \ldots, a_{m_1L})) = \prod_{t=0}^{m_1-1} \frac{1}{w}\left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})}\right)^s.$$

**Step III.** When $n = m_1L + 1 = n_1$ then for $J_{n_1}(a_1, \ldots, a_{n_1})$, define a measure

$$\mu(J_{n_1}(a_1, \ldots, a_{n_1})) = \frac{1}{2\alpha^{n_1}}\mu(J_{n_1-1}(a_1, \ldots, a_{n_1-1}))$$

In other words, the measure of $J_{n_1-1}$ is uniformly distributed on its next offspring $J_{n_1}$.

**Step IV.** When $n = n_1 + 1$.

$$\mu(J_{n_1+1}(a_1, \ldots, a_{n_1+1})) = \frac{2\alpha^{n_1}}{B^{n_1}}\mu(J_{n_1}(a_1, \ldots, a_{n_1}))$$

The measure of other fundamental intervals of level less than $n_1 - 1$ is given by the consistency of a measure. To be more precise, for any $n < n_1 - 1$, suppose

$$\mu(J_n(a_1, \cdots, a_n)) = \sum_{J_{m_1L} \subset J_n} \mu(J_{m_1L}).$$

So for any $m < m_1$, the measure of fundamental interval $J_{mL}$ is given by

$$\begin{aligned}
\mu(J_{mL}(a_{n_{k-1}+t+1} \cdots a_{n_{k-1}+(t+1)L})) &= \sum_{J_{m_1L} \subset J_{mL}} \mu(J_{m_1}L) \\
&= \prod_{t=0}^{m-1} \frac{1}{w}\left(\frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})}\right)^s.
\end{aligned}$$

The measure of fundamental intervals for other levels can be defined inductively. For $k \geq 2$ define,

$$\begin{aligned}
\mu(J_{n_k-1}(a_1, \ldots, a_{n_k-1})) &= \mu\left(J_{n_{k-1}+1}(a_1, \ldots, a_{n_{k-1}+1})\right) \cdot \\
&\quad \prod_{t=0}^{m_k-1} \frac{1}{w}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1}, \ldots, a_{n_{k-1}+(t+1)L})}\right)^s,
\end{aligned}$$

$$\mu(J_{n_k}(a_1, \ldots, a_{n_k})) = \frac{1}{2\alpha^{n_k}}\mu\left(J_{n_{k-1}}(a_1, \ldots, a_{n_{k-1}})\right),$$

and

$$\mu(J_{n_k+1}(a_1, \ldots, a_{n_k+1})) = \frac{2\alpha^{n_k}}{B^{n_k}}\mu\left(J_{n_k}(a_1, \ldots, a_{n_k})\right) \cdot$$

#### 6.1.2.4 Estimation of $\mu(J_n(a_1, \ldots, a_n))$

In this subsubsection we will estimate the measure $\mu$ of the fundamental intervals defined above. For this we split the process into several cases. Recall that $\alpha^n > 1$ for large enough $n$ which implies $\alpha^L > 1$. For sufficiently large $k_0$ choose $\epsilon_0 > \frac{n_{k-1}}{n_k} + \frac{1}{n_k}$ such that

$$\frac{m_k L}{n_k} = \frac{n_k}{n_k} - \frac{n_{k-1}}{n_k} - \frac{1}{n_k} \geq 1 - \epsilon_0, \quad \text{for all } k > k_0. \tag{6.16}$$

**Case 1.** When $n = mL$ for some $1 \leq m < m_1$.

$$\begin{aligned}
\mu(J_{mL}(a_1, \ldots, a_{mL})) &\leq \prod_{t=0}^{m-1} \left( \frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})} \right)^s \\
&\leq \prod_{t=0}^{m-1} \left( \frac{1}{q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})} \right)^s.
\end{aligned}$$

$$\begin{aligned}
\mu(J_{mL}(a_1, \ldots, a_{mL})) \leq (4^{m-1}) \left( \frac{1}{q_{mL}^2(a_1, \ldots, a_{mL})} \right)^s \quad &(\text{by } (2.5)) \\
= \left( \frac{1}{q_{mL}^2(a_1, \ldots, a_{mL})} \right)^{s - \frac{2}{L}} \quad &(\text{by } (\mathrm{P}_3) \text{ of Lemma } 2.3.3) \\
\leq 6 |J_{mL}(a_1, \ldots, a_{mL})|^{s - \frac{2}{L}} \quad &(\text{by } (6.12)).
\end{aligned}$$

**Case 2.** When $n = m_1 L = n_1 - 1$.

$$\begin{aligned}
\mu(J_{m_1 L}(a_1, \ldots, a_{m_1 L})) &\leq \prod_{t=0}^{m_1 - 1} \left( \frac{1}{\alpha^L q_L^2(a_{tL+1}, \ldots, a_{(t+1)L})} \right)^s \\
&\leq \left( \frac{1}{\alpha^{m_1 L}} \right)^s \left( \frac{1}{q_{m_1 L}^2(a_1, \ldots, a_{m_1 L})} \right)^{s - \frac{2}{L}} \\
&\leq \left( \frac{1}{\alpha^{1 - \epsilon_0}} \right)^{s n_1} \left( \frac{1}{q_{m_1 L}^2(a_1, \ldots, a_{m_1 L})} \right)^{s - \frac{2}{L}} \quad (\text{by } (6.16)) \\
&\leq \left( \frac{1}{\alpha^{n_1} q_{n_1 - 1}^2} \right)^{s - \frac{2}{L} - \epsilon_0} \qquad (6.17) \\
&\leq 12 |J_{m_1 L}(a_1, \ldots, a_{m_1 L})|^{s - \frac{2}{L} - \epsilon_0} \quad (\text{by } (6.14)).
\end{aligned}$$

**Case 3.** When $n = m_1L + 1 = n_1$.

$$\begin{aligned}
\mu(J_{n_1}(a_1,\dots,a_{n_1})) &= \frac{1}{2\alpha^{n_1}}\mu(J_{n_1-1}(a_1,\dots,a_{n_1-1}))\\
&\le \frac{1}{2\alpha^{n_1}}\left(\frac{1}{\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \quad \text{(by (6.17))}\\
&= \frac{1}{2B^{sn_1}}\left(\frac{1}{\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \quad (\ \alpha = B^s)\\
&\le \frac{1}{2}\left(\frac{1}{B^{n_1}\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0}\\
&\le 16|J_{n_1}(a_1,\dots,a_{n_1})|^{s-\frac{2}{L}-\epsilon_0} \text{ (by (6.15)).}
\end{aligned}$$

**Case 4.** When $n = n_1 + 1$.

$$\begin{aligned}
\mu(J_{n_1+1}(a_1,\dots,a_{n_1+1})) &= \frac{2\alpha^{n_1}}{B^{n_1}}\mu(J_{n_1})\\
&\le \frac{2\alpha^{n_1}}{2B^{n_1}}\left(\frac{1}{B^{n_1}\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0}\\
&\le \left(\frac{1}{B^{2n_1}\alpha^{n_1}q_{n_1-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0}\\
&\le 24|J_{n_1+1}(a_1,\dots,a_{n_1+1})|^{s-\frac{2}{L}-\epsilon_0} \text{ (by (6.13)).}
\end{aligned}$$

Here for the second inequality, we use $B/\alpha \ge (B/\alpha)^s$ which is always true for $\alpha \le B$ and $s \le 1$.

For a general fundamental interval, we only give the estimation on the measure of $J_{n_k-1}(a_1,\dots,a_{n_k-1})$. The estimation for other fundamental intervals can be carried out similarly. Recall that

$$\begin{aligned}
\mu(J_{n_k-1}(a_1,\dots,a_{n_k-1})) \quad = \quad & \mu\left(J_{n_{k-1}+1}(a_1,\dots,a_{n_{k-1}+1})\right)\cdot\\
& \prod_{t=0}^{m_k-1}\frac{1}{w}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\dots,a_{n_{k-1}+(t+1)L})}\right)^s.
\end{aligned}$$

This further implies,

$$
\begin{aligned}
&\mu\left(J_{n_k-1}(a_1,\ldots,a_{n_k-1})\right)\\
&\leq \left[\prod_{j=1}^{k-1}\left(\frac{1}{B^{n_j}}\prod_{t=0}^{m_j-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{j-1}+tL+1},\ldots,a_{n_{j-1}+(t+1)L})}\right)^s\right)\right]\\
&\qquad\cdot\prod_{t=0}^{m_k-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\ldots,a_{n_{k-1}+(t+1)L})}\right)^s\\
&\leq \left[\prod_{j=1}^{k-1}\left(\frac{1}{B^{n_j}}\prod_{t=0}^{m_j-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{j-1}+tL+1},\ldots,a_{n_{j-1}+(t+1)L})}\right)^s\right)\right]\\
&\qquad\cdot\prod_{t=0}^{m_k-1}\left(\frac{1}{\alpha^L q_L^2(a_{n_{k-1}+tL+1},\ldots,a_{n_{k-1}+(t+1)L})}\right)^s.
\end{aligned}
$$

By similar arguments as used in **Case 4** for the first product in the line above, and in **Case 2** for the second product in the line above, we obtain

$$
\begin{aligned}
&\mu\left(J_{n_k-1}(a_1,\ldots,a_{n_k-1})\right)\\
&\leq \prod_{j=1}^{k-1}\left(\frac{1}{B^{2n_j} q_{m_jL}^2(a_{n_{j-1}+tL+1},\ldots,a_{n_{j-1}+(t+1)L})}\right)^{s-\frac{2}{L}-\epsilon}\\
&\qquad\cdot\left(\frac{1}{\alpha^{n_k} q_{m_kL}^2(a_{n_{k-1}+tL+1},\ldots,a_{n_{k-1}+(t+1)L})}\right)^{s-\frac{2}{L}-\epsilon_0}\\
&\leq 4^{2k}\left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0} \leq \left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}\\
&\leq 12|J_{n_k-1}(a_1,\ldots,a_{n_k-1})|^{s-\frac{6}{L}-\epsilon_0} \qquad \text{(by (6.14))}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mu(J_{n_k}(a_1,\ldots,a_{n_k})) &= \frac{1}{2\alpha^{n_k}}\mu\left(J_{n_{k-1}}(a_1,\ldots,a_{n_{k-1}})\right)\\
&\leq \frac{1}{2(B^s)^{n_k}}\left(\frac{1}{\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}\\
&\leq \frac{1}{2}\left(\frac{1}{B^{n_k}\alpha^{n_k}q_{n_k-1}^2}\right)^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}\\
&\leq 16|J_{n_k}(a_1,\ldots,a_{n_k})|^{s-\frac{6}{L}-\epsilon_0} \qquad \text{(by (6.15))}.
\end{aligned}
$$

**Summary 6.1.1** We have shown that for any $n \geq 1$ and $(a_1,\ldots,a_n) \in D_n$

$$\mu\left(J_n\left(a_1,\ldots,a_n\right)\right) \ll \left|J_n\left(a_1,\ldots,a_n\right)\right|^{s-\frac{2}{L}-\epsilon_0-\frac{4}{L}}.$$ ■

#### 6.1.2.5 Estimation of $\mu(B(x,r))$.

First we estimate the gaps between the adjoint fundamental intervals, defined in Eq. (6.7), of the same order. This will be useful for estimating $\mu(B(x,r))$.

Let us start by assuming $n$ is even (similar steps can be followed when $n$ is odd). Then for $(a_1, a_2, \ldots, a_n) \in D_n$, given a fundamental interval $J_n(a_1, a_2, \ldots, a_n)$, represent the distance between $J_n(a_1, a_2, \ldots, a_n)$ and its left (respectively right) adjoint fundamental interval say

$$J'_n = J'_n(a_1, \cdots, a_{n-1}, a_n - 1) \text{ (if exists)}$$

(respectively, $J''_n = J''_n(a_1, \cdots, a_{n-1}, a_n + 1)$) of order $n$ by $g^l(a_1, \ldots, a_n)$ (respectively, $g^r(a_1, \ldots, a_n)$). Let

$$G_n(a_1, a_2, \ldots, a_n) = \min\left\{g^r(a_1, a_2, \ldots, a_n), g^l(a_1, a_2, \ldots, a_n)\right\}.$$

Again we will consider three different cases according to the range of $n$ as in Eq. (6.8)-Eq. (6.10) for $\mathcal{F}_M(B)$ in order to estimate the lengths of gaps on both sides of fundamental intervals $J_n(a_1, a_2, \ldots, a_n)$. The three cases are found in Table 2.

**Gap I.** When $n_{k-1} + 1 \le n \le n_k - 2$, for all $k \ge 1$.

There exists a basic interval of order $n$ contained in $I_{n-1}(a_1, a_2, \ldots, a_{n-1})$ which lies on the left of $I_n(a_1, a_2, \ldots, a_n)$, also there exists a basic interval of order $n$ contained in $I_{n-1}(a_1, a_2, \ldots, a_{n-1})$ which lies on the right of $I_n(a_1, a_2, \ldots, a_n)$. In this case, $(a_1, a_2, \ldots, a_n - 1) \in D_n, (a_1, a_2, \ldots, a_n + 1) \in D_n$, whereas $g^l(a_1, a_2, \ldots, a_n)$ is just the distance between the right endpoint of $J'_n(a_1, a_2, \ldots, a_n - 1)$ and the left endpoint of $J_n(a_1, a_2, \ldots, a_n)$.

The right endpoint of $J'_n(a_1, a_2, \ldots, a_n - 1)$ is the same as the left endpoint of $I_n(a_1, a_2, \ldots, a_n)$. Since $n$ is even, from Eq. (2.2) this has formula $\frac{p_n}{q_n}$.

Note that the left endpoint of $J_n(a_1, a_2, \ldots, a_n)$ lies on the extreme left of all the constituent intervals

$$\{I_{n+1}(a_1, a_2, \ldots, a_{n-1}, a_n, a_{n+1}) : 1 \le a_{n+1} \le M\}.$$

This tells us that $a_{n+1} = M$. Since $n+1$ is odd, again from Eq. (2.2) this has formula

$$\frac{(Mp_n + p_{n-1}) + p_n}{(Mq_n + q_{n-1}) + p_n} = \frac{(M+1)p_n + p_{n-1}}{(M+1)q_n + q_{n-1}}.$$

Therefore, we have

$$\begin{aligned} g^l(a_1, a_2, \ldots, a_n) &= \frac{(M+1)p_n + p_{n-1}}{(M+1)q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{p_{n-1}q_n - q_{n-1}p_n}{((M+1)q_n + q_{n-1})q_n} \\ &= \frac{1}{((M+1)q_n + q_{n-1})q_n}. \end{aligned}$$

Whereas in this case $g^r(a_1, a_2, \ldots, a_n)$ is just the distance between the right endpoint of $J_n(a_1, a_2, \ldots, a_n)$ and the left endpoint of $J''_n(a_1, a_2, \ldots, a_n + 1)$.

The right endpoints of $J_n(a_1, a_2, \ldots, a_n)$ and $I_n(a_1, a_2, \ldots, a_n)$ are the same. Since $n$ is even, again using equation Eq. (2.2) this has formula

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Also, the left endpoint of $J''_n(a_1, a_2, \ldots, a_n + 1)$ lies on the extreme left of all the constituent intervals $\{I_{n+1}(a_1, a_2, \ldots, a_{n-1}, a_n + 1, a_{n+1}) : 1 \leq a_{n+1} \leq M\}$. This tells us that $a_{n+1} = M$. Since $n + 1$ is odd, from ($\mathrm{P}_3$) this is given by

$$\frac{(M+1)[(a_n+1)p_{n-1} + p_{n-2}] + p_{n-1}}{(M+1)[(a_n+1)q_{n-1} + q_{n-2}] + q_{n-1}} = \frac{(M+1)(p_n + p_{n-1}) + p_{n-1}}{(M+1)(q_n + q_{n-1}) + q_{n-1}}.$$

Therefore,

$$\begin{aligned} g^r(a_1, a_2, \ldots, a_n) &= \frac{(M+1)(p_n + p_{n-1}) + p_{n-1}}{(M+1)(q_n + q_{n-1}) + q_{n-1}} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \\ &= \frac{1}{((M+1)(q_n + q_{n-1}) + q_{n-1})(q_n + q_{n-1})}. \end{aligned}$$

Hence

$$G_n(a_1, a_2, \ldots, a_n) = \frac{1}{((M+1)(q_n + q_{n-1}) + q_{n-1})(q_n + q_{n-1})}.$$

Also, by comparing $G_n(a_1, \ldots, a_n)$ with $J_n(a_1, \ldots, a_n)$ we notice that

$$G_n(a_1, \ldots, a_n) \geq \frac{1}{2M}|J_n(a_1, \ldots, a_n)|.$$

**Gap II.** When $n = n_k - 1$, we have

In this case the left gap $g^l(a_1, a_2, \ldots, a_n)$ is larger than the distance between the left endpoint of $I_n(a_1, a_2, \ldots, a_{n-1}, a_n)$ and the left endpoint of $J_n(a_1, a_2, \ldots, a_{n-1}, a_n)$ whereas the right gap $g^r(a_1, a_2, \ldots, a_n)$ is larger than the distance between the right endpoint of $I_n(a_1, a_2, \ldots, a_{n-1}, a_n)$ and the right endpoint of $J_n(a_1, a_2, \ldots, a_{n-1}, a_n)$.

Thus proceeding in the similar way as in Gap I, we obtain

$$\begin{aligned} g^l(a_1, a_2, \ldots, a_n) &\geq \frac{(2\alpha^n + 1)p_n + p_{n-1}}{(2\alpha^n + 1)q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{1}{((2\alpha^n + 1)q_n + q_{n-1})q_n}. \end{aligned}$$

and the left gap is

$$\begin{aligned} g^r(a_1, a_2, \ldots, a_n) &\geq \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{(2\alpha^n + 1)p_n + p_{n-1}}{(2\alpha^n + 1)q_n + q_{n-1}} \\ &= \frac{1}{((2\alpha^n + 1)p_n + p_{n-1})(q_n + q_{n-1})} \end{aligned}$$

Therefore,

$$g^r(a_1, a_2, \ldots, a_n) \geq \frac{2\alpha^n}{((2\alpha^n+1)q_n + q_{n-1})(q_n+q_{n-1})}.$$

Thus

$$G_n(a_1, a_2, \ldots, a_n) \geq \frac{1}{((2\alpha^n+1)q_n + q_{n-1})(q_n+q_{n-1})}.$$

Further, in this case we have

$$G_n(a_1, \ldots, a_n) \geq \frac{1}{2}|J_n(a_1, \ldots, a_n)|.$$

**Gap III.** When $n = n_k$. Following the similar arguments as in Gap II we conclude

$$\begin{aligned} g^l(a_1, a_2, \ldots, a_n) &\geq \frac{\left(\frac{B^n}{\alpha^n}+1\right)p_n + p_{n-1}}{\left(\frac{B^n}{\alpha^n}+1\right)q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{1}{\left(\left(\frac{B^n}{\alpha^n}+1\right)q_n + q_{n-1}\right)q_n}, \end{aligned}$$

and the right gap can be estimated as

$$\begin{aligned} g^r(a_1, a_2, \ldots, a_n) &\geq \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{\left(\frac{B^n}{2\alpha^n}+1\right)p_n + p_{n-1}}{\left(\frac{B^n}{2\alpha^n}+1\right)q_n + q_{n-1}} \\ &= \frac{\frac{B^n}{2\alpha^n}}{\left(\left(\frac{B^n}{2\alpha^n}+1\right)q_n + q_{n-1}\right)(q_n+q_{n-1})}. \end{aligned}$$

Thus we have,

$$G_n(a_1, a_2, \ldots, a_n) \geq \frac{1}{\left(\left(\frac{B^n}{\alpha^n}+1\right)q_n + q_{n-1}\right)(q_n+q_{n-1})},$$

and

$$G_n(a_1, \ldots, a_n) \geq \frac{1}{4}|J_n(a_1, \ldots, a_n)|.$$

#### 6.1.2.6 The Measure $\mu$ on a General Ball $B(x, r)$

Now we are in a position to estimate the measure $\mu$ on general ball $B(x, r)$. Fix $x \in \mathcal{F}_M(B)$ and let $B(x, r)$ be a ball centred at $x$ with radius $r$ small enough. There exists a unique sequence $a_1, a_2, \cdots a_n$ such that $x \in J_n(a_1, \cdots, a_n)$ for each $n \geq 1$ and

$$G_{n+1}(a_1, \ldots, a_{n+1}) \leq r < G_n(a_1, \ldots, a_n).$$

It is clear, by the definition of $G_n$ that $B(x, r)$ can intersect only one fundamental interval of order $n$, that is, $J_n(a_1, \ldots, a_n)$. The work proceeds according to the three cases described in Table 2.

**Case I.** $n = n_k$. Since in this case

$$|I_{n_k+1}(a_1, \ldots, a_{n_k+1})| = \frac{1}{q_{n_k+1}(q_{n_k+1}+q_{n_k})} \geq \frac{1}{6a^2_{n_{k+1}}\, q^2_{n_k}} \geq \frac{\alpha^{2n_k}}{6B^{2n_k}q^2_{n_k}},$$

the number of fundamental intervals of order $n_k+1$ contained in $J_{n_k}(a_1,\dots,a_{n_k})$ that the ball $B(x,r)$ intersects is at most

$$2r\frac{6B^{2n_k}}{\alpha^{2n_k}}q_{n_k}^2+2 \le 24r\frac{B^{2n_k}}{\alpha^{2n_k}}q_{n_k}^2.$$

Therefore,

$$\begin{aligned}
\mu(B(x,r)) &\le \min\Big\{\mu(J_{n_k}), 24r\frac{B^{2n_k}}{\alpha^{2n_k}}q_{n_k}^2\mu(J_{n_k+1})\Big\}\\
&\le \mu(J_{n_k})\min\Big\{1, 48r\frac{B^{n_k}}{\alpha^{n_k}}q_{n_k}^2\Big\}\\
&\le c|J_{n_k}|^{s-\frac{6}{L}-\epsilon_0}\min\Big\{1, 48r\frac{B^{n_k}}{\alpha^{n_k}}q_{n_k}^2\Big\}\\
&\le c\Big(\frac{2\alpha^{n_k}}{B^{n_k}q_{n_k}^2}\Big)^{s-\frac{6}{L}-\epsilon_0}(48r\frac{B^{n_k}}{\alpha^{n_k}}q_{n_k}^2)^{s-\frac{6}{L}-\epsilon_0}\\
&\le c_0 r^{s-\frac{6}{L}-\epsilon_0}.
\end{aligned}$$

Here we use $\min\{a,b\}\le a^{1-s}b^s$ for any $a,b>0$ and $0\le s\le 1$.

**Case II.** $n=n_k-1$. In this case, since

$$|I_{n_k}(a_1,\dots,a_{n_k})| = \frac{1}{q_{n_k}(q_{n_k}+q_{n_k-1})} \ge \frac{1}{6a_{n_k}^2\,q_{n_k-1}^2} \ge \frac{1}{24\alpha^{2n_k}q_{n_k-1}^2},$$

the number of fundamental intervals of order $n_k$ contained in $J_{n_k-1}(a_1,\dots,a_{n_k-1})$ that the ball $B(x,r)$ intersects is at most

$$48r\alpha^{2n_k}q_{n_k-1}^2+2 \le 96r\alpha^{2n_k}q_{n_k-1}^2.$$

Therefore,

$$\begin{aligned}
\mu(B(x,r)) &\le \min\Big\{\mu(J_{n_k-1}), 96r\alpha^{2n_k}q_{n_k-1}^2\mu(J_{n_k})\Big\}\\
&\le \mu(J_{n_k-1})\min\Big\{1, 48r\alpha^{n_k}q_{n_k-1}^2\Big\}\\
&\le 12|J_{n_k-1}|^{s-\frac{6}{L}-\epsilon_0}\min\Big\{1, 48r\alpha^{n_k}q_{n_k-1}^2\Big\}\\
&\le 12\Big(\frac{1}{2\alpha^{n_k}q_{n_k-1}^2}\Big)^{s-\frac{6}{L}-\epsilon_0}(48r\alpha^{n_k}q_{n_k-1}^2)^{s-\frac{6}{L}-\epsilon_0}\\
&\le c_0 r^{s-\frac{6}{L}-\epsilon_0}.
\end{aligned}$$

**Case III.** $n_{k-1}+1\le n\le n_k-2$. Since in this case $1\le a_n(x)\le M$ and $|J_n|\asymp 1/q_n^2$

thus we have

$$\begin{aligned}
\mu(B(x,r)) \leq \mu(J_n) &\leq c|J_n|^{s-\frac{6}{L}-\epsilon_0} \\
&\leq c\left(\frac{1}{q_n^2}\right)^{s-\frac{6}{L}-\epsilon 0} \\
&\leq c4M^2\left(\frac{1}{q_{n+1}^2}\right)^{s-\frac{6}{L}-\epsilon_0} \\
&\leq c24M^2|J_{n+1}|^{s-\frac{6}{L}-\epsilon 0} \\
&\leq c48M^3G_{n+1}^{s-\frac{6}{L}-\epsilon} \\
&\leq c48M^3r^{s-\frac{6}{L}-\epsilon}.
\end{aligned}$$

#### 6.1.2.7 Conclusion for the Lower Bound

Thus, combining all the above cases and applying the mass distribution principle[1], we have shown that $\dim_{\mathcal{H}}\mathcal{F}_M(B) \geq s - \frac{6}{L} - \epsilon_0$. Now letting $L \to \infty$, $M \to \infty$, by the choice of $\epsilon_0$ for all large enough $k$ and since $s < t_B$ is arbitrary, we have $s - \frac{6}{L} - \epsilon_0 \to t_B$.

Thus we have,

$$\dim_{\mathcal{H}}\mathcal{F}(B) \geq \dim_{\mathcal{H}}\mathcal{F}_M(B) \geq t_B. \tag{6.18}$$

Taken together results Eq. (6.5) and Eq. (6.18), completes the proof of the desired theorem for the case $1 < B < \infty$. ■

## 6.2 Case (ii): $B = \infty$

Next we prove Theorem 1.7.6 for the case when $B = \infty$.

**Proof:** When $B = \infty$, one notes that

$$a_n(x)a_{n+1}(x) \geq \Phi(n) \implies a_n(x) \geq \Phi(n)^{\frac{1}{2}} \text{ or } a_{n+1}(x) \geq \Phi(n)^{\frac{1}{2}}.$$

Thus

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \mathcal{G}_1(\Phi) \cup \mathcal{G}_2(\Phi), \tag{6.19}$$

where

$$\mathcal{G}_1(\Phi) := \left\{x \in [0,1) : a_n(x) \geq \Phi(n)^{1/2} \text{ for infinitely many } n \in \mathbb{N}\right\}$$

and

$$\mathcal{G}_2(\Phi) := \left\{x \in [0,1) : a_{n+1}(x) \geq \Phi(n)^{1/2} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

There are three cases.

[1] Note that in the statement of the mass distribution principle, Proposition 2.5.3, the measure of a general set is compared to its diameter. However, it can simply be tailored to compare the measure of a ball to its radius.

**(ii)a.** If $b = 1$. Then for any $\delta > 0$, $\frac{\log\log\Phi(n)}{n} \leq \log(1+\delta)$ that is $\Phi(n) \leq e^{(1+\delta)^n}$ for infinitely many $n \in \mathbb{N}$. Since

$$\left\{x \in [0,1) : a_n(x) \geq e^{(1+\delta)^n} \text{ for all sufficiently large } n \in \mathbb{N}\right\} \subset \mathcal{F}(\Phi).$$

Therefore, by using Lemma 2.3.5

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) \geq \lim_{\delta \to 0} \frac{1}{1+1+\delta} = \frac{1}{2}.$$

Note that as $B = \infty$, therefore for any $C > 1$, $\Phi(n) \geq C^n$ for all sufficiently large $n \in \mathbb{N}$. Thus by (6.19)

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \{x \in [0,1) : a_n(x) \geq C^n \text{ for infinitely many n} \in \mathbb{N}\}.$$

By Theorem 1.7.2 and Proposition 1.7.3

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) \leq \lim_{C \to \infty} s_C = \frac{1}{2}.$$

**(ii)b.** If $1 < b < \infty$. For any $\delta > 0$, $\frac{\log\log\Phi(n)}{n} \leq \log(b+\delta)$ that is $\Phi(n) \leq e^{(b+\delta)^n}$ for infinitely many $n \in \mathbb{N}$, whereas $\Phi(n) \geq e^{(b-\delta)^n}$ for all sufficiently large $n \in \mathbb{N}$. Since

$$\left\{x \in [0,1) : a_n(x) \geq e^{(1+\delta)^n} \text{ for all sufficiently large } n \in \mathbb{N}\right\} \subset \mathcal{F}(\Phi).$$

Therefore, by using Lemma 2.3.5

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) \geq \lim_{\delta \to 0} \frac{1}{1+b+\delta} = \frac{1}{1+b}.$$

Further note that from the definition of the set $\mathcal{G}_i(\Phi)$ it is clear that

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \left\{x \in [0,1) : a_n(x) \geq e^{\frac{1}{2(b-\delta)}(b-\delta)^n} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

By Lemma 2.3.5

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) \leq \lim_{\delta \to 0} \frac{1}{1+b-\delta} = \frac{1}{1+b}.$$

**(ii)c.** If $b = \infty$. Then by using the same argument as for showing the upper bound in case 2b we have for any $C > 1$, $\Phi(n) \geq e^{C^n}$ for all sufficiently large $n \in \mathbb{N}$. Thus by (6.19)

$$\mathcal{F}(\Phi) \subseteq \mathcal{E}_2(\Phi) \subset \left\{x \in [0,1) : a_n(x) \geq e^{C^n} \text{ for infinitely many n} \in \mathbb{N}\right\}.$$

By Theorem 1.7.2 and Proposition 1.7.3

$$\dim_{\mathcal{H}} \mathcal{F}(\Phi) \leq \lim_{C \to \infty} \frac{1}{C+1} = 0.$$

This completes the proof of the desired theorem for the case when $B = \infty$. ∎

With Case (i) and Case (ii) both proven, we have completed the proof of Theorem 1.7.6.

## 6.3 A Generalisation

It is possible to generalise the set $\mathcal{F}(\Phi)$ to the more general set of the form, for any $m \geq 2$

$$\mathcal{F}_m(\Phi) = \left\{ x \in [0,1) : \begin{array}{l} \displaystyle\prod_{k=1}^{m} a_{n+k-1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \textit{ and} \\ \displaystyle\prod_{k=1}^{m-1} a_{n+k-1}(x) < \Phi(n) \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

By following the same method as we have used for the proof of Theorem 1.7.6, we can show that:

**Theorem 6.3.1** *Let $\Phi : \mathbb{N} \to (1, \infty)$ be any function with $\lim_{n\to\infty} \Phi(n) = \infty$. Define $B, b$ as in Theorem 1.7.6. Then*

- $\dim_{\mathcal{H}} \mathcal{F}_m(\Phi) = \inf\{s \geq 0 : \mathsf{P}(T, -g_m \log B - s \log |T'|) \leq 0\}$ *when* $1 < B < \infty$, *where* $g_1 = s$, $g_m = \frac{s g_{m-1}(s)}{1-s+g_{m-1}(s)}$ *for* $m \geq 2$;
- $\dim_{\mathcal{H}} \mathcal{F}_m(\Phi) = 1/(1+b)$ *when* $B = \infty$. ∎

The proof of Theorem 6.3.1 involved lengthy calculations but the method of proof is the same as for Theorem 1.7.6.

# Conclusion

This thesis has made several contributions to the theory of uniform Diophantine approximation. The main contributions are three major results.

In chapter 4, we proved Theorem 1.5.9. Theorem 1.5.9 calculates the generalised Hausdorff $f$-measure of $D(\psi)^c$, the set of $\psi$-Dirichlet non-improvable numbers, for the non-essentially sub-linear dimension functions. This result is important because it completes the Hausdorff measure theory for the set of Dirichlet non-improvable numbers.

In chapter 5, we proved Theorem 1.6.2. Theorem 1.6.2 completely determines the Hausdorff dimension for the set $G(\Psi) \setminus \mathcal{K}(C\Psi)$ for any $C > 0$. This result is important because the theorem implies that there are uncountably more Dirichlet non-improvable numbers than the $\psi$-approximable numbers.

In chapter 6, we proved Theorem 1.7.6. Theorem 1.7.6 estimates the size of the set $\mathcal{F}(\Phi) = \mathcal{E}_2(\Phi) \setminus \mathcal{E}_1(\Phi)$ by calculating the $\dim_{\mathcal{H}}\mathcal{F}(\Phi)$. This result is important because it continues the work that started with Borel-Bernstein [9], and extends the research, of more than one hundred years, improving their fundamental result.

There are many open problems concerning the theory of uniform Diophantine approximation which the author aims to explore in the near future. One such problem may be to explore analogous results of this thesis in the inhomogeneous setting and over complex numbers. As can be seen from this thesis, $\psi$-Dirichlet improvability connects with the growth of the product of consecutive partial quotients. The theory of continued fractions is well developed in the one-dimensional, homogeneous setting. This makes studying the structure of related sets relatively easier. However, for the inhomogeneous settings, we have to appeal to a blend of ideas from the theory of homogeneous dynamics along with the geometry of numbers as demonstrated by [35].

Another problem to investigate, is the theory of uniform Diophantine approximation (akin to the problems discussed in this thesis) over the field of complex numbers. The theory of continued fractions is still evolving for complex numbers. This may in itself be an interesting avenue of research [12].

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