La Trobe University

Masters Dissertation

Undecidability from First Principles

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Abstract

In his 2005 paper "Undecidability without arithmetization", Andrzej Grzegorczyk proved an undecidability result for a theory of concatenation over an alphabet with at least two letters. His techniques can be likened to that of the proof of Gödel's first incompleteness theorem, but is a bit simpler.

In this thesis, we reformulate some of Grzegorczyk's results in a more clear and rigorous manner, correct some minor errors and connect some definitions he constructed from first principles to more standard definitions in the literature. Notably, Grzegorczyk relied on a notion of decidability he defined from first principles. The main result of this thesis is that we show Grzegorczyk's notion of decidability to be equivalent to decidability by a Turing machine, thereby showing that the theory of concatenation is undecidable in the sense we expect, rather than in some weaker sense.

Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person's work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

The core strategy for proving GD $\Longrightarrow$ Recursive in Chapter 5 came out of joint discussions with my supervisors, Tomasz Kowalski and Marcel Jackson. The successful implementation of this approach is my own work.

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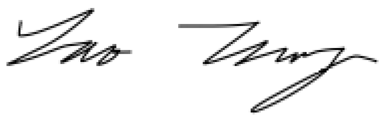

Yao Tang, 28 September 2020

Introduction

Roughly speaking, a theory is a set of statements, like "$1 + 1 = 2$", or "all primes are odd", or "Socrates was mortal". Given a context, each statement can be true or false; for instance, if "1", "2", "+" and "=" have their usual arithmetic meaning, then "$1 + 1 = 2$" is true, whereas if we are looking at the 2-element group $(\{1, 2\}; +, 1, ^{-1})$ with $1 + 1 = 1$, $1 + 2 = 2 = 2 + 1$ and $2 + 2 = 1$, then "$1 + 1 = 2$" is false. Regardless, each statement must always have a well-defined truth value in the context of a specific model.

A theory is *decidable* if there is an algorithm such that, given a statement (in a particular formal language), the algorithm tells us whether the statement is in the theory or not. In classical first order predicate logic, when a theory has a recursively enumerable axiomatization, then systematic application of a finite set of rules of logical deduction to these axioms is an algorithmic procedure. A consequence is then that if a theory is complete and is generated from a recursively enumerable set of axioms, it must be decidable. More generally, for each consistent theory T in classical first order predicate logic, at most two of the following are true:

- T is complete;
- T is undecidable;
- T has a recursively enumerable axiomatization.

For instance,

- The theory of $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ was shown by Tarski to be decidable [**16**]. Being the theory of a model, it is also complete, hence it must have a recursive axiomatisation.
- The theory of $(\mathbb{N}, +, \cdot, 0, 1, \leq)$ is undecidable, by Gödel's first incompletness theorem [**5**]. Because it is complete and undecidable, it follows that any recursive set of formulae true in $(\mathbb{N}, +, \cdot, 0, 1, \leq)$ is not complete.

In this thesis, we consider two theories: a theory TC (for Theory of Concatenation) generated by a finite set of axioms, and a theory Th(**Tx**) that contains precisely the statements that are true in an actual, concrete structure; the free semigroup on 2 generators (which we call **Tx**). We show that both of these theories are undecidable, but their undecidability says very different things about them:

- The theory TC is finitely axiomatizable, so if it is undecidable, it must be incomplete; there are some statements P (in the relevant formal language) such that neither P nor $\neg P$ are in TC. Informally, the axioms don't provide enough information to describe everything. As an analogy, you can't figure out whether or not $7 + 8 = 100$ if all you know is that "$1 + 1 = 2$".

- The theory Th(**Tx**) contains everything that is true in an actual, concrete structure, and everything must be either true or false in an actual, concrete structure, so it's complete. Thus, if it is undecidable, it follows that it has no recursively enumerable axiomatization. Informally, the theory is incredibly complicated; there is no easy way to describe all the information it contains.

1. Diagonal Arguments

This thesis is built around a proof due to Grzegorczyk [**8**]; it defines a predicate F such that $F(t)$ means "t is not the encoding of a one-argument predicate such that F holds of the encoding of the 'name' of t". In short, the predicate F describes a self-referential statement; the property t has to satisfy depends on t itself. Grzegorczyk shows that if we assume TC or Th(**Tx**) (or anything in between) is decidable, then by applying F to an encoding of F, a contradiction emerges.

Gödel's First Incompleteness Theorem [**5**] was based around a similar argument about sufficiently rich theories of arithmetic. He did not specify a particular theory of arithmetic, but theories such as Peano Arithmetic ([**11**]; see [**9**]) and the later Robinson Arithmetic are amenable to his argument. This result, along with the Second Incompleteness Theorem, dealt a fatal blow to Hilbert's Program whose goal was to axiomatize all of mathematics, thereby marking Gödel's results as arguably some of the most notable discoveries in the 20^{th} century.

Note that Gödel's result showed incompleteness, but it is a sort of "essential incompleteness". That is, all theories *containing* such an "essentially incomplete" theory are incomplete. For first order theories with a recursively enumerable axiomatization, this is equivalent to essential undecidability, so theories such as Peano arithmetic and Robinson Arithmetic are also (essentially) undecidable. Hence a standard method of proving undecidability of other theories is to interpret one such arithmetic in that theory, then referring to its essential undecidability.

Theories whose undecidability has been shown in this way include the theory of:

- rings [**17**]
- groups [**17**]
- finite graphs [**3**]
- distributive lattices [**7**]
- sets with two equivalence relations [**14**], and
- **Tx** [**17**]

For Th(**Tx**), however, using a direct approach like Grzegorczyk's would seem more natural.

2. Notions of Decidability

Grzegorczyk defined his own notion of decidability from first principles which he called General Discernibility, or GD. He never claimed that a theory being GD was equivalent to being recursive; only that it was a notion of discernibility that seemed natural. A proof of this equivalence is given in Chapter 5, and is the main original contribution of this thesis.

Showing that GD sets are recursive is a rather straightforward if cumbersome inductive argument; however, formulating an approach to prove the converse is less clear. The seed of an idea for the converse direction can be found in the earlier work of Quine [**12**] and Švejdar [**15**]. Both these authors are concerned with the interpretation of multiplication within similar theories of concatenation. They introduce definitions that rely on constructing witnesses to a step by step unfolding of the process of multiplying. While both authors work in the full first order theory rather than in a restricted framework such as General Discernibility, there are elements of their approach that can be adapted. In particular, it suggests the idea of proving that recursive sets are GD by constructing a string that witnesses the acceptance of a string by a Turing machine, and show that it relates to the input relation in a GD way.

Grzegorczyk produced his undecidability result in order to argue that results like Gödel's First Incompleteness Theorem are linguistic in nature and should not fundamentally depend on arithmetic or the properties of numbers. After studying the constructions he defined, it seems that arithmetic is *needed* only if our theory is about a language on one letter; introduce a second letter and we can avoid using arithmetic. This is because with at least 2 distinct letters, we can encode expressions into one continuous string by using the first letter as an identifier for a symbol and the second letter as an indicator of when the symbol begins and ends. Without this second letter as a separator, we need another way to mark when a symbol begins or ends, so we are forced to use results about quantity; namely, that every number has a unique prime factorization.

Grzegorczyk and Zdanowski [**6**] later proved that everything that is GD can be constructed with at most one use of complementary projections; moreover, as the final step in the GD construction process.

3. Essential Undecidability of TC

Grzegorczyk proved undecidability but not essential undecidability of TC. He conjectured that Robinson Arithmetic (Q) [**13**] isn't interpretable in TC, but Ganea [**4**] and Švejdar [**15**] have since (independently) proved that Q is interpretable in a variant of TC, which is in turn interpretable in TC since "all reasonable variants of TC are mutually interpretable." (Švejdar, 2009) [**15**].

Since Q is not only undecidable but essentially undecidable, a consequence of this is that TC is also essentially undecidable. Grzegorczyk mentioned the problem of the undecidability of extensions of TC that are consistent but not contained in Th($\mathbf{Tx}$):

"The problem of the undecidability of the extensions of the theory TC, which are consistent but not true in (Tx,!, a,b), seems to be more difficult and till now remains open." (Grzegorczyk, 2005) [**8**].

We know now that such theories are undecidable due to the essential undecidability of TC. (Grzegorczyk himself also outlined a proof of essential undecidability of TC using his method in [**6**].)

4. Other Models of TC

TC was conceived as a theory of concatenation, but it's also a theory of decorated linear order types [**2**]. TC with a *collection principle* is also a theory of sets with adjunction [**18**].

5. Notation

In this thesis, we use notation from both Grzegorczyk's paper (which we will define) and the text *A Course in Universal Algebra* by Burris and Sankappanavar [**1**] (which we will not). We use the latter mostly when we speak of interpretations of terms/functions/predicates in models. The one exception to the above is how we denote substrings (by which we mean a contiguous block of letters; for instance, we allow *bb* as a substring of *bba* but not a substring of *bab*). We define the following notations:

$s \sqsubset t \iff s$ is a substring of t;

$s \sqsubset_p t \iff s$ is a prefix of t;

$s \sqsubset_s t \iff s$ is a suffix of t;

While some efforts towards self-containment have been made, the thesis will assume some low-level background knowledge of theories and models.

6. Overview

The following is an overview of a proof of undecidability from first principles. It is by no means universal; it makes a lot of assumptions that some first order theories (e.g. ZFC) may not satisfy. This thesis covers each step in this overview in detail for the special case of TC.

We take theories to be sets of sentences closed under logical consequence.

(i) Standard Model. Let T_0 be a consistent first-order theory formalised as a set of strings over a finite alphabet Σ such that if S is the set of all instances of a particular syntactic category of T_0 (i.e. formulae, terms, constants etc), then S is a recursively enumerable subset of Σ^+.

Assume T_0 has at least one constant, and let C be a set of interpretations of the constants of T_0. Let $\mathcal{M}$ be a structure freely generated by C with respect to the equalities true in T_0. This ensures that every element of $\mathcal{M}$ is an interpretation of a definable constant. We assume the universe and all basic relations of $\mathcal{M}$ are recursive sets, and that all basic operations of $\mathcal{M}$ are computable functions.

Assume $\mathcal{M} \models T_0$. We call $\mathcal{M}$ the standard model of T_0.

In chapters 1 and 2, we show that $T_0 = TC$ and $\mathcal{M} = \mathbf{Tx}$ satisfy these assumptions.

(ii) Naming. Let Cterm be the set of constant terms in the language of T_0. Define an equivalence relation on Cterm by $s \sim t$ iff $s \approx t \in T_0$. Let $N\colon M \to \mathsf{Cterm}$ (where M is the domain of $\mathcal{M}$) be a map picking a representative from the equivalence class of constant terms whose interpretation is the input. For each $m \in M$, we call $N(m)$ the *standard name* of m.

In chapter 3, we define $N\colon Tx \to \mathsf{Cterm}$. (In chapter 8, we show that N is computable.)

(iii) Computability. *In chapters 4 and 5, we characterize computability in a way that is easier for us to work with.*

(iv) Coding. Assume there is an injective map $\ulcorner\urcorner\colon \Sigma^+ \to M$ which takes a string s to its *code* $\ulcorner s \urcorner$. (In particular, every formula φ and every term t in the language of T_0 have their codes $\ulcorner\varphi\urcorner$ and $\ulcorner t\urcorner$, and each $m \in \mathcal{M}$ has its *coded standard name* $\ulcorner N(m)\urcorner$.) Assume the map $N' : M \to M$ defined by $m \mapsto \ulcorner N(m)\urcorner$ is computable. Let $\ulcorner X\urcorner := \{\ulcorner x\urcorner : x \in X\}$ for each $X \subseteq \Sigma^+$.

Assume there is a computable map $\mathsf{deco}\colon M \to \Sigma^+$ such that $\mathsf{deco}(\ulcorner s\urcorner) = s$ for any $s \in \Sigma^+$.

In chapter 8, we define a map $\langle\!\langle\rangle\!\rangle$ *and show that* $\ulcorner\urcorner = \langle\!\langle\rangle\!\rangle$ *satisfies these assumptions.*

(v) Representability. Let T be a theory in the language of T_0. We say that an n-ary relation $R \subseteq M^n$ is *representable* in T if there is a formula $\rho(x_1, \ldots, x_n)$ such that for all $m_1, \ldots, m_n \in M$ we have

$$R(m_1, \ldots, m_n) \quad \text{if and only if} \quad \rho(N(m_1), \ldots, N(m_n)) \in T.$$

We assume that:

(*) every computable relation $R \subseteq M^n$ and every computable function $f\colon M^n \to M$ is representable in T.

In particular, this means every computable $X \subseteq M$ is representable in T. By a standard logical trick involving renaming variables, we can assume that every computable subset of M is represented by a formula with precisely one free variable.

In chapter 7, we show that consistent extensions T of TC that are contained in Th($\mathbf{Tx}$) *satisfy these assumptions.*

(vi) Substitution. We assume there exists a computable function $\mathsf{Sub}\colon \mathcal{M}^2 \to \mathcal{M}$, such that for each set S of formulae, constant term t and formula φ with precisely one free variable x which occurs only once in φ, we have:

$$\mathsf{Sub}(\ulcorner\varphi\urcorner, \ulcorner t\urcorner) \in \ulcorner S\urcorner \quad \text{if and only if} \quad \varphi(t) \in S.$$

In particular, $\mathsf{Sub}(\ulcorner\varphi\urcorner, \ulcorner t\urcorner) = \ulcorner\varphi(t)\urcorner$.

In chapter 6, we define $\mathsf{Sub}\colon Tx^2 \to Tx$. *(In chapter 9, we show that* Sub *is computable.)*

(vii) Proof of undecidability. *In chapter 9, we prove the following results for consistent extensions T of TC that are contained in Th*($\mathbf{Tx}$).

Let T be a theory in the language of T_0 satisfying (*).

LEMMA 0.1. Let $X \subseteq M$ be defined by

$$m \in X \Leftrightarrow \mathsf{Sub}\big(m, \ulcorner N(m) \urcorner\big) \notin \ulcorner T \urcorner.$$

Then, if $\ulcorner T \urcorner$ is computable, so is X.

PROOF. Let $\psi(x) := \mathsf{Sub}(x, \ulcorner N(x) \urcorner)$. Then, ψ is a composition of computable functions, so ψ is computable.

Assume $\ulcorner T \urcorner$ is computable. Since the preimage of a computable set under a computable function is a computable set, this means $M \setminus \ulcorner T \urcorner$ is computable as well, and $X = \psi^{-1}(M \setminus \ulcorner T \urcorner)$. Thus X is computable. □

LEMMA 0.2. The set $\ulcorner T \urcorner$ is not computable.

PROOF. Suppose $\ulcorner T \urcorner$ is computable. Then X of Lemma 0.1 is also computable, and so it is represented in T by a formula $\rho(x)$ with precisely one free variable x. Thus, we have

$$m \in X \Leftrightarrow \rho(N(m)) \in T$$

for any $m \in M$. Consider $m = \ulcorner \rho \urcorner$. By the equivalence above and the properties of Sub we obtain

$$\begin{aligned} \ulcorner \rho \urcorner \in X &\Leftrightarrow \rho(N(\ulcorner \rho \urcorner)) \in T \\ &\Leftrightarrow \mathsf{Sub}\big(\ulcorner \rho \urcorner, \ulcorner N(\ulcorner \rho \urcorner) \urcorner\big) \in \ulcorner T \urcorner, \end{aligned}$$

which contradicts

$$\ulcorner \rho \urcorner \in X \Leftrightarrow \mathsf{Sub}\big(\ulcorner \rho \urcorner, \ulcorner N(\ulcorner \rho \urcorner) \urcorner\big) \notin \ulcorner T \urcorner.$$

which we have by definition of X. Hence $\ulcorner T \urcorner$ is not computable. □

THEOREM 0.1. T is undecidable.

PROOF. For each $m \in M$, it is decidable whether $\mathsf{deco}(m)$ is a sentence in the language of T, since the language of T is the same as the language of T_0.

Suppose T is decidable. Then the following procedure decides membership in $\ulcorner T \urcorner$:

Let $m \in M$. If $\mathsf{deco}(m)$ is not a sentence, return NO. If $\mathsf{deco}(m)$ is a sentence and $\mathsf{deco}(m) \in T$, return YES, otherwise return NO.

But $\ulcorner T \urcorner$ is not computable by Lemma 0.2, so T is not decidable. □

CHAPTER 1

The Elementary Theory of Concatenation

In this chapter, we define a theory of concatenation which we will eventually prove to be undecidable. We also give some examples of theorems in this theory. This chapter is based on Sections 1 and 3 under Part One of 'Undecidability without Arithmetization' [**8**], and is referenced by part (i) of the overview.

1. Defining the theory

The Elementary Theory of Concatenation (TC) is the set of all sentences (in a particular first order language) provable from the following axioms:

(Associativity) $(\forall x, y, z)\, x * (y * z) = (x * y) * z$

(Editor Axiom) $(\forall x, y, z, u)$

$$x * y = z * u \implies \Big((x = z \wedge y = u) \quad \text{or} \quad (\exists w)\big((x * w = z \wedge w * u = y) \\ \quad \text{or} \quad (z * w = x \wedge w * y = u)\big)\Big)$$

(Existence of 'atom' a) $(\forall x, y)\, \neg(\alpha = x * y)$

(Existence of 'atom' b) $(\forall x, y)\, \neg(\beta = x * y)$

(Non-equality of atoms) $\neg(\alpha = \beta)$

using a deduction system for first order logic. We leave the precise system unspecified; we only note that all logical tautologies (in the appropriate language) are provable and standard rules of proof (e.g., extensionality (which refers to the rule that if $x = y$ and $P(x)$, then $P(y)$), modus ponens, quantifier rules etc) apply. We may also think of TC as the set defined inductively with the axioms listed above as the initial cases and the laws of the deductive system as the inductive conditions.

We introduce such a language to express theorems in TC for two main reasons:

- We will often treat sentences in TC as objects in and of themselves, rather than statements. The notation of this language helps distinguish the sentences from (meta) statements written in more conventional mathematical notation.
- We will introduce an encoding scheme which requires the encoded sentences to be over a finite alphabet.

For our purposes, the particular language in question is the set of all well-formed formulae over the following 14-symbol alphabet:

$$A := \{\alpha, \quad \beta, \quad [, \quad], \quad *, \quad x, \quad /, \quad \approx, \quad E, \quad \sqsubset, \quad \rightarrow, \quad \wedge, \quad \vee, \quad \neg\}$$

which give us:

- The variables x, $x/$, $x//$, $x///\ldots$
- The logical operators of conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) and implication ($\rightarrow$)
- The nullary function symbols α and β
- The binary function symbol $*$
- The binary relation symbols $\approx$ and $\sqsubset$, and
- The existential quantifier (E).

The universal quantifier is denoted by having a single variable in brackets; for instance, $[x]$ means $(\forall x)$. Furthermore, variables are quantified one by one; for instance, $(\exists x, y)$ would be written as $[Ex][Ex/]$.

For readability, we will often denote x as x_0, $x/$ as x_1, $x//$ as x_2, etc. Furthermore, we will denote sentences of the form $[P \rightarrow Q] \wedge [Q \rightarrow P]$ as $P \leftrightarrow Q$.

Definition 1.1. TC is the set of all sentences provable from the following axioms:

$$[x_0][x_1][x_2]\, x_0 * [x_1 * x_2] \approx [x_0 * x_1] * x_2 \tag{A1}$$

$$\begin{aligned} &[x_0][x_1][x_2][x_3]\, x_0 * x_1 \approx x_2 * x_3 \\ &\qquad \rightarrow \Big[[x_0 \approx x_2 \wedge x_1 \approx x_3] \vee \\ &\qquad\quad [Ex_4]\big[[x_0 * x_4 \approx x_2 \wedge x_4 * x_3 \approx x_1] \\ &\qquad\qquad \vee [x_2 * x_4 \approx x_0 \wedge x_4 * x_1 \approx x_3]\big]\Big] \end{aligned} \tag{A2}$$

$$[x_0][x_1]\, \neg[\alpha \approx x_0 * x_1] \tag{A3}$$

$$[x_0][x_1]\, \neg[\beta \approx x_0 * x_1] \tag{A4}$$

$$\neg[\alpha \approx \beta] \tag{A5}$$

$$\begin{aligned} &[x_0][x_1]\, x_0 \sqsubset x_1 \leftrightarrow \big[x_0 \approx x_1 \vee \\ &\qquad [Ex_2]\,[x_1 \approx x_0 * x_2 \vee x_1 \approx x_2 * x_0] \\ &\qquad \vee [Ex_2][Ex_3]\, x_1 \approx x_2 * [x_0 * x_3]\big] \end{aligned} \tag{A6}$$

The axioms (A1)–(A5) are merely the five axioms at the beginning written in the introduced language, and (A6) is the definition of the substring relation $\sqsubset$, which can be expressed in the language without introducting the symbol $\sqsubset$, but is defined nonetheless for the sake of readability.

2. Theorems of TC

The following are some results that show certain theorems to belong in TC, by proving them from the axioms and other theorems in TC. When we cite a theorem in TC in these proofs, we shall simply cite them as if they were true; for example, instead of saying "since $p \in$ TC", we simply say "by p". Hence when we conclude a theorem to be "true", we only conclude that it belongs in TC. We state the (meta)theorems in the form $p \in$ TC to remind ourselves what we are really proving, and to prevent confusion when they are cited later in this thesis.

LEMMA 1.1. The following are theorems of TC:

(T0$_A$) $[x_0]\,[x_0 \sqsubset \alpha \leftrightarrow x_0 \approx \alpha]$

(T0$_B$) $[x_0]\,[x_0 \sqsubset \beta \leftrightarrow x_0 \approx \beta]$

PROOF. Let $x_0 \sqsubset \alpha$. Then by (A6), we have

- $x_0 \approx \alpha$, or
- $[Ex_1]\,[\alpha \approx x_0 * x_1 \vee \alpha \approx x_1 * x_0]$, or
- $[Ex_1][Ex_2]\,\alpha \approx x_1 * [x_0 * x_2]$.

But by (A4), we have

- $[x_1]\,[\neg[\alpha \approx x_0 * x_1] \wedge \neg[\alpha \approx x_1 * x_0]]$, and
- $[x_1][x_2]\,\neg[\alpha \approx x_1 * [x_0 * x_2]]$,

which contradict $[Ex_1]\,[\alpha \approx x_0 * x_1 \vee \alpha \approx x_1 * x_0]$ and $[Ex_1][Ex_2]\,\alpha \approx x_1 * [x_0 * x_2]$.

Thus $[x_0]\,[x_0 \sqsubset \alpha \rightarrow x_0 \approx \alpha]$. By (A6), the converse is also true. Hence $[x_0]\,[x_0 \sqsubset \alpha \leftrightarrow x_0 \approx \alpha]$. By a similar argument, $[x_0]\,[x_0 \sqsubset \beta \leftrightarrow x_0 \approx \beta]$. □

LEMMA 1.2. The following is a theorem of TC:

(T1) $[x_0][x_1][x_2]\,[x_0 \sqsubset [x_1 * x_2]$

$$\rightarrow [x_0 \sqsubset x_1 \vee x_0 \sqsubset x_2 \vee [Ex_3][Ex_4]\,[x_0 \approx x_3 * x_4 \wedge x_3 \sqsubset x_1 \wedge x_4 \sqsubset x_2]]]$$

PROOF. Suppose $x_0 \sqsubset [x_1 * x_2]$. Then by (A6), either:

- $x_0 \approx [x_1 * x_2]$, in which case $[Ex_3][Ex_4]\,[x_1 \approx x_3 * x_4 \wedge x_3 \sqsubset x_1 \wedge x_4 \sqsubset x_2]]$ since $x_0 \approx x_1 * x_2$, $x_1 \sqsubset x_1$ and $x_2 \sqsubset x_2$, or

- $[Ex_3] x_1 * x_2 \approx x_3 * x_0$, in which case by (A2), either:
  - $x_1 \approx x_3$ and $x_2 \approx x_0$ (and thus $x_0 \sqsubset x_2$), or
  - $[Ex_4]\,[x_1 * x_4 \approx x_3 \wedge x_4 * x_0 \approx x_2]$ (and thus $x_0 \sqsubset x_2$), or
  - $[Ex_4]\,[x_3 * x_4 \approx x_1 \wedge x_4 * x_2 \approx x_0]$ (and thus $x_4 * x_2 \approx x_0$ with $x_4 \sqsubset x_1$ and $x_2 \sqsubset x_2$), or
- $[Ex_3] x_1 * x_2 \approx x_0 * x_3$, in which case by (A2), either:
  - $x_1 \approx x_0$ and $x_2 \approx x_3$ (and thus $x_0 \sqsubset x_1$), or
  - $[Ex_4]\,[x_1 * x_4 \approx x_0 \wedge x_4 * x_3 \approx x_2]$ (and thus $x_1 * x_4 \approx x_0$ with $x_1 \sqsubset x_1$ and $x_4 \sqsubset x_2$), or
  - $[Ex_4]\,[x_0 * x_4 \approx x_1 \wedge x_4 * x_2 \approx x_3]$ (and thus $x_0 \sqsubset x_1$), or
- $[Ex_3][Ex_4]\, x_1 * x_2 \approx x_3 * [x_0 * x_4]$, in which case by (A2), either:
  - $x_1 \approx x_3$ and $x_2 \approx x_0 * x_4$ (and thus $x_0 \sqsubset x_2$), or
  - $[Ex_5]\, x_1 * x_5 \approx x_3 \wedge x_5 * [x_0 * x_4] \approx x_2$ (and thus $x_0 \sqsubset x_2$), or
  - $[Ex_5]\, x_3 * x_5 \approx x_1 \wedge x_5 * x_2 \approx x_0 * x_4$, in which case by (A2), either:
    * $x_5 \approx x_0$ and $x_2 \approx x_4$ (and thus $x_0 \sqsubset x_1$), or
    * $[Ex_6]\ x_5 * x_6 \approx x_0 \wedge x_6 * x_4 \approx x_2$ (and thus $x_5 * x_6 \approx x_0$ with $x_5 \sqsubset x_1$ and $x_6 \sqsubset x_2$), or
    * $[Ex_6]\ x_0 * x_6 \approx x_5 \wedge x_6 * x_2 \approx x_4$ (and thus $x_3 * [x_0 * x_6] \approx x_1$, so $x_0 \sqsubset x_1$).

In each case we have $x_0 \sqsubset x_1$, or $x_0 \sqsubset x_2$, or $[Ex_3][Ex_4]\,[x_0 \approx x_3 * x_4 \wedge x_3 \sqsubset x_1 \wedge x_4 \sqsubset x_2]$. $\square$

LEMMA 1.3. The following are theorems of TC:

(T2$_A$) $[x_0][x_1]$

$$x_0 \sqsubset [x_1 * \alpha] \leftrightarrow \big[x_0 \sqsubset x_1 \vee x_0 \approx \alpha \vee [Ex_2][Ex_3][x_1 \approx x_2 * x_3 \wedge x_0 \approx x_2 * \alpha] \vee x_0 \approx x_1 * \alpha\big]$$

(T2$_B$) $[x_0][x_1]$

$$x_0 \sqsubset [x_1 * \beta] \leftrightarrow \big[x_0 \sqsubset x_1 \vee x_0 \approx \beta \vee [Ex_2][Ex_3][x_1 \approx x_2 * x_3 \wedge x_0 \approx x_2 * \beta] \vee x_0 \approx x_1 * \beta\big]$$

PROOF. (Sketch) By (T1), we have

$$[x_0][x_1]\, x_0 \sqsubset [x_1 * \alpha] \to \big[x_0 \sqsubset x_1 \vee x_0 \sqsubset \alpha \vee [Ex_2][Ex_3]\,[x_0 \approx x_2 * x_3 \wedge x_2 \sqsubset x_1 \wedge x_3 \sqsubset \alpha]\big].$$

We can then use (T0$_A$) to show that $[x_0] x_0 \sqsubset \alpha \to x_0 \approx \alpha$ and

$$[x_0][x_1]\,[Ex_2][Ex_3]\,[x_0 \approx x_2 * x_3 \wedge x_2 \sqsubset x_1 \wedge x_3 \sqsubset \alpha]$$
$$\to [Ex_2][Ex_3][x_1 \approx x_2 * x_3 \wedge x_0 \approx x_2 * \alpha] \vee x_0 \approx x_1 * \alpha].$$

For the converse, we can check directly from the axioms that each of the conditions $x_0 \sqsubset x_1$, $x_0 \approx \alpha$, $[Ex_2][Ex_3][x_1 \approx x_2 * x_3 \wedge x_0 \approx x_2 * \alpha]$ and $x_0 \approx x_1 * \alpha$ implies $x_0 \sqsubset [x_1 * \alpha]$.

The proof for (T2$_B$) is analogous. $\square$

LEMMA 1.4. The following is a theorem of TC:

$$\textbf{(T3)}\ [x_0][x_1]\,\neg[x_0 * \alpha \approx x_1 * \beta]$$

PROOF. Suppose $[Ex_0][Ex_1]\,[x_0 * \alpha \approx x_1 * \beta]$. Then by (A2),

$$[Ex_0][Ex_1]\,[x_0 \approx x_1 \wedge \alpha \approx \beta] \vee \big[[Ex_2][x_0 * x_2 \approx x_1 \wedge x_2 * \beta \approx \alpha] \vee [x_1 * x_2 \approx x_0 \wedge x_2 * \alpha \approx \beta]\big].$$

But $\neg[x_0 \approx x_1 \wedge \alpha \approx \beta]$ by (A5) and $\neg\big[[Ex_2][x_0 * x_2 \approx x_1 \wedge x_2 * \beta \approx \alpha] \vee [x_1 * x_2 \approx x_0 \wedge x_0 * \alpha \approx \beta]\big]$ by (A3) and (A4), so $[x_0][x_1]\,\neg[x_0 * \alpha \approx x_1 * \beta]$. □

LEMMA 1.5. The following are theorems of TC:

(T4$_A$) $[x_0][x_1]\,[\alpha * x_0 \approx \alpha * x_1] \to [x_0 \approx x_1]$

(T4$_B$) $[x_0][x_1]\,[x_0 * \alpha \approx x_1 * \alpha] \to [x_0 \approx x_1]$

(T4$_C$) $[x_0][x_1]\,[\beta * x_0 \approx \beta * x_1] \to [x_0 \approx x_1]$

(T4$_D$) $[x_0][x_1]\,[x_0 * \beta \approx x_1 * \beta] \to [x_0 \approx x_1]$

PROOF. Suppose $\alpha * x_0 \approx \alpha * x_1$. Then by (A2),

$$[\alpha \approx \alpha \wedge x_0 \approx x_1] \vee \big[[Ex_2][\alpha * x_2 \approx \alpha \wedge x_2 * x_1 \approx x_0] \vee [\alpha * x_2 \approx \alpha \wedge x_2 * x_0 \approx x_1]\big].$$

But $\neg\big[[Ex_2][\alpha * x_2 \approx \alpha \wedge x_2 * x_1 \approx x_0] \vee [\alpha * x_2 \approx \alpha \wedge x_2 * x_0 \approx x_1]\big]$ by (A3), so $\alpha \approx \alpha \wedge x_0 \approx x_1$, so $x_0 \approx x_1$. Hence (a) holds. The proofs for (b), (c) and (d) use similar arguments. □

CHAPTER 2

The standard model of TC

In this chapter, we introduce the standard model of TC, which we will use to prove undecidability of TC by encoding into it theorems of TC. This chapter is based on Section 5 under Part Two of 'Undecidability without Arithmetization' [**8**], and is referenced by part (i) of the overview.

Definition 2.1. Consider the set of strings $\{a, b\}$. We define the set Tx $:= \bigcap \mathcal{A}$ where $\mathcal{A}$ is the $\big($ class of (sets X of strings on any alphabet of any size) $\big)$ which satisfies the following:

$$\mathcal{A} := \{X \mid a, b \in X \text{ and } (\forall s, t \in X)\, st \in X\}.$$

We call Tx the set of standard texts. Note that Tx, as well as all sets in $\mathcal{A}$, are just sets of strings on some alphabet $\Sigma \supseteq \{a, b\}$.

1. Properties of Tx

Lemma 2.1. $\mathrm{Tx} = \bigcap \mathcal{B} = \bigcap \mathcal{C} = \{a, b\}^+$, where

$$\mathcal{B} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\} \text{ and}$$
$$\mathcal{C} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, as, bs \in X\}.$$

Proof. Let $X \in \mathcal{A}$ and $s \in X$. Then $a, b, s \in X$. Then $sa, sb, as, bs \in X$, since $tu \in X$ for all $t, u \in X$. Hence $X \in \mathcal{B}$ and $X \in \mathcal{C}$, so $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{C}$. Thus $\bigcap \mathcal{B} \subseteq \bigcap \mathcal{A} = \mathrm{Tx}$ and $\bigcap \mathcal{C} \subseteq \bigcap \mathcal{A} = \mathrm{Tx}$.

Now let $x \in \mathrm{Tx}$, $Y \in \mathcal{B}$ and $Z \in \mathcal{C}$. We will show that $x \in Y$ and $x \in Z$.

Let $T := \{a, b\}^+$. Then $a, b \in T$. Furthermore, let $s, t \in T$. Then $s = s_1 \dots s_n$ and $t = t_1 \dots t_m$ for some $m, n \in \mathbb{N}$ and $s_1, \dots, s_n, t_1, \dots, t_n \in \{a, b\}$. Then
$st = s_1 \dots s_n t_1 \dots t_m \in T$. Hence $T \in \mathcal{A}$.

Let $y \in T$. Then $y = t_1 \dots t_n$ for some $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \{a, b\}$. Since $Y \in \mathcal{B}$, we have $a, b \in Y$, and thus $t_1 \in Y$. Now suppose $t_1 \dots t_i \in Y$ for some $i \in \{1, \dots, n-1\}$. Then $t_1 \dots t_i a, t_1 \dots t_i b \in Y$, since $Y \in \mathcal{B}$. Then $t_1 \dots t_{i+1} \in Y$, since $t_{i+1} \in \{a, b\}$. Hence by induction, $y = t_1 \dots t_n \in Y$. Similarly, since $Z \in \mathcal{C}$, we have $a, b \in Z$, and thus $t_n \in Z$. Now suppose $t_i \dots t_n \in Z$ for some $i \in \{2, \dots, n\}$. Then $at_i \dots t_n, bt_i \dots t_n \in Z$, since $Z \in \mathcal{C}$.

Then $t_{i-1} \dots t_n \in Y$, since $t_{i-1} \in \{a, b\}$. Hence by induction, $y = t_1 \dots t_n \in Z$. Hence $T \subseteq Y$ and $T \subseteq Z$, so $x \in Y$ and $x \in Z$ since $x \in \bigcap \mathcal{A} \subseteq T$.

Thus $x \in Y$ for all $Y \in \mathcal{B}$ and $x \in Z$ for all $Z \in \mathcal{C}$, so $x \in \bigcap \mathcal{B}$ and $x \in \bigcap \mathcal{C}$. Hence Tx $\subseteq \bigcap \mathcal{B}$ and Tx $\subseteq \bigcap \mathcal{C}$.

Thus $\bigcap \mathcal{B} = \mathrm{Tx} = \bigcap \mathcal{C}$. Furthermore, we have shown above that $T \in \mathcal{A}$, so $\mathrm{Tx} = \bigcap \mathcal{A} \subseteq T$, but $T \subseteq Y$ for all $Y \in \mathcal{B}$, so $T \subseteq \bigcap \mathcal{B}$. Hence $\mathrm{Tx} = T = \bigcap \mathcal{B} = \bigcap \mathcal{C}$, as required. $\square$

Note that it's not the case that $\mathcal{A} = \mathcal{B} = \mathcal{C}$; for instance, a set $\mathrm{Tx} \cup \{cx \mid x \in \mathrm{Tx}\}$ (where $c \notin \{a, b\}^*$) would belong to $\mathcal{B}$ but not to $\mathcal{A}$. Also note that during the above proof, we have shown the following result:

Lemma 2.2. $\mathrm{Tx} \in \mathcal{A}$.

Furthermore, each member of $\mathcal{A}$ is the universe of a model of TC:

Lemma 2.3. For all $X \in \mathcal{A}$, the structure $\langle X; *^{\mathbf{X}}, \alpha^{\mathbf{X}}, \beta^{\mathbf{X}}, \sqsubset^{\mathbf{X}} \rangle$ where:

- $\alpha^{\mathbf{X}} = a$
- $\beta^{\mathbf{X}} = b$
- $x *^{\mathbf{X}} y = xy$ for all $x, y \in X$, and
- $x \sqsubset^{\mathbf{X}} y \iff (\ (x = y)$ or $((\exists z \in X)\, y = xz$ or $y = zx)$ or $((\exists z, w \in X)\, y = zxw)\ )$ for all $x, y \in X$

is a model of TC.

The proof of this lemma amounts to checking that (A1) - (A6) are satisfied, and we shall not include it here. We call the structure $\mathbf{Tx} := \langle \mathrm{Tx}; *^{\mathbf{Tx}}, \alpha^{\mathbf{Tx}}, \beta^{\mathbf{Tx}}, \sqsubset^{\mathbf{Tx}} \rangle$ the standard model of TC.

Conversely, if $\langle X; *^{\mathbf{X}}, \alpha^{\mathbf{X}}, \beta^{\mathbf{X}}, \sqsubset^{\mathbf{X}} \rangle$ is a model of TC, there should be a renaming map $\varphi : X \to (X \cup \{a, b\})^+$ with $\varphi(\alpha^{\mathbf{X}}) = a$, $\varphi(\beta^{\mathbf{X}}) = b$ and $\varphi(x *^{\mathbf{X}} y) = \varphi(x)\varphi(y)$ for all $x, y \in X$ such that $\varphi(X) \in \mathcal{A}$.

(Note that by Lemma 2.2, there are rather 'obvious' non-standard models of TC satisfying sentences false in the standard model; like the set of strings on a 3-letter alphabet $\{a, b, c\}$, which satisfies

$$[Ex_2][x_1][x_0] \neg[z \approx xy] \wedge \neg[x \approx a] \wedge \neg[x \approx b].$$

Thus TC is incomplete. Decidability, on the other hand, can be slightly more complicated.)

Corollary 2.1. TC is consistent and TC $\subseteq$ Th($\mathbf{Tx}$).

CHAPTER 3

Discussing elements of Tx in the context of TC

In this chapter, we introduce a "naming" map, which in effect lets us talk about elements of Tx in the context of TC. This chapter is based on Sections 6 and 7 under Part Two of 'Undecidability without Arithmetization' [**8**], and is referenced by part (ii) of the overview.

DEFINITION 3.1. The map $N : \mathrm{Tx} \rightharpoonup A^+$ is defined inductively as follows:

$$N(a) = \alpha \text{ and } N(b) = \beta \tag{1}$$

and if $s \in \mathrm{Tx}$ and $N(s)$ is well-defined, then

$$N(sa) = [N(s) * N(a)] \text{ and } N(sb) = [N(s) * N(b)] \tag{2}$$

DEFINITION 3.2. The set $\mathrm{Cterm} \subseteq A^+$ is defined inductively as follows:

$$\alpha \in \mathrm{Cterm} \text{ and } \beta \in \mathrm{Cterm} \tag{3}$$

$$s, t \in \mathrm{Cterm} \implies [s * t] \in \mathrm{Cterm} \tag{4}$$

1. Properties of the naming map

LEMMA 3.1. N is well-defined on Tx, and for all $s \in \mathrm{Tx}$, $N(s) \in \mathrm{Cterm}$.

PROOF. Let $x \in \mathrm{Tx}$. Then by Lemma 2.1, $x = t_1 \dots t_n$ for some $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \{a, b\}$. Also by Lemma 2.1, we have $t_1 \dots t_m \in$ Tx for all $m \in \{1, \dots, n\}$. Now $N(t_1)$ is well-defined by (1), since $t_1 \in \{a, b\}$. Furthermore, for all
$m \in \{1, \dots, n-1\}$, if $N(t_1 \dots t_m)$ is well-defined, then $N(t_1 \dots t_{m+1})$ is well-defined by (2). Hence by induction, $N(t_1 \dots t_n)$ is well-defined.

Let $X := \{x \in \mathrm{Tx} \,|\, N(x) \in \mathrm{Cterm}\}$.
We have $N(a) = \alpha \in \mathrm{Cterm}$ and $N(b) = \beta \in \mathrm{Cterm}$, so $a, b \in X$.

Let $s \in X$. Then $N(s) \in \mathrm{Cterm}$, so by (4), we have
$N(sa) = [N(s) * N(a)] \in \mathrm{Cterm}$ and $N(sb) = [N(s) * N(b)] \in \mathrm{Cterm}$ since $a, b \in X$ so $N(a), N(b) \in \mathrm{Cterm}$. Hence $sa, sb \in X$.

Thus $X \in \mathcal{B} := \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have Tx $= \bigcap \mathcal{B} \subseteq X$, so
$s \in \mathrm{Tx} \implies s \in X \iff N(s) \in \mathrm{Cterm}$, as required. □

This basically shows that the naming map maps elements of Tx to terms in the language of TC, so statements like $N(s) \approx x_0$ or $N(s) \approx \alpha$ are syntactically valid sentences in the language of TC. In particular, the naming map maps elements of Tx to terms constructible from α and β from applying $*$ a finite number of times.

Lemma 3.2. For all $s, t \in \mathrm{Tx}$, the sentence $N(st) \approx [N(s) * N(t)]$ is in TC. In other words,

$$N(st) \approx [N(s) * N(t)] \;\in\; \mathrm{TC}.$$

Proof. Let $X := \{t \in \mathrm{Tx} \mid (\forall s \in \mathrm{Tx})\,(N(st) \approx [N(s) * N(t)] \;\in\; \mathrm{TC})\}$.

Let $s \in \mathrm{Tx}$. By (2), we have $N(sa) = [N(s) * N(a)]$ and $N(sb) = [N(s) * N(b)]$. Now all tautologies in the appropriate language are contained in TC, so
$(N(sa) \approx [N(s) * N(a)]) \in \mathrm{TC}$ and $(N(sb) \approx [N(s) * N(b)]) \in \mathrm{TC}$, since
$N(sa) \approx [N(s) * N(a)]$ and $N(sb) \approx [N(s) * N(b)]$ are tautologies of the form $x \approx x$. Hence $a, b \in X$.

Now let $t \in X$. Then $(N(st) \approx [N(s) * N(t)]) \;\in\; \mathrm{TC}$ for all $s \in \mathrm{Tx}$.

Let $s \in \mathrm{Tx}$. By (2), we have $N(sta) = [N(st) * N(a)]$. Then
$N(sta) \approx [N(st) * N(a)]$ is a tautology of the form $x = x$, so

$$N(sta) \approx [N(st) * N(a)] \;\in\; \mathrm{TC}. \tag{$f1$}$$

But $N(st) \approx [N(s) * N(t)] \;\in\; \mathrm{TC}$, so

$$[N(st) * N(a)] \approx [[N(s) * N(t)] * N(a)] \;\in\; \mathrm{TC}, \tag{$f2$}$$

since concatenating on the right by the same thing on both sides of an equality preserves the equality in TC.

By the axiom (A1), we have

$$[[N(s) * N(t)] * N(a)] \approx [N(s) * [N(t) * N(a)]] \ \in \ \text{TC}. \tag{$f3$}$$

By (2), we have $N(ta) = [N(t) * N(a)]$. Then $N(ta) \approx [N(t) * N(a)]$ is a tautology of the form $x \approx x$, so $N(ta) \approx [N(t) * N(a)] \ \in \ \text{TC}$, so we can substitute $N(ta)$ for $[N(t) * N(a)]$ in the tautology $[N(s) * [N(t) * N(a)]] \approx [N(s) * [N(t) * N(a)]]$ to get

$$[N(s) * [N(t) * N(a)]] \approx [N(s) * N(ta)] \ \in \ \text{TC}. \tag{$f4$}$$

Now $(f1) - (f4)$ form a chain of equalities in TC, so by transitivity of equality in TC,

$$N(sta) \approx [N(s) * N(ta)] \ \in \ \text{TC}.$$

By a similar argument,

$$N(stb) \approx [N(s) * N(tb)] \ \in \ \text{TC}.$$

Hence $t \in X \implies ta, tb \in X$, so
$X \in \mathcal{B} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have
$\text{Tx} = \bigcap \mathcal{B} \subseteq X$, so
$t \in \text{Tx} \implies t \in X \iff (\forall s \in \text{ Tx})\ (N(st) \approx [N(s) * N(t)] \ \in \ \text{TC})$.

Hence for all $t \in \text{Tx}$, we have $(N(st) \approx [N(s) * N(t)]) \in$ TC for all $s \in \text{Tx}$, so $N(st) \approx [N(s) * N(t)] \ \in$ TC for all $s, t \in \text{Tx}$, as required. □

Lemma 3.3. For all $t \in \text{Tx}$, we have

$$[x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \mid s \in X_t\}] \ \in \ \text{TC}.$$

where $X_t := \{s \in \text{Tx} \mid s \sqsubset^{\mathbf{Tx}} t\}$.

Note that when a set X is finite, $\bigvee\{x_0 \approx N(s) \mid s \in X\} = [x_0 \approx N(s_1)] \vee \cdots \vee [x_0 \approx N(s_1)]$ for some $s_1, \ldots, s_n \in \text{Tx}$, so $\bigvee\{x_0 \approx N(s) \mid s \in X\}$ is a valid first-order formula in the language of TC. From Lemma 2.1 and the definition of $\sqsubset^{\mathbf{Tx}}$ in Lemma 2.3, we can see that X_t is finite for all $t \in \text{Tx}$; in particular, if $t = t_1 \ldots t_n$ with $t_1, \ldots, t_n \in \{a, b\}$, then $|X_t| = \frac{n(n+1)}{2}$.

PROOF. (Sketch)
Let $Y := \{t \in \mathrm{Tx} \,|\, [x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X_t\}] \ \in \ \mathrm{TC}\}$.

By Theorem 1.1, we have
$[x_0]\,[x_0 \sqsubset \alpha \leftrightarrow x_0 \approx \alpha] \ \in \ \mathrm{TC}$ and $[x_0]\,[x_0 \sqsubset \beta \leftrightarrow x_0 \approx \beta] \ \in \ \mathrm{TC}$, so
$[x_0]\,[x_0 \sqsubset N(a) \leftrightarrow x_0 \approx N(a)] \ \in \ \mathrm{TC}$ and $[x_0]\,[x_0 \sqsubset N(b) \leftrightarrow x_0 \approx N(b)] \ \in \ \mathrm{TC}$ since $N(a) = \alpha$ and $N(b) = \beta$. Furthermore, $x_0 \approx N(a) = \bigvee\{x \approx N(s) \,|\, s \in \{a\}\}$ and $x_0 \approx N(b) = \bigvee\{x \approx N(s) \,|\, s \in \{b\}\}$, and $\{a\} = X_a$ and $\{b\} = X_b$, so $a, b \in Y$.

Suppose $t \in Y$. Then

$$[x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X_t\}] \ \in \ \mathrm{TC}.$$

By Theorem 1.3(a), we have

$$\begin{aligned}
[x_0]\,[\ & x_0 \sqsubset [N(t) * \alpha] \leftrightarrow \\
& [\,x_0 \sqsubset N(t) \vee \\
& \ x_0 \approx \alpha \vee [Ex_1][Ex_2][N(t) \approx x_1 * x_2 \wedge x_0 \approx x_1 * \alpha] \vee \\
& \ x_0 \approx N(t) * \alpha \qquad\qquad\qquad\qquad\qquad\qquad] \quad] \ \in \ \mathrm{TC},
\end{aligned}$$

so

$$\begin{aligned}
[x_0]\,[\ & x_0 \sqsubset [N(t) * \alpha] \leftrightarrow \\
& [\bigvee\{x_0 \approx N(s) \,|\, s \in X\} \vee \\
& x_0 \approx N(a) \vee [Ex_1][Ex_2][N(t) \approx x_1 * x_2 \wedge x_0 \approx x_1 * \alpha] \vee \\
& x_0 \approx N(ta) \qquad\qquad\qquad\qquad\qquad\qquad] \quad] \ \in \ \mathrm{TC},
\end{aligned}$$

since $[x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X\}] \ \in \ \mathrm{TC}$ by assumption, $N(a) = \alpha$ and $N(sa) = [N(s) * N(a)]$ for all $s \in \mathrm{Tx}$. Furthermore,
$x_0 \approx N(a) = \bigvee\{x \approx N(s) \,|\, s \in \{a\}\}$, and $x_0 \approx N(ta) = \bigvee\{x \approx N(s) \,|\, s \in \{ta\}\}$, and

$$\begin{aligned}
[\quad & [Ex_1][Ex_2][N(t) \approx x_1 * x_2 \wedge x_0 \approx x_1 * \alpha] \\
\leftrightarrow & \bigvee\{x_0 \approx N(s) \,|\, s \in \{s \in \mathrm{Tx} \,|\, (\exists u, w \in Tx)\,(t = uw \wedge s = wa)\}\}\,] \ \in \ \mathrm{TC},
\end{aligned}$$

so

$$[x_0]\,[x_0 \sqsubset [N(ta)] \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X'\}] \ \in \ \mathrm{TC},$$

with $X' = X \cup \{a\} \cup \{s \in \mathrm{Tx} \,|\, (\exists u, w \in Tx)\,(t = uw \wedge s = wa)\} \cup \{ta\}$. Now $X' = X_{ta}$, so $ta \in Y$. By a similar argument, $tb \in Y$. Thus
$Y \in \mathcal{B} = \{X \,|\, a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have
$\mathrm{Tx} = \bigcap \mathcal{B} \subseteq Y$, so

$$t \in \ \mathrm{Tx} \implies t \in Y \iff [x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X_t\}] \ \in \ \mathrm{TC}.$$

Hence for all $t \in \mathrm{Tx}$, we have $[x_0][x_0 \sqsubset N(t) \leftrightarrow \bigvee\{x_0 \approx N(s) \,|\, s \in X_t\}] \ \in \ \mathrm{TC}$, as required. □

In a sense, the map N is "bijective" with respect to TC. The following result shows the "surjectivity". The proof of "injectivity" won't be shown here, since a far simpler proof can be given later once we've constructed some more machinery.

LEMMA 3.4. For all $c \in \mathrm{Cterm}$, there exists $s \in \mathrm{Tx}$ such that $N(s) \approx c \ \in \ \mathrm{TC}$.

PROOF. We have $a, b \in \mathrm{Tx}$, $N(a) = \alpha$ and $N(b) = \beta$, so $N(a) \approx \alpha \ \in \ \mathrm{TC}$ and $N(b) \approx \beta \ \in \ \mathrm{TC}$ as $N(a) \approx \alpha$ and $N(b) \approx \beta$ are tautologies. Hence there exist
$s, t \in \mathrm{Tx}$ such that $N(s) \approx \alpha \ \in \ \mathrm{TC}$ and $N(t) \approx \beta \ \in \ \mathrm{TC}$.

Let $c, d \in \mathrm{Cterm}$ and suppose there exist $s, t \in \mathrm{Tx}$ such that

$$N(s) \approx c \ \in \ \mathrm{TC} \tag{*}$$

$$\text{and } N(t) \approx d \ \in \ \mathrm{TC}. \tag{**}$$

By Lemma 2.2, we have $st \in \mathrm{Tx}$, so by Lemma 3.2,

$$N(st) \approx [N(s) * N(t)] \ \in \ \mathrm{TC} \tag{***}$$

Now TC is closed under logical operations, so by (*) and (**), we may substitute c for $N(s)$ and d for $N(t)$ in $(***)$ to get $N(st) \approx [c * d] \in \ \mathrm{TC}$. Hence there exists $u = st \in \mathrm{Tx}$ such that $N(u) \approx [c * d] \ \in \ \mathrm{TC}$.

Hence by induction, for all $c \in \mathrm{Cterm}$, there exists $s \in \mathrm{Tx}$ such that
$N(s) \approx c \ \in \ \mathrm{TC}$. □

CHAPTER 4

Discernibility of relations on Tx

In this chapter, we introduce a set $\mathbf{GD} \subseteq \{X \subseteq \mathrm{Tx}^n \mid n \in \mathbb{N}\}$, which we call the set of General Discernible relations. Membership of an n-ary relation R in $\mathbf{GD}$ is meant to be a way to define whether or not there is an algorithmic procedure that decides whether or not an arbitrary $t \in \mathrm{Tx}^n$ is in R. This chapter is based on Section 8 under Part Three of 'Undecidability without Arithmetization' [**8**], and is referenced by part (iii) of the overview.

Each $R \in \mathbf{GD}$ is an n-ary relation on Tx for some $n \in \mathbb{N}$, and instead of writing $(t_1, \ldots, t_n) \in R$, we often write $R(t_1, \ldots, t_n)$. In addition, if R is an n-ary relation on Tx, then we denote $\neg R := \mathrm{Tx}^n \backslash R$. Note that for all
$(t_1, \ldots, t_n) \in \mathrm{Tx}^n$, we have $(\neg R)(t_1, \ldots, t_n) \iff \neg(R(t_1, \ldots, t_n))$, so the statement $\neg R(t_1, \ldots, t_n)$ is unambiguous.

Definition 4.1 (General Discernible (GD) Relations). A relation R on Tx is GD if and only if it can be constructed from the following base cases by applying the following inductive conditions a finite number of times. We denote the class of GD relations by $\mathbf{GD}$.

Base cases:

(GD1) Letters
$\{t \in \mathrm{Tx} \mid t = a\} \in \mathbf{GD}$ and $\{t \in \mathrm{Tx} \mid t = b\} \in \mathbf{GD}$

(GD2) Equality
$\{(t, y) \in \mathrm{Tx}^2 \mid t = y\} \in \mathbf{GD}$

(GD3) Concatenation
$\{(t, y, z) \in \mathrm{Tx}^3 \mid t = yz\} \in \mathbf{GD}$

Inductive conditions:

(GD4) Adding a parameter
if $R \in \mathbf{GD}$, then $\{(y, t_1, \ldots, t_n) \in \mathrm{Tx}^{n+1} \mid R(t_1, \ldots, t_n)\} \in \mathbf{GD}$

(GD5) Eliminating duplicates
if $R \in \mathbf{GD}$, then $\{(t_1, t_3, \ldots, t_n) \in \mathrm{Tx}^{n-1} \mid R(t_1, t_1, t_3, \ldots, t_n)\} \in \mathbf{GD}$

(GD6) Swapping coordinates

if $R \in \mathbf{GD}$, then for all $k \in \{1, \ldots, n-1\}$, we have

$\{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid R(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n)\} \in \mathbf{GD}$

(GD7) Relative complement

if $R \in \mathbf{GD}$, then $\{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid \neg R(t_1, \ldots, t_n)\} \in \mathbf{GD}$

(GD8) Direct product

if $R, S \in \mathbf{GD}$, then

$\{(t_1, \ldots, t_{n+k}) \in \mathrm{Tx}^{n+k} \mid R(t_1, \ldots, t_n) \text{ and } S(t_{n+1}, \ldots, t_{n+k})\} \in \mathbf{GD}$

(GD9) Substring-closed interior

if $R \in \mathbf{GD}$, then

$\{(y, t_2, \ldots, t_n) \in \mathrm{Tx}^n \mid (\forall t_1 \in \mathrm{Tx})\, t_1 \sqsubset^{\mathbf{Tx}} y \implies R(t_1, t_2, \ldots, t_n)\} \in \mathbf{GD}$

(GD10) Complementary projections

if $R \subseteq \mathrm{Tx}^n$, and there exist $S, T \in \mathbf{GD}$ such that:

$$R(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+k} \in \mathrm{Tx})\, S(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}) \text{ and}$$
$$R(t_1, \ldots, t_n) \iff (\forall t_{n+1}, \ldots, t_{n+l} \in \mathrm{Tx})\, T(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}),$$

then $R \in \mathbf{GD}$.

Equivalently, by De Morgan's Laws and (GD7), we can state (GD10) as

- if $R \subseteq \mathrm{Tx}^n$, and there exist $S, T' \in \mathbf{GD}$ such that:

$$R(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+k} \in \mathrm{Tx})\, S(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}) \text{ and}$$
$$\neg R(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+l} \in \mathrm{Tx})\, T'(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}),$$

then $R \in \mathbf{GD}$,

where we may think of T' as $\neg T$ from the original statement of condition 10.

We denote by **ED** the set generated from the initial elements (GD1)-(GD3) by the inductive conditions (GD4)-(GD9), but not (GD10). We call **ED** the set of Elementary Discernible relations.

(Note that in the definition of GD relations only the cases (GD3) and (GD9) make use of the fact that Tx is a set of strings. The other notions are purely logical.)

DEFINITION 4.2. A function $f : \mathrm{Tx}^n \to \mathrm{Tx}$ is General Discernible (GD) if and only if there exists $R \in \mathbf{GD}$ such that the following hold:

(1) $(\forall s_1, \ldots, s_n, t, v \in \mathrm{Tx})\,(R(s_1, \ldots, s_n, t) \text{ and } R(s_1, \ldots, s_n, v)) \implies t = v$
(2) $(\forall s_1, \ldots, s_n \in \mathrm{Tx})(\exists t \in \mathrm{Tx})\,(R(s_1, \ldots, s_n, t)$
(3) $(\forall s_1, \ldots, s_n, t \in \mathrm{Tx})\, f(s_1, \ldots, s_n) = t \iff R(s_1, \ldots, s_n, t)$

For each GD function $f : \mathrm{Tx}^n \to \mathrm{Tx}$, we shall denote the corresponding $(n+1)$-ary relation satisfying the above conditions for f by R_f.

1. Properties of GD relations and functions

The preimage of a **GD** relation under a GD function is also **GD**:

LEMMA 4.1. Suppose $f : \mathrm{Tx}^n \to \mathrm{Tx}$ is GD and $X \in \mathbf{GD}$. Then $f^{-1}(X) \in \mathbf{GD}$.

PROOF. Let $Y = f^{-1}(X)$. Then for all $s_1, \ldots, s_n \in \mathrm{Tx}$,

$$\begin{aligned}
(s_1, \ldots, s_n) \in Y &\iff f(s_1, \ldots, s_n) \in X \\
&\qquad\qquad\qquad \text{(by definition of } Y) \\
&\iff (\exists t \in \mathrm{Tx})\,(f(s_1, \ldots, s_n) = t \text{ and } t \in X) \\
&\qquad\qquad\qquad \text{(introducing } t := f(s_1, \ldots, s_n) \text{ as a variable)} \\
&\iff (\exists t \in \mathrm{Tx})\,(R_f(s_1, \ldots, s_n, t) \text{ and } t \in X) \\
&\qquad\qquad\qquad \text{(by condition 3 of Definition 4.2),} \\
\text{and } (s_1, \ldots, s_n) \notin Y &\iff f(s_1, \ldots, s_n) \notin X \\
&\qquad\qquad\qquad \text{(by definition of } Y) \\
&\iff (\exists t \in \mathrm{Tx})\,(f(s_1, \ldots, s_n) = t \text{ and } t \notin X) \\
&\qquad\qquad\qquad \text{(introducing } t := f(s_1, \ldots, s_n) \text{ as a variable)} \\
&\iff (\exists t \in \mathrm{Tx})\,(R_f(s_1, \ldots, s_n, t) \text{ and } t \notin X) \\
&\qquad\qquad\qquad \text{(by condition 3 of Definition 4.2).}
\end{aligned}$$

Now $R_f, X \in \mathbf{GD}$ by assumption, so by condition 8 of Definition 4.1, we have

$$S_1 := \{(t_1, \ldots, t_{n+2}) \in \mathrm{Tx}^{n+2} \mid R_f(t_1, \ldots, t_{n+1}) \text{ and } X(t_{n+2})\} \in \mathbf{GD}.$$

Then by a finite number of applications of condition 6 of Definition 4.1, we have

$$\begin{aligned}
S_2 :&= \{(t_{n+1}, t_{n+2}, t_1, \ldots, t_n) \in \mathrm{Tx}^{n+2} \mid S_1(t_1, \ldots, t_{n+2})\} \\
&= \{(t_{n+1}, t_{n+2}, t_1, \ldots, t_n) \in \mathrm{Tx}^{n+2} \mid R_f(t_1, \ldots, t_{n+1}) \text{ and } X(t_{n+2})\} \in \mathbf{GD}.
\end{aligned}$$

Then by condition 5 of Definition 4.1, we have

$$\begin{aligned} S_3 :&= \{(t_{n+1}, t_1, \dots, t_n) \in \mathrm{Tx}^{n+1} \mid S_2(t_{n+1}, t_{n+1}, t_1, \dots, t_n)\} \\ &= \{(t_{n+1}, t_1, \dots, t_n) \in \mathrm{Tx}^{n+1} \mid R_f(t_1, \dots, t_{n+1}) \text{ and } X(t_{n+1})\} \in \mathbf{GD}. \end{aligned}$$

Then by a finite number of applications of condition 6 of Definition 4.1, we have

$$\begin{aligned} S_4 :&= \{(t_1, \dots, t_{n+1}) \in \mathrm{Tx}^{n+1} \mid S_3(t_{n+1}, t_1, \dots, t_n)\} \\ &= \{(t_1, \dots, t_{n+1}) \in \mathrm{Tx}^{n+1} \mid R_f(t_1, \dots, t_{n+1}) \text{ and } X(t_{n+1})\} \\ &= \{(t_1, \dots, t_{n+1}) \in \mathrm{Tx}^{n+1} \mid R_f(t_1, \dots, t_{n+1}) \text{ and } t_{n+1} \in X\} \in \mathbf{GD}. \end{aligned}$$

Furthermore, $\neg X \in \mathbf{GD}$ by condition 7 of Definition 4.1, so by a similar argument,

$$\begin{aligned} T_4' :&= \{(t_1, \dots, t_{n+1}) \in \mathrm{Tx}^{n+1} \mid R_f(t_1, \dots, t_{n+1}) \text{ and } t_{n+1} \in \neg X\} \\ &= \{(t_1, \dots, t_{n+1}) \in \mathrm{Tx}^{n+1} \mid R_f(t_1, \dots, t_{n+1}) \text{ and } t_{n+1} \notin X\} \in \mathbf{GD}. \end{aligned}$$

Hence

$$\begin{aligned} Y(s_1, \dots, s_n) &\iff (s_1, \dots, s_n) \in Y \iff (\exists t \in \mathrm{Tx})\, S_4(s_1, \dots, s_n, t) \\ \text{and } \neg Y(s_1, \dots, s_n) &\iff (s_1, \dots, s_n) \notin Y \iff (\exists t \in \mathrm{Tx})\, T_4'(s_1, \dots, s_n, t), \end{aligned}$$

with $S_4, T_4' \in \mathbf{GD}$, so by condition 10 of Definition 4.1, $Y = f^{-1}(X) \in \mathbf{GD}$, as required. $\square$

When composing two GD functions in some way, the resulting function is also GD:

LEMMA 4.2 ((SFF) Substituting Functions into Functions). Let $f : \mathrm{Tx}^m \to \mathrm{Tx}$ and $g : \mathrm{Tx}^n \to \mathrm{Tx}$ be GD functions. Then the function $h : \mathrm{Tx}^{m+n-1} \to \mathrm{Tx}$ defined by $h(t_1, \dots, t_{m+n-1}) = f(g(t_1, \dots, t_n), t_{n+1}, \dots, t_{m+n-1})$ is also a GD function.

PROOF. Define $R \in \mathrm{Tx}^{m+n}$ by $R(t_1, \dots, t_{m+n}) \iff h(t_1, \dots, t_{m+n-1}) = t_{m+n}$. Note that for all $t_1, \dots, t_{m+n-1}, v \in \mathrm{Tx}$,

$$\begin{aligned} h(t_1, \dots, t_{m+n-1}) = v &\iff f(g(t_1, \dots, t_n), t_{n+1}, \dots, t_{m+n-1}) = v \quad \text{(by definition of } h\text{)} \\ &\iff (\exists u \in \mathrm{Tx})\, (g(t_1, \dots, t_n) = u \text{ and } f(u, t_{n+1}, \dots, t_{m+n-1}) = v) \\ &\qquad\qquad \text{(introducing } u := g(t_1, \dots, t_n) \text{ as a variable)} \\ &\iff (\exists u \in \mathrm{Tx})\, (R_g(t_1, \dots, t_n, u) \text{ and } R_f(u, t_{n+1}, \dots, t_{m+n-1}, v)) \\ &\qquad\qquad \text{(by definition of } R_f \text{ and } R_g\text{)}. \end{aligned}$$

Now $R_f, R_g \in \mathbf{GD}$ since f and g are GD by assumption, so by conditions 8, 6 and 5 of Definition 4.1,

$$S := \{\ (t_1, \ldots, t_{m+n+1}) \in \ \mathrm{Tx}^{m+n+1} \mid$$
$$R_g(t_1, \ldots, t_n, t_{m+n+1}) \text{ and } R_f(t_{m+n+1}, t_{n+1}, \ldots, t_{m+n-1}, t_{m+n})\ \} \in \mathbf{GD}.$$

Then for all $t_1, \ldots, t_{m+n-1}, v \in \ \mathrm{Tx}$,

$$R(t_1, \ldots, t_{m+n-1}, v) \iff h(t_1, \ldots, t_{m+n-1}) = v \iff (\exists u \in \mathrm{Tx})\, S(t_1, \ldots, t_{m+n-1}, v, u) \quad (*)$$

with $S \in \mathbf{GD}$. Furthermore,

$$\begin{aligned}
\neg R(t_1, \ldots, t_{m+n-1}, v) &\iff h(t_1, \ldots, t_{m+n-1}) \neq v \qquad \text{(by definition of } R\text{)}\\
&\iff (\exists v' \in \mathrm{Tx})\, (h(t_1, \ldots, t_{m+n-1}) = v' \text{ and } v \neq v')\\
&\qquad\qquad \text{(as } h \text{ is a function)}\\
&\iff (\exists v' \in \mathrm{Tx})\, (((\exists u \in \mathrm{Tx})\, S(t_1, \ldots, t_{m+n-1}, v', u)) \text{ and } v \neq v')\\
&\qquad\qquad \text{(by } (*)\text{)}\\
&\iff (\exists u, v' \in \mathrm{Tx})\, (S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v').
\end{aligned}$$

Now for all $t_1, \ldots, t_{m+n-1}, v \in \ \mathrm{Tx}$,

$$\begin{aligned}
&(\exists u, v' \in \mathrm{Tx})\, (S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')\\
\iff &(\exists z \in \mathrm{Tx})(\exists u, v' \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z \text{ and } v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v').
\end{aligned}$$

The backward implication is trivial and the forward implication holds by taking z to be uv'.

Now $S \in \mathbf{GD}$, and $\{(t, y) \in \ \mathrm{Tx} \mid t \neq y\} \in \mathbf{GD}$ by conditions 2 and 7 of Definition 4.1, so by conditions 8, 6 and 5 of Definition 4.1,

$$T_1' := \{(v', u, t_1, \ldots, t_{m+n-1}, v) \in \ \mathrm{Tx}^{m+n+2} \mid S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v'\} \in \mathbf{GD}.$$

By conditions 9 and 7 of Definition 4.1, for all $T \subseteq \mathrm{Tx}^n$,

$$T \in \mathbf{GD} \implies \{(y, t_2, \ldots, t_n) \in \mathrm{Tx}^n \mid (\exists t_1 \in \mathrm{Tx})\, (t_1 \sqsubset^{\mathbf{Tx}} y \text{ and } \neg T(t_1, t_2, \ldots, t_n))\} \in \mathbf{GD}. \quad (**)$$

As $\neg T_1' \in \mathbf{GD}$ by condition 7 of Definition 4.1, we have

$$\begin{aligned}
T_2' :&= \{(z, u, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists v' \in \mathrm{Tx})\, (v' \sqsubset^{\mathbf{Tx}} z \text{ and } \neg(\neg T_1')(v', u, t_1, \ldots, t_{m+n-1}, v))\} \\
&\qquad \text{(by } (**)\text{)} \\
&= \{(z, u, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists v' \in \mathrm{Tx})\, (v' \sqsubset^{\mathbf{Tx}} z \text{ and } T_1'(v', u, t_1, \ldots, t_{m+n-1}, v))\} \\
&= \{(z, u, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists v' \in \mathrm{Tx})\, (v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')\} \\
&\in \mathbf{GD}.
\end{aligned}$$

By condition 6 of Definition 4.1,

$$\begin{aligned}
T_3' :&= \{(u, z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid T_2'(z, u, t_1, \ldots, t_{m+n-1}, v)\} \\
&= \{(u, z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists v' \in \mathrm{Tx})\, (v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v'\} \\
&\in \mathbf{GD}.
\end{aligned}$$

As $\neg T_3' \in \mathbf{GD}$ by condition 7 of Definition 4.1, by $(**)$, we have

$$\begin{aligned}
T_4' :&= \{(z', z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists u \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z' \text{ and } \neg(\neg T_3')(u, z, t_1, \ldots, t_{m+n-1}, v))\} \\
&= \{(z', z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid (\exists u \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z' \text{ and } T_3'(u, z, t_1, \ldots, t_{m+n-1}, v))\} \\
&= \{(z', z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid \\
&\qquad (\exists u \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z' \text{ and } (\exists v' \in \mathrm{Tx})\, (v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')\} \\
&= \{(z', z, t_1, \ldots, t_{m+n-1}, v) \in \mathrm{Tx}^n \mid \\
&\qquad (\exists u, v' \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z' \text{ and } v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')\} \\
&\in \mathbf{GD}.
\end{aligned}$$

By conditions 5 and 6 of Definition 4.1,

$$\begin{aligned}
T_5' :&= \{(t_1, \ldots, t_{m+n-1}, v, z) \in \mathrm{Tx}^n \mid T_4'(z, z, t_1, \ldots, t_{m+n-1}, v)\} \\
&= \{(t_1, \ldots, t_{m+n-1}, v, z) \in \mathrm{Tx}^n \mid \\
&\qquad (\exists u, v' \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z \text{ and } v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')\} \\
&\in \mathbf{GD}.
\end{aligned}$$

Then

$$\begin{aligned}
\neg R(t_1, \ldots, t_{m+n-1}, v) &\iff (\exists u, v' \in \mathrm{Tx})\, (S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v') \\
&\iff (\exists z \in \mathrm{Tx}) \\
&\quad ((\exists u, v' \in \mathrm{Tx})\, (u \sqsubset^{\mathbf{Tx}} z \text{ and } v' \sqsubset^{\mathbf{Tx}} z \text{ and } S(t_1, \ldots, t_{m+n-1}, v', u) \text{ and } v \neq v')) \\
&\iff (\exists z \in \mathrm{Tx})\, T_5'(t_1, \ldots, t_{m+n-1}, v, z)
\end{aligned}$$

with $T_5' \in \mathbf{GD}$. Hence by condition 10 of Definition 4.1, $R \in \mathbf{GD}$.

Finally, $h(s_1, \ldots, s_n) = t \iff R(s_1, \ldots, s_n, t)$ for all $s_1, \ldots, s_n, t \in \mathrm{Tx}$ by definition of R, and R satisfies conditions 1 and 2 of Definition 4.2 since h is a function. Thus h is a GD function. □

LEMMA 4.3 (Derived GD rule (GD9_5)). If $R \subseteq \mathrm{Tx}^{n+1}$ is GD, then $T := \{(y, t_2, \ldots, t_n) \in \mathrm{Tx}^n \mid (\forall t_0 \sqsubset^{\mathrm{Tx}} y)\, R(t_0, y, t_2, \ldots, t_n)\}$ is GD.

PROOF. We have $S := \{(y, t_1, t_2, \ldots, t_n) \in \mathrm{Tx}^n \mid (\forall t_0 \sqsubset^{\mathrm{Tx}} y)\, R(t_0, t_1, t_2, \ldots, t_n)\}$ is GD by (GD9).

Then $T(y, t_2, \ldots, t_n) \iff S(y, y, t_2, \ldots, t_n)$, so T is GD by (GD5). □

LEMMA 4.4 (SPS GD). The substring, prefix and suffix relations are GD.

PROOF. $s \sqsubset^{\mathrm{Tx}} t \iff (\exists x \sqsubset^{\mathrm{Tx}} t)\, x = s$, so $\sqsubset^{\mathrm{Tx}}$ is GD by (GD6), ($\overline{9}$), (GD7) and (GD2).

$s \sqsubset_p^{\mathrm{Tx}} t \iff (s = t \vee (\exists x \sqsubset^{\mathrm{Tx}} t)\, sx = t)$, so $\sqsubset_p^{\mathrm{Tx}}$ is GD by (GD7), (GD8), (GD6), ($\overline{9}_5$) and (GD2).

$s \sqsubset_s^{\mathrm{Tx}} t \iff (s = t \vee (\exists x \sqsubset^{\mathrm{Tx}} t)\, xs = t)$, so $\sqsubset_s^{\mathrm{Tx}}$ is GD by (GD7), (GD8), (GD6), ($\overline{9}_5$) and (GD2). □

LEMMA 4.5 (RGD $\iff$ CFGD). Let R be an n-ary relation and let the characteristic function χ_R be defined by $\chi_R(x_1, \ldots, x_n) = a \iff R(x_1, \ldots, x_n)$ and $\chi_R(x_1, \ldots, x_n) = b \iff \neg R(x_1, \ldots, x_n)$. Then R is GD iff χ_R is GD.

PROOF. Suppose R is GD. Then $R_{\chi_R} = \{(x_1, \ldots, x_n, x_0) \in \mathrm{Tx}^{n+1} \mid (R(x_1, \ldots, x_n) \wedge x_0 = a) \vee (\neg(R(x_1, \ldots, x_n)) \wedge x_0 = b)\}$ is GD by (GD1), (GD7), (GD8), (GD6) and (GD5).

Conversely, suppose χ_R is GD. Then
$R_1 = \{(x_1, \ldots, x_n, x_0) \in \mathrm{Tx}^{n+1} \mid R_{\chi_R}(x_1, \ldots, x_n, x_0) \wedge x_0 = a\}$ and
$R_2 = \{(x_1, \ldots, x_n, x_0) \in \mathrm{Tx}^{n+1} \mid R_{\chi_R}(x_1, \ldots, x_n, x_0) \wedge x_0 = b\}$ are GD by (GD1), (GD8) and (GD5). Then for all $x_1, \ldots, x_n \in \mathrm{Tx}$, we have $R(x_1, \ldots, x_n) \iff (\exists x \in \mathrm{Tx})\, R_1 = \{(x_1, \ldots, x_n, x)$ and $\neg R(x_1, \ldots, x_n) \iff (\exists x \in \mathrm{Tx})\, R_2 = \{(x_1, \ldots, x_n, x)$, so by (GD10), we have $R \in \mathrm{GD}$. □

LEMMA 4.6 ((SFR) Substituting Functions into Relations). Let R be an n-ary GD relation and f an m-ary GD function. Then the $(m + n - 1)$-ary relation S defined by $S(x_1, \ldots, x_{m+n-1}) \iff R(f(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$ is GD.

PROOF. Since R is GD, χ_R is GD by **(RGD $\iff$ CFGD)**. By Lemma 4.2, the $(m+n-1)$-ary function g defined by $g(x_1, \dots, x_{m+n-1}) = \chi_R(f(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})$ is GD. Now $g = \chi_S$, so S is GD by **(RGD $\iff$ CFGD)**. □

LEMMA 4.7. By (GD2) and Lemma 4.2, $\varphi_n : (x_1, x_2, \dots, x_n) \mapsto x_1 x_2 \dots x_n$ is GD for all $n \geqslant 2$.

LEMMA 4.8 ((B) Strings of b are GD). Let $B(x) := \{x \in \mathrm{Tx} \,|\, (\forall t \in \mathrm{Tx})\, t \sqsubset^{\mathrm{Tx}} x \implies b \sqsubset^{\mathrm{Tx}} t\}$. Then B is GD.

PROOF. We have $b \sqsubset^{\mathrm{Tx}} t \iff \neg((\forall s \in \mathrm{Tx})\, s \sqsubset^{\mathrm{Tx}} t \implies \neg(s = b))$, so $B \in \mathbf{GD}$ by (GD1), (GD7) and (GD9) of Definition 4.1. □

LEMMA 4.9 (C). Constant functions are GD

PROOF. By (GD1) and (GD4), constant functions of all arities that return a, b or c are all GD.

Suppose constant n-ary functions that return s or t are GD.

Then $S(x_1, \dots, x_n, x_0) := \{(x_1, \dots, x_n, x_0) \in \mathrm{Tx}^{n+1} \,|\, x_0 = s\}$ and $T(x_1, \dots, x_n, x_0) := \{(x_1, \dots, x_n, x_0) \in \mathrm{Tx}^{n+1} \,|\, x_0 = t\}$ are GD.

Let f and g be functions such that $S = R_f$ and $T = R_g$.

Then by (GD3), (GD6), Lemma 4.2 and (GD5), the n-ary function h defined by

$$h(x_1, \dots, x_n) = f(x_1, \dots,)g(x_1, \dots, x_n)$$

is GD, so the n-ary constant function returning st is GD.

Hence by induction, all constant functions are GD. □

LEMMA 4.10 (P). Projections are GD

PROOF. Let the function f be the n-th projection on k coordinates. Then $R_f = \{(s_1, \dots, s_k, t) \in \mathrm{Tx}^{k+1} \,|\, s_n = t\}$, which is GD by (GD2), (GD4) and (GD6). □

LEMMA 4.11 (LSGD). Taking the longest substring of "b"s is GD.

PROOF. Define the relation R by $R(w, z) \iff (\forall v \sqsubset^{\mathrm{Tx}} w)\, (B(v) \iff v \sqsubset^{\mathrm{Tx}} z)$. Then $R(w, z) \iff z$ is the longest substring of w consisting only of "b"s, and R is GD by (GD9), (GD7), (GD8) and **(B)**.

It may help to make this a total function so we can apply Lemma 4.2. Define the relation LS_b by $LS_b(w, z) \iff (\neg(b \sqsubset^{\mathrm{Tx}} w) \wedge z = b) \vee R(w, z))$. The LS_b is GD by (GD8), (GD7), **(C)** and **(SFR)**. For convenience, we denote $ls_b(w) = z \iff LS_b(w, z)$. $\square$

CHAPTER 5

Equivalence of GD to Turing machine decidability

In this chapter, we prove that the GD relations are precisely the set of relations on Tx that are also recursive sets. This chapter is new material and does not correspond to any part of 'Undecidability without Arithmetization' [**8**]. This chapter is referenced by part (iii) of the overview.

Throughout this chapter we will frequently refer to the inductive defining conditions of GD relations from Definition 4.1, as well as to lemmas in Chapter 4, Section 1: "Properties of GD relations and functions".

1. GD $\implies$ Recursive

THEOREM 5.1. For all $R \in \mathbf{GD}$, there exists a Turing machine that takes an input $x \in \Sigma^*$ (where Σ is some alphabet containing at least the symbols "a", "b", "", ",", "\$" and "&") and halts, giving an output that indicates whether or not $x \in R$.

PROOF. (Sketch) We give rough steps describing the functioning of a Turing machine recognizing each of the initial relations and preserving each of the conditions. Each of these steps should be a constructible Turing machine process. All except perhaps (GD9) and (GD10) are straightforwrd, but we give details for completeness.

To accommodate the processes specified, we assume each Turing machine constructed below has 3 tapes. This is an inessential modification, since multi-tape Turing machines are equivalent to single-tape Turing machines [**10**, Theorem 2.1].

(GD1) For $\{t \in \text{ Tx} \,|\, t = a\}$ and $\{t \in \text{ Tx} \,|\, t = b\}$:

(a) Check that input is of length 1; if not, reject.
 (i) (For $\{t \in \text{ Tx} \,|\, t = a\}$.) Read starting letter of input; accept if "a" is read, reject otherwise; or
 (ii) (For $\{t \in \text{ Tx} \,|\, t = b\}$.) Read starting letter of input; accept if "b" is read, reject otherwise.

(GD2) For $\{(t, y) \in \text{ Tx}^2 \,|\, t = y\}$:

(a) Check that input is of the form $s_i \dots s_n, t_1 \dots t_n$ for some $n \in \mathbb{N}$ and $s_1, \dots, s_n, t_1, \dots, t_n \in \{a, b\}$; if not, reject.

(b) Check that $s_i = t_i$ for all $i \in \{1, \dots, n\}$; if not, reject, otherwise accept.

(GD3) For $\{(t, y, z) \in \text{ Tx}^3 \,|\, t = yz\}$:

(a) Check that input is of the form $t_i \dots t_l, y_1 \dots y_m, z_1 \dots z_n$ for some $l, m, n \in \mathbb{N}$ and

$t_1, \dots, t_n, y_1, \dots, y_n, z_1, \dots, z_n \in \{a, b\}$; if not, reject.

(b) Shift $z_1, \dots, z_n$ one space to the left, overwriting the "," after "y_m".

(c) Run the machine for $\{(t, y) \,|\, t = y\}$ on the resultant string.

(GD4) Suppose there exists a machine determining membership in R. Then a machine for $\{(y, t_1, \dots, t_n) \in \text{Tx}^{n+1} \,|\, R(t_1, \dots, t_n)\}$ may run as follows:

(a) Check that input is of the form $t_1, \dots, t_{n+1}$ (where n is the arity of R and $t_1, \dots, t_{n+1} \in \text{Tx}$); if not, reject.

(b) Copy $t_2, \dots, t_{n+1}$ to the second tape.

(c) Run the machine for R on the string on the 2^{nd} tape.

(GD5) Suppose there exists a machine determining membership in R. Then a machine for $\{(t_1, t_3, \dots, t_n) \in \text{Tx}^{n-1} \,|\, R(t_1, t_1, t_3, \dots, t_n)\}$ may run as follows:

(a) Check that input is of the form $t_1, \dots, t_{m-1}$ (where m is the arity of R and $t_1, \dots, t_{m-1} \in \text{Tx}$); if not, reject.

(b) Copy "t_1," to the second tape.

(c) Copy "$t_1, \dots, t_{m-1}$" immediately after "t_1," on the second tape.

(d) Run the machine for R on the string on the 2^{nd} tape.

(GD6) Suppose there exists a machine determining membership in R. Then for all $k \in \{1, \dots, n-1\}$ (where n is the arity of R), a machine for $\{(t_1, \dots, t_n) \in \text{Tx}^n \,|\, R(t_1, \dots, t_{k+1}, t_k, \dots, t_n)\}$ may run as follows:

(a) Check that input is of the form $t_1, \dots, t_n$ (with $t_1, \dots, t_n \in \text{Tx}$); if not, reject.

(b) Move (the possibly empty) "$t_1, \dots, t_{k-1}$," from the first tape to the second tape, and move "t_{k+1}," from the first tape to directly after it on the second tape.

(c) move "t_k," from the first tape to directly after "t_{k+1}," on the second tape.

(d) move the remainder of the first tape to directly after "t_k," it on the second tape.

(e) Run the machine for R on the string on the 2^{nd} tape.

(GD7) Suppose there exists a machine determining membership in R. Then a machine for $\{(t_1, \dots, t_n) \in \mathrm{Tx}^n \mid \neg R(t_1, \dots, t_n)\}$ may run as follows:

(a) Check that input is of the form $t_1, \dots, t_n$ (with $t_1, \dots, t_n \in \mathrm{Tx}$); if not, reject.

(b) Run the machine for R on resultant string, but accept at the rejecting states and reject at the accepting states.

(GD8) Suppose there exists a machine determining membership in R, and there exists a machine determining membership in S. Then a machine for

$$\{(t_1, \dots, t_{n+k}) \in \mathrm{Tx}^{n+k} \mid R(t_1, \dots, t_n) \text{ and } S(t_{n+1}, \dots, t_{n+k})\}$$

may run as follows:

(a) Check that input is of the form $t_1, \dots, t_{n+k}$ (where n is the arity of R, k is the arity of S and
$t_1, \dots, t_{n+k} \in \mathrm{Tx}$); if not, reject.

(b) Copy "$t_{n+1}, \dots, t_{n+k}$" to the second tape.

(c) Delete the rightmost "," on the first tape.

(d) Run the machine for R on the string on the first tape and the machine for S on the string on the second tape.

(e) If both machines reach an accepting state, accept. Otherwise, reject.

(GD9) Suppose there exists a machine determining membership in R. Then a machine for $\{(y, t_2, \dots, t_n) \in \mathrm{Tx}^n \mid (\forall t_1 \in \mathrm{Tx})\, t_1 \sqsubset^{\mathbf{Tx}} y \implies R(t_1, t_2, \dots, t_n)\}$ may run as follows:

(a) Check that input is of the form $t_1, \dots, t_n$ (where n is the arity of R and $t_1, \dots, t_n \in \mathrm{Tx}$); if not, reject.

(b) Suppose t_1 is of length m. Enumerate all the substrings of t_1 on the second tape; i.e., write $s_1, \dots, s_{\frac{m(m+1)}{2}}$ (where $s_i \mathsf{c}^{\mathbf{Tx}} t_1$ for all $i \in \{1, \dots, \frac{m(m+1)}{2}\}$ and $s_i \neq s_j$ for all $i \neq j$) on the second tape.

(c) For each $i \in \{1, \dots, \frac{m(m+1)}{2}\}$,

(i) Clear the third tape.

(ii) Write "$s_i, t_2, \dots, t_n$" on the third tape.

(iii) Run the machine for R on the contents of the third tape, but instead of accepting, move onto the next pass of this loop.

(d) If the machine has not yet halted, accept.

(GD10) Let $R \subseteq \mathrm{Tx}^n$, and suppose there exist $S, T' \in \mathbf{GD}$ such that:

$$R(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+k} \in \ \mathrm{Tx})\, S(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}) \text{ and}$$

$$\neg R(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+l} \in \ \mathrm{Tx})\, T'(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}),$$

Suppose there exists a machine determining membership in S, and there exists a machine determining membership in T'. Then a machine for R may run as follows:

(a) Check that input is of the form $t_1, \ldots, t_n$ (where n is the arity of R and $t_1, \ldots, t_n \in \mathrm{Tx}$); if not, reject.

(b) Let $u_1, u_2, \ldots$ be an enumeration of all strings of the form "$s_1, \ldots, s_k$" (where k is the arity of S minus the arity of R, and $s_1, \ldots, s_k \in \mathrm{Tx}$).
Let $v_1, v_2, \ldots$ be an enumeration of all strings of the form "$s_1, \ldots, s_l$" (where l is the arity of T' minus the arity of R, and $s_1, \ldots, s_l \in \mathrm{Tx}$).
For each $i \in \mathbf{N}$,
  (i) Clear the second and third tapes.
  (ii) Write "$t_1, \ldots, t_n, u_i$" on the second tape.
  (iii) Write "$t_1, \ldots, t_n, v_i$" on the third tape.
  (iv) Simultaneously run the machine for S on the string on the first tape and run the machine for T' on the string on the second tape. Instead of rejecting, move to a state that specifies that the relevant machine has reached a rejecting state. If the machine for T' accepts, reject.
  (v) If both machines have reached a rejecting state, move onto the next pass of this loop.

It is clear that the machines in 1–3 halt from their explicit constructions. Furthermore, the machines in 4–9 halt, providing the machines they are constructed from are halting. As for the machine in 10, providing the machines it is constructed from are halting, there is only one scenario where it may not halt; when an input $t_1, \ldots, t_n$ is given such that

$$(\forall t_{n+1}, \ldots, t_{n+k} \in \ \mathrm{Tx})\, \neg S(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k})$$

and

$$(\forall t_{n+1}, \ldots, t_{n+l} \in \ \mathrm{Tx})\, \neg T'(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}).$$

If this is the case, then $\neg R(t_1, \ldots, t_n)$ and $R(t_1, \ldots, t_n)$, which cannot happen. Hence this scenario cannot happen, and thus the machine in 10 always halts. $\square$

2. Recursive $\implies$ GD: The Idea

Suppose $R \subseteq \mathrm{Tx}^n$ is recursive. Then there exist Turing machines, M and N such that for all $x \in \mathrm{Tx}^n$:

- M accepts x if and only if $x \in R$, and
- N accepts x if and only if $x \notin R$.

If we show that the $(n+1)$-ary relations

- $S_M := \{(x; c_M) \mid c_M$ is an accepting computation of x on $M\}$ and
- $S_N := \{(x; c_N) \mid c_N$ is an accepting computation of x on $N\}$

are GD, then we can apply (GD10) of Definition 4.1 to S_M and S_N to obtain a GD relation. This relation is R.

In order for c_M and c_N to make sense, we need a way to encode computations (i.e. sequences of Turing machine configurations) as elements of Tx.

Definition 5.1. Given a Turing machine with a set $Q = \{q_y, q_0, q_1, \ldots, q_n\}$ of states (where q_0 is the initial state and q_Y is the unique accepting state), and tape alphabet $\Gamma = \{a, b, \text{␣}\} \cup \{,\}$ (i.e., with Γ containing exactly the symbols 'a', 'b', ',' and '␣'), we can define a function
$\varphi : (\Gamma \cup Q \cup \{\rightarrow\}) \rightarrow \mathrm{Tx}$ by

x	a	b	,	␣	%	$\rightarrow$	q_0	q_Y	q_i
$\varphi(x)$	aba	$abba$	ab^3a	ab^4a	ab^5a	ab^6a	ab^7a	ab^8a	$ab^{8+i}a$

The % symbol denotes an infinite sequence of blank tape symbols.

If s is a Turing machine configuration depicted as a string of elements of $\Gamma \cup Q$, then we denote by $\varphi(s)$ the string with φ applied to each of the elements. For instance,

$$\varphi(\%aq_0\%) = \varphi(\%)\varphi(a)\varphi(q_0)\varphi(\%) = ab^5aabaab^6aab^5a.$$

We want to say that c_M is an accepting computation of x on machine M if and only if:

(i) c_M is of the form $\varphi(\rightarrow c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_k \rightarrow)$, where c_i are Turing machine configurations,

(ii) $c_1 = \varphi(\%q_0x\%)$

(iii) $c_k = \varphi(\%q_Y\%)$, and

(iv) each c_i is the configuration obtained by doing one step of computation on M with configuration c_{i-1}.

- Condition (ii) shows that we need the encoded version of x to construct c_M.
- We want a relation E such that $E(u, v)$ if and only if v is the encoding of u.
- In order for our "encoding relation" to be GD, we need another parameter w that witnesses the encoding of u as v.

- So we want instead a relation E such that $E(u, v, w)$ if and only if $v = \varphi(u)$ and w "describes" the steps taken to encode u.

We will not formally construct E just yet, but first give some idea of what we want E to be. For this, we need extra symbols (whose roles will ultimately be played by special strings in $\{a, b\}^*$); we use a "separator" σ and a "marker" δ.

Example: let $u = abb$ and $v = abaabbaabba$. Then $v = \varphi(u)$, so there should be some $w \in \mathrm{Tx}$ such that $E(u, v, w)$. We want $w = \sigma\ \ \delta abb\ \ \sigma\ \ aba\delta bb\ \ \sigma\ \ abaabba\delta b\ \ \sigma\ \ abaabbaabba\delta\ \ \sigma$.

In general, for $E(u, v, w)$ to hold, we want

- $w = \sigma w_0 \sigma w_1 \sigma \ldots \sigma w_k \sigma$
- $w_i = w_i^L \delta w_i^R$ (for some w_i^L and w_i^R that avoid δ and σ)
- w_i is obtained from w_{i-1} by applying one of the following transformations: $(\delta a \mapsto aba\delta), (\delta b \mapsto abba\delta)$,

where σ acts as a "separator" and δ acts as a "marker". Note that $\delta = als_b(u)bba$ and $\sigma = als_b(u)bbba$ will always work, as the longest string of consecutive b's in u is shorter, so a separator will never be confused with a substring of u.

Proposition 5.1 (E is GD).

Sketch. In essence, the construction of E relies only on the following:

- concatenating GD functions (GD by (GD2) and (SFF)):
  - constants (GD by (C))
  - $ls_b(u)$ (GD by (LSGD))
- substituting the above into GD relations (GD by (SFR))
- taking conjunctions (GD by (GD7) and (GD8))
- taking substrings, prefixes and suffixes (GD by (SPS GD))
- quantifying over substrings of w (GD by (GD9))

$\square$

The n-ary case:

$$E_n := \{\, (u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v) \in \mathrm{Tx}^{3n+1} \mid \left(\bigwedge_{i=1^n} E(u_i, v_i, w_{(i)})\right) \wedge (v = v_1 \varphi(,) \ldots \varphi(,) v_n) \,\} \text{ is GD.}$$

(Recall that $\varphi(,) = ab^3a$.)

If we already have $v := \varphi(x)$, then its relation to its accepting computation c_M is analogous to the relation between u and w. We say $C_M(v, c_M)$ if and only if v and c_M have the following relationship (even if v is not the encoding of any word on $a, b*$).

- $c_M = \varphi(\rightarrow)c_0\varphi(\rightarrow)c_1\varphi(\rightarrow)\ldots\varphi(\rightarrow)c_k\varphi(\rightarrow)$
- $c_i = c_i^L\varphi(q_i)c_i^R$ (for some c_i^L and c_i^R that avoid $\varphi(q_i)$ and $\varphi(\rightarrow)$)
- c_i is obtained from c_{i-1} by applying one of a finite list of transformations (determined by the transition function of M)

Example: A Turing machine transition $(q_4, a) \mapsto (q_2, b, L)$ *would give a transformation* $\varphi(\gamma q_4 a) \mapsto \varphi(q_2\gamma b)$ *for each* $\gamma \in \Gamma$.

C_M is a GD relation by an argument analogous to the proof of $E \in GD$.

The proof of Recursive $\implies$ GD shown below relies on the following proposition, which will be proved formally in the remainder of the chapter.

Proposition 5.2. Given a Turing machine M on the requisite alphabet, the relation

$$S_{M,n} := \{ (u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v, c_M) \in \mathrm{Tx}^{3n+2} \mid \\ E_n(u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v) \wedge C_M(v, c_M) \}$$

is GD.

We have now (informally) introduced all the concepts we need to prove the main result.

Theorem 5.2. For all $n \in \mathbb{N}$, all recursive subsets of Tx^n are GD.

Proof. Suppose $R \subseteq \mathrm{Tx}^n$ is recursive. Then there exist Turing machines, M and N such that for all $x \in \mathrm{Tx}^n$:

- $(\exists(\varphi(x); w; v, c_M) \in \mathrm{Tx}^{2n+2})\, S_{M,n}(x; \varphi(x); w; v, c_M)$
  $\iff x \in R$, and
- $(\exists(\varphi(x); w; v, c_N) \in \mathrm{Tx}^{2n+2})\, S_{N,n}(x; \varphi(x); w; v, c_N)$
  $\iff x \notin R$.

Then R is GD by (GD10). $\square$

3. Recursive $\implies$ GD: The Missing Details

First, we need to show that the encoding and computation relations are GD. The proofs are recursive in style; we argue that one relation is GD because it can be obtained via GD axioms/rules from some simpler relations. Then we show these simpler relations are GD by obtaining them via GD rules from relations that are simpler still, and so on.

The GD axioms (GD6), (GD7) and (GD8) tell us that permuting coordinates and apply Boolean operators (conjunction, disjunction, negation, implication etc) preserve GD relations. To simplify the proof, we will perform these actions without reference to (GD6), (GD7) and (GD8).

LEMMA 5.1 (Encoding is GD). There exists a GD relation $E \subseteq \mathrm{Tx}^3$ such that for all $u, v \in \mathrm{Tx}$, there exists $z \in \mathrm{Tx}$ such that $E(u, v, z)$ if and only if $v = \varphi(u)$, with φ being the encoding map from Definition 5.1.

PROOF. Let $\sigma : \mathrm{Tx} \to \mathrm{Tx}$ and $\delta : \mathrm{Tx} \to \mathrm{Tx}$ be defined by $\delta(u) = als_b(u)bba$ and $\sigma(u) = als_b(u)bbba$. Then σ and δ are GD functions by **(LSGD)**, **(C)**, Lemma 4.7 and **(SFF)**. Define the relation $E \subseteq \mathrm{Tx}^3$ by $E(u, v, z) \iff S_1(u, v, z) \wedge S_2(u, v, z) \wedge S_3(u, v, z)$, with

- $S_1(u, v, z) \iff \sigma(u)\delta(u)u\sigma(u) \sqsubset_p^{\mathrm{Tx}} z$ and
- $S_2(u, v, z) \iff \sigma(u)v\delta(u)\sigma(u) \sqsubset_s^{\mathrm{Tx}} z$ and
- $S_3(u, v, z) \iff (\forall w \sqsubset^{\mathrm{Tx}} z)\, T_1(w, u, v, z)$. ($T_1$ will be defined later.)

(Recall that we want z to be a 'sequence' of words.
S_1 says "the sequence starts with u (i.e. the original, completely unencoded word)".
S_2 says "the sequence ends with v (i.e. the completely encoded word)".
S_3 says "each word in the sequence can be obtained from the previous word by encoding one letter". Or in slightly more detail, if w is the word immediately after w_{-1} in the sequence, then there exist (possibly empty) strings x and y such that $w_{-1} = xuy$ and $w = xvy$ (where $u = \delta(u)a$ and $v = \varphi(a)\delta(u)$, or $u = \delta(u)b$ and $v = \varphi(b)\delta(u)$). Roughly speaking, $T_1 - T_3$ introduce the fact that w is the word immediately after w_{-1} in the sequence, and $T_4 - T_{6,i}$ introduce the fact that w can be obtained from w_{-1} by encoding one letter.)

Now S_1 is GD since:

- $S_1(u, v, z) \iff R_1(u, z)$, where $R_1(u, z) \iff \sigma(u)\delta(u)u\sigma(u) \sqsubset_p^{\mathrm{Tx}} z$. Hence if R_1 is GD, then S_1 is GD by (GD4).
- The relation R_2 defined by $R_2(x, z) \iff x \sqsubset_p^{\mathrm{Tx}} z$ is GD by **(SPS GD)**.
- The function $(x_1, x_2, x_3, x_4) \mapsto x_1x_2x_3x_4$ is GD by Lemma 4.7.
- Hence the function $(x_1, x_2, x_3, x_4) \mapsto \sigma(x_1)\delta(x_2)x_3\sigma(x_4)$ is GD by **(SFF)**.
- The relation R_3 defined by $R_3(x_1, x_2, x_3, x_4, z) \iff \sigma(x_1)\delta(x_2)x_3\sigma(x_4) \sqsubset_p^{\mathrm{Tx}} z$ is GD by **(SFR)**.
- The relation R_4 defined by $R_4(u, z) \iff R_3(u, u, u, u, z)$ os GD by repeated application of (GD5).
- Now $R_4 = R_1$, so R_1 is GD (and hence S_1 is GD as mentioned above).

Similarly, S_2 is GD by (GD4), **(SPS GD)**, Lemma 4.7, **(SFF) (SFR)**, and (GD5). If T_1 is GD, then S_3 will be GD by (GD9_5), which in turn makes E GD by (GD8). (We will go through the rest of the proof in less granular detail, and instead refer to a list of results needed like we have done with S_2.)

$T_1(w,u,v,z) \iff (\neg(ls_b(u)bbb \sqsubset^{\mathrm{Tx}} w) \implies (\forall s \sqsubset^{\mathrm{Tx}} z)\, T_2(s,w,u,v,z))$. If T_2 is GD, then T_1 will be GD by **(LSGD)**, **(C)**, Lemma 4.7, (GD5), **(SPS GD)**, **(SFR)**, **(SFF)**, (GD1) and (GD9_5).

$T_2(s,w,u,v,z) \iff (s\sigma(u)w\sigma(u) \sqsubset_p^{\mathrm{Tx}} z \implies (\exists w_{-1} \sqsubset^{\mathrm{Tx}} z)\, T_3(w_{-1},s,w,u,v,z))$. If T_3 is GD, then T_2 will be GD by (GD5), **(SPS GD)**, **(SFR)**, **(SFF)**, (GD3), (GD1) and (GD9_5).

$$T_3(w_{-1},s,w,u,v,z) \iff$$
$$(\neg(ls_b(u)bbb \sqsubset^{\mathrm{Tx}} w_{-1}) \wedge \sigma(u)w_{-1}\sigma(u) \sqsubset_s^{\mathrm{Tx}} s\sigma(u) \wedge T_4(w_{-1},s,w,u,v,z)).$$

If T_4 is GD, then T_3 will be GD by **(LSGD)**, **(C)**, Lemma 4.7, (GD5), **(SPS GD)**, **(SFR)**, **(SFF)**, (GD3) and (GD1).

$T_4(w_{-1},s,w,u,v,z) \iff \bigvee_{i=1}^{2}(\exists x \sqsubset^{\mathrm{Tx}} z)T_{5,i}(x,w_{-1},s,w,u,v,z)$ with $T_{5,i}(x,w_{-1},s,w,u,v,z) \iff (\exists y \sqsubset^{\mathrm{Tx}} z)T_{6,i}(y,x,w_{-1},s,w,u,v,z)$. If $T_{6,i}$ is GD for all $i \in \{1,2\}$, then T_4 will be GD by (GD9_5).

$$T_{6,i}(y,x,w_{-1},s,w,u,v,z) \iff$$
$$(w_{-1} = u_i \wedge w = v_i) \vee (w_{-1} = xu_i \wedge w = xv_i) \vee (w_{-1} = u_iy \wedge w = v_iy) \vee (w_{-1} = xu_iy \wedge w = xv_iy).$$

with $u_1 = \delta(u)a$, $v_1 = aba\delta(u)$, $u_2 = \delta(u)b$ and $v_2 = abba\delta(u)$.

For all $i \in \{1,2\}$, $T_{6,i}$ is GD by (GD5), **(SFR)**, **(SFF)**, (GD3) and **(C)**.

Hence E is a GD relation such that for all $u,v \in \mathrm{Tx}$, there exists $z \in \mathrm{Tx}$ such that $E(u,v,z)$ if and only if $v = \varphi(u)$, with φ being the encoding map from Definition 5.1. □

LEMMA 5.2 (Computation is GD). Let M be a Turing machine and $\varphi : \mathrm{Tx} \to \mathrm{Tx}$ be the corresponding encoding map from Definition 5.1. There exists a relation $C_M \subseteq \mathrm{Tx}^2$ such that for all $u \in \mathrm{Tx}$ with $u = \varphi(x)$ for some $x \in \Gamma^+$, there exists $c_u \in \mathrm{Tx}$ such that $C_M(u,c_u)$ if and only if x is accepted by M.

PROOF. Define the relation $C_M \subseteq \mathrm{Tx}^2$ by $C(u,c_u) \iff S_1(u,c_u) \wedge S_2(u,c_u) \wedge S_3(u,c_u)$, with

- $S_1(u, c_u) \iff \varphi(\rightarrow)\varphi(\%)u\varphi(\%)\varphi(\rightarrow) \sqsubset_p^{\mathrm{Tx}} z$ and
- $S_2(u, c_u) \iff \varphi(\rightarrow)\varphi(\%q_Y\%)\varphi(\rightarrow) \sqsubset_s^{\mathrm{Tx}} z$.
- $S_3(u, c_u) \iff (\forall w \sqsubset^{\mathrm{Tx}} z)\, T_1(w, u, c_u)$. ($T_1$ will be defined later.)

(Recall that we want c_u to be a 'sequence' of Turing machine configurations.

S_1 says "the sequence starts with u (i.e. the input)".

S_2 says "the sequence ends with the accepting state".

S_3 says "each configuration in the sequence can be obtained from the previous configuration by performing one step of computation". Or in slightly more detail, if w is the configuration immediately after w_{-1} in the sequence, then there exist (possibly empty) strings x and y such that $w_{-1} = xuy$ and $w = xvy$, where u and v are the local changes caused by some particular Turing machine transition. (For instance, the Turing machine transition $(q_4, a) \mapsto (q_2, b, R)$ would give $u = \varphi(q_4a)$ and $v = \varphi(bq_2)$.) Roughly speaking, $T_1 - T_3$ introduce the fact that w is the configuration immediately after w_{-1} in the sequence, and $T_4 - T_{6,i}$ introduce the fact that w can be obtained from w_{-1} by doing one step of computation.)

S_1 and S_2 are GD by (GD4), **(SPS GD)**, **(SFR)**, (GD5), **(SFF)** and (GD1). If T_1 is GD, then S_3 will be GD by (GD9$_5$), which in turn makes C GD by (GD8).

$T_1(w, u, c_u) \iff (\neg(b^6 \sqsubset^{\mathrm{Tx}} w) \implies (\forall s \sqsubset^{\mathrm{Tx}} z)\, T_2(s, w, u, c_u))$. If T_2 is GD, then T_1 will be GD by **(C)**, (GD5), **(SPS GD)**, **(SFR)**, (GD1) and (GD9$_5$).

$T_2(s, w, u, c_u) \iff (s\varphi(\rightarrow)w\varphi(\rightarrow) \sqsubset_p^{\mathrm{Tx}} z \implies (\exists w_{-1} \sqsubset^{\mathrm{Tx}} z)\, T_3(w_{-1}, s, w, u, c_u))$. If T_3 is GD, then T_2 will be GD by (GD5), **(SPS GD)**, **(SFR)**, **(SFF)**, (GD3), (GD1) and (GD9$_5$).

$T_3(w_{-1}, s, w, u, c_u) \iff (\neg(b^6 \sqsubset^{\mathrm{Tx}} w_{-1}) \wedge \varphi(\rightarrow)w_{-1}\varphi(\rightarrow) \sqsubset_s^{\mathrm{Tx}} s\varphi(\rightarrow) \wedge T_4(w_{-1}, s, w, u, c_u))$. If T_4 is GD, then T_3 will be GD by **(C)**, (GD5), **(SPS GD)**, **(SFR)**, **(SFF)**, (GD3) and (GD1).

The Turing machine M has $|Q \times \Gamma| = 4|Q|$ transitions. We assume $q_i, q_k \in Q$ and $\gamma_j, \gamma_l \in \Gamma \backslash \{\textvisiblespace\}$. For each transition, we assign to it some number of ordered pairs $(u, v) \in \mathrm{Tx}^2$ as follows:

- If the transition is of the form $(q_i, \gamma_j) \mapsto (q_k, \gamma_l, R)$, we assign to it the ordered pair $(\varphi(q_i\gamma_j), \varphi(\gamma_l q_k))$.
- If the transition is of the form $(q_i, \textvisiblespace) \mapsto (q_k, \gamma_l, R)$, we assign to it 2 ordered pairs: $(\varphi(q_i\textvisiblespace), \varphi(\gamma_l q_k))$ and $(\varphi(q_i\%), \varphi(\gamma_l q_k\%))$.

- If the transition is of the form $(q_i, \gamma_j) \mapsto (q_k, \llcorner\!\lrcorner, R)$, we assign to it 5 ordered pairs: $(\varphi(\%q_i\gamma_j), \varphi(\%q_k))$, and $(\varphi(\gamma q_i\gamma_j), \varphi(\gamma\llcorner\!\lrcorner q_k))$ for each $\gamma \in \Gamma$.
- If the transition is of the form $(q_i, \llcorner\!\lrcorner) \mapsto (q_k, \llcorner\!\lrcorner, R)$, we assign to it 13 ordered pairs: $(\varphi(\%q_i\%), \varphi(\%q_k\%))$, and $(\varphi(\gamma q_i\%), \varphi(\gamma \llcorner\!\lrcorner q_k\%))$, $(\varphi(\%q_i \llcorner\!\lrcorner), \varphi(\%q_k))$ and $(\varphi(\gamma q_i \llcorner\!\lrcorner), \varphi(\gamma \llcorner\!\lrcorner q_k))$ for each $\gamma \in \Gamma$.
- If the transition is of the form $(q_i, \gamma_j) \mapsto (q_k, \gamma_l, L)$, we assign to it 5 ordered pairs: $(\varphi(\%q_i\gamma_j), \varphi(q_k \llcorner\!\lrcorner\gamma_l))$ and $(\varphi(\gamma q_i\gamma_j), \varphi(q_k\gamma\gamma_l))$ for each $\gamma \in \Gamma$.
- If the transition is of the form $(q_i, \llcorner\!\lrcorner) \mapsto (q_k, \gamma_l, L)$, we assign to it 16 ordered pairs: $(\varphi(\gamma q_i\llcorner\!\lrcorner), \varphi(q_k\gamma\gamma_l))$, $(\varphi(\%q_i\llcorner\!\lrcorner), \varphi(\%q_k\llcorner\!\lrcorner\gamma_l))$, $(\varphi(\gamma q_i\%), \varphi(q_k\gamma\gamma_l\%))$ and $(\varphi(\%q_i\%), \varphi(\%q_k\llcorner\!\lrcorner\gamma_l\%))$ for each $\gamma \in \Gamma$.
- If the transition is of the form $(q_i, \gamma_j) \mapsto (q_k, \llcorner\!\lrcorner, L)$, we assign to it 5 ordered pairs: $(\varphi(\%q_i\gamma_j), \varphi(\%q_k\llcorner\!\lrcorner\llcorner\!\lrcorner))$ and $(\varphi(\gamma q_i\gamma_j), \varphi(q_k\gamma\llcorner\!\lrcorner))$ for each $\gamma \in \Gamma$.
- If the transition is of the form $(q_i, \llcorner\!\lrcorner) \mapsto (q_k, \llcorner\!\lrcorner, L)$, we assign to it 13 ordered pairs: $(\varphi(\%q_i\%), \varphi(\%q_k\%))$, and $(\varphi(\gamma q_i\%), \varphi(q_k\gamma\%))$, $(\varphi(\%q_i \llcorner\!\lrcorner), \varphi(\%q_k\llcorner\!\lrcorner\llcorner\!\lrcorner))$ and $(\varphi(\gamma q_i \llcorner\!\lrcorner), \varphi(q_k\gamma \llcorner\!\lrcorner))$ for each $\gamma \in \Gamma$.

The total number of ordered pairs assigned to the transitions is at most $64|Q|$. Let n be the total number of ordered pairs assigned to transitions of M, and let $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ be the set of all such ordered pairs.

Now $T_4(w_{-1}, s, w, u, c_u) \iff \bigvee_{i=1}^{n} (\exists x \sqsubset^{\mathrm{Tx}} z) T_{5,i}(x, w_{-1}, s, w, u, c_u)$ with $T_{5,i}(x, w_{-1}, s, w, u, c_u) \iff (\exists y \sqsubset^{\mathrm{Tx}} z) T_{6,i}(y, x, w_{-1}, s, w, u, c_u)$. If $T_{6,i}$ is GD for all $i \in \{1, \ldots, n\}$, then T_4 will be GD by (GD9$_5$).

$$T_{6,i}(y, x, w_{-1}, s, w, u, c_u) \iff$$

$$(w_{-1} = u_i \wedge w = v_i) \vee (w_{-1} = xu_i \wedge w = xv_i) \vee (w_{-1} = u_iy \wedge w = v_iy) \vee (w_{-1} = xu_iy \wedge w = xv_iy).$$

For all $i \in \{1, \ldots, n\}$, $T_{6,i}$ is GD by (GD5), **(SFR)**, **(SFF)**, (GD3) and **(C)**.

Hence C_M is a GD relation such that for all $u, c_u \in \mathrm{Tx}$ with $u = \varphi(x)$ for some $x \in \Gamma^+$, $C_M(u, c_u)$ if and only if c_u is an accepting computation of x in machine M; and thus, for all $u, c_u \in \mathrm{Tx}$ with $u = \varphi(x)$ for some $x \in \Gamma^+$, there exists $c_u \in \mathrm{Tx}$ such that $C_M(u, c_u)$ if and only if x is accepted by M. □

Now $S_{M,n}$ being GD follows easily, which completes the proof of Theorem 5.2.

Lemma 5.3 (S_M is GD). Let M be a Turing machine and $n \in \mathbb{N}$. Then the relation

$$S_M := \{ (u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v, c_M) \in \mathrm{Tx}^{3n+2} \mid E_n(u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v) \wedge C_M(v, c_M) \},$$

where

$$E_n := \{ (u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v) \in \mathrm{Tx}^{3n+1} \mid \\ (\bigwedge_{i=1^n} E(u_i, v_i, w_{(i)})) \wedge (v = v_1 \varphi(,) \ldots \varphi(,) v_n) \},$$

is GD.

Proof. Since E is GD by Lemma 5.1, equality is GD by (GD2) and

$$(u_1, \ldots, u_n, v_1, \ldots, v_n, w_{(1)}, \ldots, w_{(n)}, v) \mapsto v_1 \varphi(,) \ldots \varphi(,) v_n$$

is a GD function by Lemma 4.7, (GD4) and **(C)**, it follows that E_n is GD by (GD8), (GD5) and **(SFR)**.

Then $S_{M,n}$ is GD by Lemma 5.2, (GD8) and (GD5). $\square$

CHAPTER 6

Representability of ED relations

In this chapter, we introduce the notion of representability, which describes whether or not a predicate relating elements of Tx is 'simple' enough to be described (i.e. represented) in a theory. One can also fix a relation on Tx and think of representability of that relation in a theory as describing whether the theory is rich enough to capture what that relation is saying.

Furthermore, we show that all ED relations are 'strongly represented' in theories that contain TC. This will be useful for the main result of the next chapter, which concerns representability of GD relations. This chapter is based on Section 9 under Part Three of 'Undecidability without Arithmetization' [**8**], and is referenced by part (vi) of the overview.

1. Representability

We denote the set of all well-formed formulae in the language of TC by wff, and the set of all sentences in the language of TC by Sent. From here on, when we discuss formulae, sentences, theories etc, we refer to those in the language of TC unless explicitly stated otherwise.

Definition 6.1. If $t_0, \ldots, t_n$ are terms in the language of TC and $F \in \text{wff}$ is a formula with at least the free variables $x_0, \ldots, x_n$, we denote by

$$\text{sub}[F; x_0/t_0, \ldots, x_n/t_n]$$

the formula obtained by (simultaneously) substituting all instances of x_i with the term t_i for all $i \in \{0, \ldots, n\}$. (For instance, $\big[sub[x_0 \approx x_1; x_0/x_1, x_1/x_2]\big] = [x_1 \approx x_2]$, as opposed to $[x_2 \approx x_2]$ (which one would obtain via substituting x_1 for x_0 then substituting x_2 for all instances of x_1 in the result).)

In particular, we often have some $c_0, \ldots, c_n \in \text{Tx}$ and denote by
$\text{sub}[F; x_0/N(c_0), \ldots, x_n/N(c_n)]$ the formula obtained by substituting all instances of x_i with the constant term $N(c_i)$ for all $i \in \{0, \ldots, n\}$. In general, however, the terms $t_0, \ldots, t_n$ are not necessarily constant, and may in fact be variables themselves.

Definition 6.2. A relation $R \subseteq \mathrm{Tx}^n$ is **represented** in a set $T \subseteq \mathrm{wff}$ by some $F \in \mathrm{wff}$ with at least n free variables $x_0, \dots, x_{n-1}$ if and only if for all $c_0, \dots, c_{n-1} \in \mathrm{Tx}$,

$$R(c_0, \dots, c_{n-1}) \iff (\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T).$$

A relation $R \subseteq \mathrm{Tx}^n$ is **strongly represented** in a set $T \subseteq \mathrm{wff}$ by some $F \in \mathrm{wff}$ with at least n free variables $x_0, \dots, x_{n-1}$ if and only if R and $\neg R$ are represented in T by F and $\neg F$ respectively.

The formula F itself need not be in T, and is in fact most often not, since we often take T to be a theory.

We have $\neg\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] = \mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})]$ for all $F \in \mathrm{wff}$ and $c_0, \dots, c_{n-1} \in \mathrm{Tx}$, but be wary that

$$\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \notin \ T$$

does not imply

$$\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T$$

for incomplete theories T. Hence being represented is most often a strictly weaker property than being strongly represented.

Lemma 6.1. Let $T \subseteq \mathrm{wff}$ be a consistent theory which is closed under logical operations, $F, G \in \mathrm{wff}$ be formulae with precisely the free variables $x_0, \dots, x_{n-1}$ and $R \in \mathrm{Tx}^n$. Suppose for all $c_0, \dots, c_{n-1} \in \mathrm{Tx}$, we have the following:

(1) $R(c_0, \dots, c_{n-1}) \implies (\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T)$
(2) $\neg R(c_0, \dots, c_{n-1}) \implies (\mathrm{sub}[G; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T)$
(3) $[x_0] \dots [x_{n-1}] \, [G \leftrightarrow \neg F] \ \in \ T$

Then for all $c_0, \dots, c_{n-1} \in \mathrm{Tx}$,

- $R(c_0, \dots, c_{n-1}) \iff (\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T)$ and
- $\neg R(c_0, \dots, c_{n-1}) \iff (\mathrm{sub}[G; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T)$

(Because T is closed under logical operations and we have assumption 3, this means R is strongly represented in T by F.)

Proof. By assumption 3, we have

$$[\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \leftrightarrow \mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)]] \ \in \ T. \qquad (*)$$

Thus

$$
\begin{aligned}
& \mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T \\
\Longrightarrow\ & \neg\mathrm{sub}[F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T \qquad \text{(as } T \text{ is consistent)} \\
\Longleftrightarrow\ & \mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T \\
& \qquad \text{(as } \neg\mathrm{sub}[F; x_0/N(c_0), \dots, x_n/N(c_n)] = \mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)]) \\
\Longrightarrow\ & \mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T,
\end{aligned}
$$

since if $\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \in \ T$, then by $(*)$, $\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \in \ T$ as T is closed under logical operations. This contradicts $\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T$, hence $\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T$. Furthermore,

$$
(\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T) \implies \neg(\neg R(c_0, \dots, c_{n-1})) \iff R(c_0, \dots, c_{n-1})
$$

by assumption 2, so $(\mathrm{sub}[F; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T) \implies R(c_0, \dots, c_{n-1})$. The converse is true by assumption 1.

Now

$$
\begin{aligned}
& \mathrm{sub}[G; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T \\
\Longrightarrow\ & \neg\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T \qquad \text{(as } T \text{ is consistent)} \\
\Longrightarrow\ & \neg\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T,
\end{aligned}
$$

since if $\neg\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \in \ T$, then by $(*)$, $\neg\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \in \ T$ as T is closed under logical operations. This contradicts $\neg\mathrm{sub}[G; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T$, hence $\neg\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T$. Furthermore,

$$
\begin{aligned}
& \neg\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T \\
\Longleftrightarrow\ & \mathrm{sub}[F; x_0/N(c_0), \dots, x_n/N(c_n)] \ \notin \ T \qquad (\text{as } \neg\mathrm{sub}[\neg F; x_0/N(c_0), \dots, x_n/N(c_n)] \\
& \qquad = \mathrm{sub}[\neg(\neg F); x_0/N(c_0), \dots, x_n/N(c_n)] \\
& \qquad = \mathrm{sub}[F; x_0/N(c_0), \dots, x_n/N(c_n)] \quad) \\
\Longrightarrow\ & \neg R(c_0, \dots, c_{n-1}) \qquad \text{by assumption 1,}
\end{aligned}
$$

so $(\mathrm{sub}[G; x_0/N(c_0), \dots, x_{n-1}/N(c_{n-1})] \ \in \ T) \implies \neg R(c_0, \dots, c_{n-1})$. The converse is true by assumption 2.

Hence $R(c_0,\dots,c_{n-1}) \iff (\mathrm{sub}[F;x_0/N(c_0),\dots,x_{n-1}/N(c_{n-1})] \in T)$ and $\neg R(c_0,\dots,c_{n-1}) \iff (\mathrm{sub}[G;x_0/N(c_0),\dots,x_{n-1}/N(c_{n-1})] \in T)$, as required. $\square$

2. ED Relations

LEMMA 6.2. Let $T \subseteq \mathrm{wff}$ be a consistent theory which is closed under logical operations with $\mathrm{TC} \subseteq T$. Then for all $R \in \mathbf{ED}$, R is strongly represented in T by some $F \in \mathrm{wff}$.

PROOF. As T is a consistent theory which is closed under logical operations, Lemma 6.1 can be applied if assumptions 1, 2 and 3 are shown to hold. Furthermore, since T contains TC and is closed under logical operations, we can cite results about formulae in TC (which will also be in T) and deduce from them formulae in T.

(GD1) For $A := \{t \in \mathrm{Tx} \mid t = a\}$ and $B := \{t \in \mathrm{Tx} \mid t = b\}$:
Let $t \in \mathrm{Tx}$ and $F_A = [x_0 \approx \alpha]$. Suppose $A(t)$. Then $t = a$. Then

$$\begin{aligned}
\mathrm{sub}[F_A;x_0/N(t)] &= [N(t) \approx \alpha] && (\text{as } F_A = [x_0 \approx \alpha])\\
&= [N(a) \approx \alpha] && (\text{as } t = a)\\
&= [\alpha \approx \alpha] && (\text{as } N(a) = \alpha)\\
&\in T,
\end{aligned}$$

as T is closed under logical operations, and thus contains all tautologies.

Now suppose $\neg A(t)$. Then $t \neq a$. By Lemma 2.1, $t = t_1 \dots t_n$ for some $n \in \mathbf{N}$ and $t_1,\dots,t_n \in \{a,b\}$, so $n > 1$ or $t = t_1 = b$.

Suppose $t = b$. Then

$$\begin{aligned}
\mathrm{sub}[\neg F_A;x_0/N(t)] &= \neg[N(t) \approx \alpha] && (\text{as } \neg F_A = \neg[x_0 \approx \alpha])\\
&= \neg[N(b) \approx \alpha] && (\text{as } t = b)\\
&= \neg[\beta \approx \alpha] && (\text{as } N(b) = \beta)\\
&\in T && (\text{by the axiom A5 of TC})
\end{aligned}$$

Suppose $n > 1$. Then $t = uv$ for some $u = t_1 \dots t_m$ and $v = t_{m+1} \dots t_n$ with $m \in \{1,\dots,n-1\}$ and
$t_1,\dots,t_n \in \{a,b\}$. By Lemma 2.1, $u, v \in \mathrm{Tx}$. Then

$$\begin{aligned}
\mathrm{sub}[\neg F_A;x_0/N(t)] &= \neg[N(uv) \approx \alpha] && (\text{as } \neg F_A = \neg[x_0 \approx \alpha])\\
&= \neg[[N(u) * N(v)] \approx \alpha] && (\text{by Lemma 3.2})\\
&\in T && (\text{by the axiom A3 of TC})
\end{aligned}$$

Let $G_A = \neg F_A$. Then for all $t \in$ Tx,

- $A(t) \implies (\text{sub}[F_A; x_0/N(t)] \;\in\; T)$,
- $\neg A(t) \implies (\text{sub}[G_A; x_0/N(t)] \;\in\; T)$ and
- $[x_0][G_A \leftrightarrow \neg F_A] \;\in\; T$.

Hence by Lemma 6.1, A is strongly repersented in T by $F_A = [x_0 \approx \alpha]$. By a similar argument, B is strongly repersented in T by $F_B = [x_0 \approx \beta]$.

(GD2) For $R_2 := \{(t, y) \in \text{Tx}^2 \mid t = y\}$:
Let $t, y \in$ Tx and $F_2 = [x_0 \approx x_1]$. Suppose $R_2(t, y)$. Then $t = y$. Then

$$\begin{aligned}\text{sub}[F_2; x_0/N(t), x_1/N(y)] &= [N(t) \approx N(y)] && (\text{as } F_2 = [x_0 \approx x_1])\\ &= [N(y) \approx N(y)] && (\text{as } t = y)\\ &\in T\end{aligned}$$

as T is closed under logical operations, and thus contains all tautologies.

We now define a set $X :=$
$\{y \in \text{Tx} \mid (\forall t \in \text{Tx})\,(\neg R_2(t, y) \implies (\text{sub}[\neg F_2; x_0/N(t), x_1/N(y)] \;\in\; T))\}$.

Let $t \in$ Tx and suppose $\neg R_2(t, a)$. Then $t \neq a$, so $\neg A(t)$. Then

$$\begin{aligned}\text{sub}[\neg F_2; x_0/N(t), x_1/N(a)] &= \neg[N(t) \approx N(a)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1])\\ &= \neg[N(t) \approx \alpha] && (\text{as } N(a) = \alpha)\\ &= \neg\text{sub}[F_A; x_0/N(t)] && (\text{as } \neg F_A = \neg[x_0 \approx \alpha])\\ &\in T\end{aligned}$$

since $\neg A(t)$, and A is strongly represented in T by F_A.

By a similar argument, if we suppose $\neg R_2(t, b)$, then
$\text{sub}[\neg F_2; x_0/N(t), x_1/N(b)] \;\in\; T$. Hence $a, b \in X$.

Now let $y \in X$. Then for all $t \in$ Tx, we have

$$\neg R_2(t, y) \implies (\text{sub}[\neg F_2; x_0/N(t), x_1/N(y)] \;\in\; T).$$

Let $t \in$ Tx and suppose $\neg R_2(t, ya)$. Then $t \neq ya$. By Lemma 2.1, $t = t_1 \dots t_n$ for some $n \in \mathbf{N}$ and $t_1, \dots, t_n \in \{a, b\}$, so $n = 1$, or ($n > 1$ and $t_n = b$) or ($n > 1$ and $t_n = a$ and $u := t_1 \dots t_{n-1} \neq y$).

Suppose $n = 1$. Then $t = t_1 \in \{a, b\}$, so

$$\begin{aligned}
\mathrm{sub}[\neg F_2; x_0/N(t), x_1/N(ya)] &= \neg[N(t) \approx N(ya)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1]) \\
&= \neg[N(t) \approx [N(y) * N(a)]] && (\text{by Definition 3.1 (GD2)}) \\
&= \neg[N(t) \approx [N(y) * \alpha]] && (\text{as } N(a) = \alpha) \\
&\in T && (\text{by the axioms A3 and A4} \\
& && \text{of TC, as } t \in \{a, b\}, \text{ so} \\
& && N(t) = \alpha \text{ or } N(t) = \beta).
\end{aligned}$$

Suppose $n > 1$ and $t_n = b$. Then $t = vb$ for some $v = t_1 \dots t_{n-1}$ with $t_1, \dots, t_{n-1} \in \{a, b\}$.
By Lemma 2.1, $v \in \mathrm{Tx}$. Then

$$\begin{aligned}
&\mathrm{sub}[\neg F_2; x_0/N(t), x_1/N(ya)] \\
=&\neg[N(t) \approx N(ya)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1]) \\
=&\neg[N(vb) \approx N(ya)] && (\text{as } t = vb) \\
=&\neg[[N(v) * N(b)] \approx [N(y) * N(a)]] && (\text{by Definition 3.1 (2)}) \\
=&\neg[[N(v) * \beta] \approx [N(y) * \alpha]] && (\text{as } N(a) = \alpha \text{ and } N(b) = \beta) \\
\in& T && (\text{by Theorem 1.4}).
\end{aligned}$$

Suppose $n > 1$ and $t_n = a$ and $u := t_1 \dots t_{n-1} \neq y$. Then $t = ua$. As $t_1, \dots, t_{n-1} \in \{a, b\}$, we have $u \in \mathrm{Tx}$ by Lemma 2.1. Then

$$\begin{aligned}
&\mathrm{sub}[\neg F_2; x_0/N(t), x_1/N(ya)] \\
=&\neg[N(t) \approx N(ya)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1]) \\
=&\neg[N(ua) \approx N(ya)] && (\text{as } t = ua) \\
=&\neg[[N(u) * N(a)] \approx [N(y) * N(a)]] && (\text{by Definition 3.1 (2)}) \\
=&\neg[[N(u) * \alpha] \approx [N(y) * \alpha]] && (\text{as } N(a) = \alpha).
\end{aligned}$$

By Theorem 1.5(b), we have $[[N(u) * \alpha] \approx [N(y) * \alpha]] \to [N(u) \approx N(y)] \;\in\; T$, so the contrapositive $\neg[N(u) \approx N(y)] \to \neg[[N(u) * \alpha] \approx [N(y) * \alpha]]$ is also in T. Now $u \neq y$, so $\neg R_2(u, y)$, so we have

$$\begin{aligned}
\neg[N(u) \approx N(y)] &= \mathrm{sub}[\neg F_2; x_0/N(t), x_1/N(y)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1]) \\
&\in T && (\text{by inductive assumption}).
\end{aligned}$$

Thus $\text{sub}[\neg F_2; x_0/N(t), x_1/N(ya)] = \neg[[N(u) * \alpha] \approx [N(y) * \alpha]] \;\in\; T$.

Hence $ya \in X$. By a similar argument, $yb \in X$.

Thus $X \in \mathcal{B} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have $\mathrm{Tx} = \bigcap \mathcal{B} \subseteq X$, so

$$\begin{aligned} & y \in \mathrm{Tx} \\ \Longrightarrow\ & y \in X \\ \Longleftrightarrow\ & (\forall t \in \mathrm{Tx})\,(\neg R_2(t, y) \\ \Longrightarrow\ & (\text{sub}[\neg F_2; x_0/N(t), x_1/N(y)] \;\in\; T)). \end{aligned}$$

Hence for all $y \in \mathrm{Tx}$, we have
$\neg R_2(t, y) \implies (\text{sub}[\neg F_2; x_0/N(t), x_1/N(y)] \;\in\; T)$ for all $t \in \mathrm{Tx}$, so
$\neg R_2(t, y) \implies (\text{sub}[\neg F_2; x_0/N(t), x_1/N(y)] \;\in\; T)$ for all $t, y \in \mathrm{Tx}$.

Let $G_2 = \neg F_2$. Then for all $t, y \in \mathrm{Tx}$,

- $R_2(t, y) \implies (\text{sub}[F_2; x_0/N(t), x_1/N(y)] \;\in\; T)$,
- $\neg R_2(t, y) \implies (\text{sub}[G_2; x_0/N(t), x_1/N(y)] \;\in\; T)$ and
- $[x_0][x_1]\,[G_2 \leftrightarrow \neg F_2] \;\in\; T$.

Hence by Lemma 6.1, R_2 is strongly repersented in T by $F_2 = [x_0 \approx x_1]$.

(GD3) For $R_3 := \{(t, y, z) \in \mathrm{Tx}^3 \mid t = yz\}$:
Let $t, y, z \in \mathrm{Tx}$ and $F_3 = [x_0 \approx [x_1 * x_2]]$. Suppose $R_3(t, y, z)$. Then $t = yz$. Then

$$\begin{aligned} \text{sub}[F_3; x_0/N(t), x_1/N(y), x_2/N(z)] &= [N(t) \approx [N(y) * N(z)]] && (\text{as } F_3 = [x_0 \approx [x_1 * x_2]]) \\ &= [N(t) \approx N(yz)] && (\text{by Lemma 3.2}) \\ &= [N(yz) \approx N(yz)] && (\text{as } t = yz) \\ &\in T \end{aligned}$$

as T is closed under logical operations, and thus contains all tautologies.

Now suppose $\neg R_3(t,y,z)$. Then $t \neq yz$. Then $\neg R_2(t,yz)$, so

$$\begin{aligned}
&\mathrm{sub}[\neg F_3; x_0/N(t), x_1/N(y), x_2/N(z)] \\
=&\neg[N(t) \approx [N(y) * N(z)]] && (\text{as } \neg F_3 = \neg[x_0 \approx [x_1 * x_2]]) \\
=&\neg[N(t) \approx N(yz)] && (\text{by Lemma 3.2}) \\
=&\mathrm{sub}[\neg F_2; x_0/N(t), x_1/N(yz)] && (\text{as } \neg F_2 = \neg[x_0 \approx x_1]) \\
\in& T && (\text{since } \neg R_2(t,yz), \text{ and } R_2 \\
& && \text{is strongly represented in } T \text{ by } F_2).
\end{aligned}$$

Let $G_3 = \neg F_3$. Then for all $t, y, z \in \mathrm{Tx}$,

- $R_3(t,y,z) \implies (\mathrm{sub}[F_3; x_0/N(t), x_1/N(y), x_2/N(z)] \ \in \ T)$,
- $\neg R_3(t,y,z) \implies (\mathrm{sub}[F_3; x_0/N(t), x_1/N(y), x_2/N(z)] \ \in \ T)$ and
- $[x_0][x_1][x_2]\,[G_3 \leftrightarrow \neg F_3] \ \in \ T$.

Hence by Lemma 6.1, R_3 is strongly repersented in T by $F_3 = [x_0 \approx [x_1 * x_2]]$.

(GD4) Suppose R is strongly represented by F in T. Let

$$S := \{(y, t_1, \ldots, t_n) \in \mathrm{Tx}^{n+1} \mid R(t_1, \ldots, t_n)\}$$

and

$$G := [x_0 \approx x_0] \wedge \mathrm{sub}[F; x_0/x_1, \ldots, x_{n-1}/x_n].$$

Then for all $y, t_1, \ldots, t_n \in \mathrm{Tx}$,

$$\begin{aligned}
&(\mathrm{sub}[G; x_0/N(y), x_1/N(t_1), \ldots, x_n/N(t_n)] \ \in \ T) \\
\iff &([N(y) \approx N(y)] \wedge \mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &(\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &R(t_1, \ldots, t_n) \iff S(y, t_1, \ldots, t_n),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{sub}[\neg G; x_0/N(y), x_1/N(t_1), \ldots, x_n/N(t_n)] \ \in \ T) \\
\iff &(\neg[N(y) \approx N(y)] \vee \neg\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &(\neg\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &(\mathrm{sub}[\neg F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &\neg R(t_1, \ldots, t_n) \iff \neg S(y, t_1, \ldots, t_n),
\end{aligned}$$

so S is strongly represented in T by G.

(GD5) Suppose R is strongly represented by F in T.
Let $S := \{(t_1, t_3, \ldots, t_n) \in \mathrm{Tx}^{n-1} \mid R(t_1, t_1, t_3, \ldots, t_n)\}$ and
$G := \mathrm{sub}[F; x_0/x_0, x_1/x_0, x_2/x_1, \ldots, x_{n-1}/x_{n-2}]$. Then for all $y, t_1, \ldots, t_n \in \mathrm{Tx}$,

$$\begin{aligned} & (\mathrm{sub}[G; x_0/N(t_1), x_1/N(t_3), \ldots, x_{n-2}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\mathrm{sub}[F; x_0/N(t_1), x_1/N(t_1), x_2/N(t_3), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & R(t_1, t_1, t_3, \ldots, t_n) \iff S(t_1, t_3, \ldots, t_n), \end{aligned}$$

and

$$\begin{aligned} & (\mathrm{sub}[\neg G; x_0/N(t_1), x_1/N(t_3), \ldots, x_{n-2}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\neg\mathrm{sub}[F; x_0/N(t_1), x_1/N(t_1), x_2/N(t_3), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\mathrm{sub}[\neg F; x_0/N(t_1), x_1/N(t_1), x_2/N(t_3), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & \neg R(t_1, t_1, t_3, \ldots, t_n) \iff \neg S(t_1, t_3, \ldots, t_n), \end{aligned}$$

so S is strongly represented in T by G.

(GD6) Suppose R is strongly represented by F in T and let $k \in \{1, \ldots, n-1\}$ (where n is the arity of R). Let $S := \{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid R(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n)\}$ and $G := \mathrm{sub}[F; x_0/x_0, \ldots, x_{k-1}/x_k, x_k/x_{k-1}, \ldots, x_{n-1}/x_{n-1}]$. Then for all $t_1, \ldots, t_n \in \mathrm{Tx}$,

$$\begin{aligned} & (\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{k-1}/N(t_{k+1}), x_k/N(t_k), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & R(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n) \iff S(t_1, \ldots, t_n), \end{aligned}$$

and

$$\begin{aligned} & (\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\neg\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{k-1}/N(t_{k+1}), x_k/N(t_k), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & (\mathrm{sub}[\neg F; x_0/N(t_1), \ldots, x_{k-1}/N(t_{k+1}), x_k/N(t_k), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\ \Longleftrightarrow\ & \neg R(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n) \iff \neg S(t_1, \ldots, t_n), \end{aligned}$$

so S is strongly represented in T by G.

(GD7) Suppose R is strongly represented by F in T. Let
$S := \{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid \neg R(t_1, \ldots, t_n)\}$ and $G := \neg F$.

Then for all $t_1, \ldots, t_n \in \mathrm{Tx}$,

$$
\begin{aligned}
& (\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\Longleftrightarrow\ & (\mathrm{sub}[\neg F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\Longleftrightarrow\ & \neg R(t_1, \ldots, t_n) \iff S(t_1, \ldots, t_n),
\end{aligned}
$$

and

$$
\begin{aligned}
& (\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\Longleftrightarrow\ & (\mathrm{sub}[\neg(\neg F); x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\Longleftrightarrow\ & (\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\Longleftrightarrow\ & R(t_1, \ldots, t_n) \iff \neg(\neg R(t_1, \ldots, t_n)) \iff \neg S(t_1, \ldots, t_n),
\end{aligned}
$$

so S is strongly represented in T by G.

(GD8) Suppose R is strongly represented by F and R' is strongly represented by F' and in T. Let $S := \{(t_1, \ldots, t_{n+k}) \in \mathrm{Tx}^{n+k} \mid R(t_1, \ldots, t_n) \text{ and } R'(t_{n+1}, \ldots, t_{n+k})\}$ and $G := [F \wedge \mathrm{sub}[F'; x_0/x_n, \ldots, x_{k-1}/x_{n+k-1}]]$. Then for all $t_1, \ldots, t_{n+k} \in \mathrm{Tx}$,

$$
\begin{aligned}
& S(t_1, \ldots, t_{n+k}) \\
\Longleftrightarrow\ & (R(t_1, \ldots, t_n) \text{ and } R'(t_{n+1}, \ldots, t_{n+k})) \\
\Longleftrightarrow\ & (\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \text{ and} \\
& (\mathrm{sub}[F'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T) \\
\Longrightarrow\ & ([\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \wedge \mathrm{sub}[F'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})]] \ \in \ T) \\
\Longleftrightarrow\ & (\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T)
\end{aligned}
$$

and

$$
\begin{aligned}
& \neg S(t_1, \ldots, t_{n+k}) \\
\Longleftrightarrow\ & \neg(R(t_1, \ldots, t_n) \text{ and } R'(t_{n+1}, \ldots, t_{n+k})) \\
\Longleftrightarrow\ & \neg R(t_1, \ldots, t_n) \text{ or } \neg R'(t_{n+1}, \ldots, t_{n+k}) \\
\Longleftrightarrow\ & (\mathrm{sub}[\neg F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \text{ or} \\
& (\mathrm{sub}[\neg F'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T) \\
\Longleftrightarrow\ & (\neg\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \text{ or} \\
& (\neg\mathrm{sub}[F'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T) \\
\Longrightarrow\ & ([\neg\mathrm{sub}[F; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \vee \neg\mathrm{sub}[F'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})]] \ \in \ T) \\
\Longleftrightarrow\ & (\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&(\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T) \\
\implies &(\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \notin \ T) \\
\implies &\neg(\neg S(t_1, \ldots, t_{n+k})) \implies S(t_1, \ldots, t_{n+k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T) \\
\implies &(\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \notin \ T) \\
\implies &\neg S(t_1, \ldots, t_{n+k}),
\end{aligned}
$$

so $(\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T) \iff S(t_1, \ldots, t_{n+k})$ and $(\mathrm{sub}[\neg G; x_0/N(t_1), \ldots, x_{n+k-1}/N(t_{n+k})] \ \in \ T) \iff \neg S(t_1, \ldots, t_{n+k})$.

Hence S is strongly represented in T by G.

(GD9) Suppose R is strongly represented by F in T.
Let $S := \{(y, t_2, \ldots, t_n) \in \mathrm{Tx}^n \mid (\forall t_1 \in \mathrm{Tx})\, t_1 \sqsubset^{\mathbf{Tx}} y \implies R(t_1, t_2, \ldots, t_n)\}$, and let $G := [x_n][x_n \sqsubset x_0 \to \mathrm{sub}[F; x_0/x_n, x_1/x_1, \ldots, x_{n-1}/x_{n-1}]]$.

Let $y, t_2, \ldots, t_n \in \mathrm{Tx}$ and suppose $S(y, t_2, \ldots, t_n)$.
Then $(\forall t_1 \in \mathrm{Tx})\, (t_1 \sqsubset^{\mathbf{Tx}} y \implies R(t_1, t_2, \ldots, t_n))$. Then

$$
(\forall t_1 \in \mathrm{Tx})\, (t_1 \sqsubset^{\mathbf{Tx}} y \implies (\mathrm{sub}[F; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T)).
$$

Then

$$
\begin{aligned}
X_y :=& \{s \in \mathrm{Tx} \mid s \sqsubset^{\mathbf{Tx}} y\} \\
\subseteq& \{s \in \mathrm{Tx} \mid \mathrm{sub}[F; x_0/N(s), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
N(X_y) \subseteq& N(\{s \in \mathrm{Tx} \mid \mathrm{sub}[F; x_0/N(s), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\}) \\
=& \{c \in \mathrm{Cterm} \mid \mathrm{sub}[F; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\}.
\end{aligned}
$$

by Lemma 3.1.

In particular, for every $c \in N(X_y)$, the sentence $\mathrm{sub}[F; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]$ is in T.

Now T is closed under logical operations, so for every $c \in N(X_y)$, the sentence $[x_n][x_n \approx c \to \mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]$ is in T.

Then the (finite) conjunction of all such sentences is in T, which implies

$$[x_n][\bigvee\{x_n \approx c \,|\, c \in N(X_y)\} \to \mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T.$$

But $\{x_n \approx c \,|\, c \in N(X_y)\} = \{x_n \approx N(t_1) \,|\, t_1 \in X_y\}$, so

$$[x_n][\bigvee\{x_n \approx N(t_1) \,|\, t_1 \in X_y\} \to \mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T,$$

so $[x_n][x_n \sqsubset N(y) \to \mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$

as $[x_n][x_n \sqsubset N(y) \leftrightarrow \bigvee\{x_n \approx N(t_1) \,|\, t_1 \in X_y\} \ \in \ T$ by Lemma 3.3.

But

$$\begin{aligned} &[x_n][x_n \sqsubset N(y) \to \mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \\ =\,&\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)], \end{aligned}$$

so $\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$.

Now suppose $\neg S(y, t_2, \ldots, t_n)$. Then there exists $t_1 \in \mathrm{Tx}$ such that $t_1 \sqsubset^{\mathbf{Tx}} y$ and $\neg R(t_1, t_2, \ldots, t_n)$.

We have $t_1 \sqsubset^{\mathbf{Tx}} y \iff (t_1 = y$ or $(\exists z \in \mathrm{Tx})\,(y = t_1 z$ or $y = z t_1)$ or $(\exists z, w \in \mathrm{Tx})\,(y = z t_1 w))$, by definition of $\sqsubset^{\mathbf{Tx}}$.

We have

$$\begin{aligned} t_1 = y &\iff R_2(t_1, y) && \text{(by definition of } R_2)) \\ &\iff ([N(t_1) \approx N(y)] \ \in \ T) \\ & && \text{(since } R_2 \text{ is strongly represented in } T \text{ by } [x_0 \approx x_1]). \end{aligned}$$

We have

$$
\begin{aligned}
& (\exists z \in \mathrm{Tx})\,(y = t_1 z \text{ or } y = z t_1) \\
\Longleftrightarrow\ & (\exists z \in \mathrm{Tx})\,(R_3(y, t_1, z) \text{ or } R_3(y, z, t_1)) && \text{(by definition of } R_3)) \\
\Longleftrightarrow\ & (\exists z \in \mathrm{Tx})\,([N(y) \approx N(t_1) * N(z)] \ \in\ T \text{ or } [N(y) \approx N(z) * N(t_1)] \ \in\ T) \\
& && \text{(since } R_3 \text{ is strongly represented in } T \text{ by } [x_0 \approx [x_1 * x_2]]) \\
\Longrightarrow\ & (\exists z \in \mathrm{Tx})\,([N(y) \approx N(t_1) * N(z)] \vee [N(y) \approx N(z) * N(t_1)] \ \in\ T) \\
& && \text{(since } T \text{ is closed under logical operations)} \\
\Longrightarrow\ & ([Ex_0][N(y) \approx N(t_1) * x_0] \vee [N(y) \approx x_0 * N(t_1)] \ \in\ T),
\end{aligned}
$$

by quantifier rules in T, since there exists $c = N(z) \in \mathrm{Cterm}$ such that $[N(y) \approx N(t_1) * c] \vee [N(y) \approx c * N(t_1)] \ \in\ T$.

We have

$$
\begin{aligned}
& (\exists z, w \in \mathrm{Tx})\,(y = z t_1 w) \\
\Longleftrightarrow\ & (\exists z, w \in \mathrm{Tx})\, R_3(y, z, t_1 w) && \text{(by definition of } R_3)) \\
\Longleftrightarrow\ & (\exists z, w \in \mathrm{Tx})\,([N(y) \approx N(z) * N(t_1 w)] \ \in\ T) \\
& && \text{(since } R_3 \text{ is strongly represented in } T \text{ by } [x_0 \approx [x_1 * x_2]]) \\
\Longleftrightarrow\ & (\exists z, w \in \mathrm{Tx})\,([N(y) \approx N(z) * [N(t_1) * N(w)]] \ \in\ T) \\
& && \text{(since } N(t_1 w) = [N(t_1) * N(w)]) \\
\Longrightarrow\ & ([Ex_0][Ex_1][N(y) \approx x_0 * [N(t_1) * x_1]] \ \in\ T),
\end{aligned}
$$

by quantifier rules in T, since there exist $c, d \in \mathrm{Cterm}$ (with $c = N(z)$ and $d = N(w)$) such that $[N(y) \approx c * [N(t_1) * d]] \ \in\ T$.

Thus $[N(t_1) \approx N(y)] \ \in\ T$, or $[Ex_0][N(y) \approx N(t_1) * x_0] \vee [N(y) \approx x_0 * N(t_1)] \ \in\ T$, or $[Ex_0][Ex_1][N(y) \approx x_0 * [N(t_1) * x_1]] \ \in\ T$, so their disjunction is in T, as T is closed under logical operations. Hence by (A6), $N(t_1) \sqsubset N(y) \ \in\ T$.

Furthermore, since $\neg R(t_1, t_2, \ldots, t_n)$, we have $\mathrm{sub}[\neg F; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in\ T$ as R is strongly represented by F in T. Hence $[N(t_1) \ \sqsubset\ N(y) \wedge \mathrm{sub}[\neg F; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]] \ \in\ T$, as T is closed under logical operations.

Then

$$[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[\neg F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]] \;\; \in \;\; T,$$

by quantifier rules in T, since there exists $c = N(t_1) \in$ Cterm such that $[c \sqsubset N(y) \wedge \mathrm{sub}[\neg F; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]] \;\; \in \;\; T$.

But

$$\begin{aligned} &[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[\neg F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]] \\ =&[Ex_n][x_n \sqsubset N(y) \wedge \neg\mathrm{sub}[F; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]] \\ =&\mathrm{sub}[\neg G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)], \end{aligned}$$

so $\mathrm{sub}[\neg G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \;\; \in \;\; T$.

Let $G' = \neg G$. Then for all $y, t_2, \ldots, t_n \in$ Tx,

- $S(y, t_2, \ldots, t_n) \implies (\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \;\; \in \;\; T)$,
- $\neg S(y, t_2, \ldots, t_n) \implies (\mathrm{sub}[G'; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \;\; \in \;\; T)$ and
- $[x_0] \ldots [x_{n-1}]\,[G' \leftrightarrow \neg G] \;\; \in \;\; T$.

Hence by Lemma 6.1, S is strongly repersented in T by $G = [x_n][x_n \sqsubset x_0 \rightarrow \mathrm{sub}[F; x_0/x_n, x_1/x_1, \ldots, x_{n-1}/x_{n-1}]]$.

Hence by induction, each $R \in \mathbf{ED}$ is strongly represented in T by some $F \in$ wff. □

LEMMA 6.3. Let $T \subseteq$ wff be a consistent theory which is closed under logical operations with TC $\subseteq T$. Then the relation $\{(s,t) \in \mathrm{Tx}^2 \mid s \sqsubset^{\mathbf{Tx}} t\}$ is strongly represented in T by the formula $[x_0 \sqsubset x_1]$.

The proof of this result is similar to parts 1, 2 and 3 of the proof of Lemma 6.2, and we shall not include it here.

CHAPTER 7

Representability of GD relations

In this chapter, we show that all GD relations are represented in theories T that contain TC and are contained in Th(**Tx**). This is a key result that will be used in the proof of undecidability of T, which means the undecidability result depends on T being rich enough to describe recursive relations. This chapter is based on Sections 10 and 11 under Part Three of 'Undecidability without Arithmetization' [**8**], and is referenced by part (v) of the overview.

1. Existential Quantifiers

Using the notion of representability of relations, we now have a shortcut for proving the "injectivity" of the map $N : \mathrm{Tx} \to \mathrm{Cterm}$ with respect to TC:

LEMMA 7.1. For all $s, t \in \mathrm{Tx}$, if $N(s) \approx N(t) \;\; \in \;\mathrm{TC}$, then $s = t$.

PROOF. Let $s, t \in \mathrm{Tx}$ and suppose $N(s) \approx N(t) \;\; \in \; \mathrm{TC}$. As TC is closed under logical operations and $\mathrm{TC} \subseteq \mathrm{TC}$, by part 2 of the proof of Lemma 6.2, the relation $R := \{(s,t) \in \mathrm{Tx}^2 \,|\, s = t\}$ is strongly represented in TC by $F := [x_0 \approx x_1]$. Now $N(s) \approx N(t) = \mathrm{sub}[F; x_0/N(s), x_1/N(t)]$, so $\mathrm{sub}[F; x_0/N(s), x_1/N(t)] \;\; \in \; \mathrm{TC}$, so $R(s,t)$, so $s = t$, as required. $\square$

DEFINITION 7.1. Let $c \in \mathrm{Cterm}$. Then

$$\mathrm{De}(c) = t \iff (c \approx N(t) \in \mathrm{TC}).$$

By Lemma 3.4 and Lemma 7.1, De: $\mathrm{Cterm} \to \mathrm{Tx}$ is a well-defined function.

LEMMA 7.2. The map De has the following properties:

(1) $(\forall t \in \mathrm{Tx})\, t = \mathrm{De}(N(t))$
(2) $(\forall c \in \mathrm{Cterm})\, (c \approx N(\mathrm{De}(c)) \;\; \in \; \mathrm{TC})$

The above properties can be obtained from the definition in a straightforward way.

LEMMA 7.3. For all $c \in \mathrm{Cterm}$, $\mathrm{De}(c) = c^{\mathbf{Tx}}$.

PROOF. We have $N(a) = \alpha$ and $N(b) = \beta$, so $\mathrm{De}(\alpha) = a$ and $\mathrm{De}(\beta) = b$. By the definitions of $\alpha^{\mathbf{Tx}}$ and $\beta^{\mathbf{Tx}}$ listed in Lemma 2.3, we have $\alpha^{\mathbf{Tx}} = a$ and $\beta^{\mathbf{Tx}} = b$. Hence $\mathrm{De}(\alpha) = \alpha^{\mathbf{Tx}}$ and $\mathrm{De}(\beta) = \beta^{\mathbf{Tx}}$.

Let $c, d \in \mathrm{Cterm}$ and suppose $\mathrm{De}(c) = s = c^{\mathbf{Tx}}$ and $\mathrm{De}(d) = t = d^{\mathbf{Tx}}$. Then $N(s) \approx c \ \in$ TC and $N(t) \approx d \ \in$ TC. By Lemma 3.2, we have $N(st) \approx [N(s) * N(t)] \ \in$ TC, so by substituting c for $N(s)$ and d for $N(t)$ we can get $N(st) \approx [c*d] \ \in$ TC, so $\mathrm{De}([c*d]) = st$. Now $[c*d]^{\mathbf{Tx}} = (c^{\mathbf{Tx}} *^{\mathbf{Tx}} d^{\mathbf{Tx}}) = (s *^{\mathbf{Tx}} t) = st$, so $\mathrm{De}([c*d]) = [c*d]^{\mathbf{Tx}}$.

Hence by induction, $\mathrm{De}(c) = c^{\mathbf{Tx}}$ for all $c \in \mathrm{Cterm}$. □

Recall that for all $F \in \mathrm{Sent}$, we have $F \in \mathrm{Th}(\mathbf{Tx}) \iff \mathbf{Tx} \models F$. So for all $c, d \in$ Cterm,

$$c \approx d \in \mathrm{Th}(\mathbf{Tx}) \iff c^{\mathbf{Tx}} = d^{\mathbf{Tx}} \iff \mathrm{De}(c) = \mathrm{De}(d)$$

by Lemma 7.2.

LEMMA 7.4. If $F \in \mathrm{wff}$ has precisely n free variables $x_0, \ldots, x_{n-1}$, then

$$\begin{aligned} &([Ex_0] \ldots [Ex_{n-1}]F \in \mathrm{Th}(\mathbf{Tx})) \\ \iff &(\exists c_0, \ldots, c_{n-1} \in \mathrm{Cterm})(\mathrm{sub}[F; x_0/c_0, \ldots, x_{n-1}/c_{n-1}] \in \mathrm{Th}(\mathbf{Tx})). \end{aligned}$$

PROOF. The backward implication holds due to by quantifier rules in $\mathrm{Th}(\mathbf{Tx})$. We shall now prove the forward implication:

Suppose $[Ex_0] \ldots [Ex_{n-1}]F \in \mathrm{Th}(\mathbf{Tx})$. Then
$\mathbf{Tx} \models [Ex_0] \ldots [Ex_{n-1}]F$, so $(\exists t_0, \ldots, t_{n-1} \in \mathrm{Tx})\, F^{\mathbf{Tx}}(t_0, \ldots, t_{n-1})$.

By Lemma 3.4, there exist $c_0, \ldots, c_{n-1} \in \mathrm{Cterm}$ such that $N(t_i) = c_i$ for all $i \in \{0, \ldots, n-1\}$. Then $(\exists c_0, \ldots, c_{n-1} \in \mathrm{Cterm})\, F^{\mathbf{Tx}}(\mathrm{De}(c_0), \ldots, \mathrm{De}(c_{n-1}))$, so
$(\exists c_0, \ldots, c_{n-1} \in \mathrm{Cterm})\, F^{\mathbf{Tx}}(c_0^{\mathbf{Tx}}, \ldots, c_{n-1}^{\mathbf{Tx}})$ by Lemma 7.2.

But

$$\begin{aligned} &F^{\mathbf{Tx}}(c_0^{\mathbf{Tx}}, \dots, c_{n-1}^{\mathbf{Tx}}) \\ \iff &\mathbf{Tx} \models \mathrm{sub}[F; x_0/c_0, \dots, x_{n-1}/c_{n-1}] \\ \iff &(\mathrm{sub}[F; x_0/c_0, \dots, x_{n-1}/c_{n-1}] \in \mathrm{Th}(\mathbf{Tx})), \end{aligned}$$

so $(\exists c_0, \dots, c_{n-1} \in \mathrm{Cterm})\,(\mathrm{sub}[F; x_0/c_0, \dots, x_{n-1}/c_{n-1}] \in \mathrm{Th}(\mathbf{Tx}))$, as required. $\square$

2. GD Relations

LEMMA 7.5. For all $R \in \mathbf{GD}$, there exists $F \in$ wff such that for all theories $T \subseteq$ wff such that $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations, R is represented in T by F.

PROOF. We shall prove by induction the seemingly stronger property that for all $R \in \mathbf{GD}$, there exist $F, G \in$ wff such that for all theories $T \subseteq$ wff where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) R is represented by F in T, and
(2) $\neg R$ is represented by G in T.

Let $T \subseteq$ wff be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Then we may cite results about formulae in TC (which will also be in T) and deduce from them formulae in T. Furthermore, T is consistent since $T \subseteq \mathrm{Th}(\mathbf{Tx})$ and $\mathrm{Th}(\mathbf{Tx})$ is consistent.

Base cases: Since T contains TC, is consistent and is closed under logical operations, by Lemma 6.2,

- $A := \{t \in \mathrm{Tx} \mid t = a\}$ is represented by $[x_0 \approx \alpha]$ in T,
- $\neg A$ is represented by $\neg[x_0 \approx \alpha]$ in T,
- $B := \{t \in \mathrm{Tx} \mid t = b\}$ is represented by $[x_0 \approx \beta]$ in T,
- $\neg B$ is represented by $\neg[x_0 \approx \beta]$ in T,
- $R_2 := \{(t, y) \in \mathrm{Tx}^2 \mid t = y\}$ is represented by $[x_0 \approx x_1]$ in T,
- $\neg R_2$ is represented by $\neg[x_0 \approx x_1]$ in T,
- $R_3 := \{(t, y, z) \in \mathrm{Tx}^3 \mid t = yz\}$ is represented by $[x_0 \approx [x_1 * x_2]]$ in T, and
- $\neg R_3$ is represented by $\neg[x_0 \approx [x_1 * x_2]]$ in T.

Hence if R is one of the initial relations, then there exist $F, G \in$ wff such that for all theories $T \subseteq$ wff where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) R is represented by F in T, and
(2) $\neg R$ is represented by G in T.

Inductive conditions: Let R_1 and R_2 be relations in Tx and let $F_1, F_1', F_2, F_2' \in$ wff. Suppose for all theories $T \subseteq$ wff where TC $\subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

- R_1 is represented by F_1 in T and in $\mathrm{Th}(\mathbf{Tx})$,
- $\neg R_1$ is represented by F_1' in T and in $\mathrm{Th}(\mathbf{Tx})$,
- R_2 is represented by F_2 in T and in $\mathrm{Th}(\mathbf{Tx})$, and
- $\neg R_2$ is represented by F_2' in T and in $\mathrm{Th}(\mathbf{Tx})$.

(GD4). Let $T \subseteq$ wff be a theory closed under logical operations with TC $\subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Let

- $S := \{(y, t_1, \ldots, t_n) \in \mathrm{Tx}^{n+1} \mid R_1(t_1, \ldots, t_n)\}$,
- $G := [x_0 \approx x_0] \wedge \mathrm{sub}[F_1; x_0/x_1, \ldots, x_{n-1}/x_n]$ and
- $G' := [x_0 \approx x_0] \wedge \mathrm{sub}[F_1'; x_0/x_1, \ldots, x_{n-1}/x_n]$.

Then for all $y, t_1, \ldots, t_n \in$ Tx,

$$\begin{aligned}
&(\mathrm{sub}[G; x_0/N(y), x_1/N(t_1), \ldots, x_n/N(t_n)] \ \in \ T)\\
\iff &([N(y) \approx N(y)] \wedge \mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T)\\
\iff &(\mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T)\\
\iff &R_1(t_1, \ldots, t_n) \iff S(y, t_1, \ldots, t_n),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{sub}[G'; x_0/N(y), x_1/N(t_1), \ldots, x_n/N(t_n)] \ \in \ T)\\
\iff &([N(y) \approx N(y)] \wedge \mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T)\\
\iff &(\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T)\\
\iff &\neg R_1(t_1, \ldots, t_n) \iff \neg S(y, t_1, \ldots, t_n),
\end{aligned}$$

so for all theories $T \subseteq$ wff where TC $\subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and
(2) $\neg S$ is represented by G' in T.

(GD5). Let $T \subseteq$ wff be a theory closed under logical operations with TC $\subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Let

- $S := \{(t_1, t_3, \ldots, t_n) \in \mathrm{Tx}^{n-1} \mid R_1(t_1, t_1, t_3, \ldots, t_n)\}$,
- $G := \mathrm{sub}[F_1; x_0/x_0, x_1/x_0, x_2/x_1, \ldots, x_{n-1}/x_{n-2}]$ and
- $G' := \mathrm{sub}[F_1'; x_0/x_0, x_1/x_0, x_2/x_1, \ldots, x_{n-1}/x_{n-2}]$.

Then for all $t_1, t_3, \ldots, t_n \in \mathrm{Tx}$,

$$\begin{aligned} & (\mathrm{sub}[G; x_0/N(t_1), x_1/N(t_3), \ldots, x_{n-2}/N(t_n)] \;\in\; T) \\ \iff & (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_1), x_2/N(t_3), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & R_1(t_1, t_1, t_3, \ldots, t_n) \iff S(t_1, t_3, \ldots, t_n), \\ \text{and} \qquad & (\mathrm{sub}[G'; x_0/N(t_1), x_1/N(t_3), \ldots, x_{n-2}/N(t_n)] \;\in\; T) \\ \iff & (\mathrm{sub}[F_1'; x_0/N(t_1), x_1/N(t_1), x_2/N(t_3), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & \neg R_1(t_1, t_1, t_3, \ldots, t_n) \iff \neg S(t_1, t_3, \ldots, t_n), \end{aligned}$$

so for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and
(2) $\neg S$ is represented by G' in T.

(GD6). Let $T \subseteq \mathrm{wff}$ be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Let

- $k \in \{1, \ldots, n-1\}$ (where n is the arity of R_1),
- $S := \{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid R_1(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n)\}$,
- $G := \mathrm{sub}[F_1; x_0/x_0, \ldots, x_{k-1}/x_k, x_k/x_{k-1}, \ldots, x_{n-1}/x_{n-1}]$ and
- $G' := \mathrm{sub}[F_1'; x_0/x_0, \ldots, x_{k-1}/x_k, x_k/x_{k-1}, \ldots, x_{n-1}/x_{n-1}]$.

Then for all $t_1, \ldots, t_n \in \mathrm{Tx}$,

$$\begin{aligned} & (\mathrm{sub}[G; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & (\mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{k-1}/N(t_{k+1}), x_k/N(t_k), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & R_1(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n) \iff S(t_1, \ldots, t_n), \\ \text{and} \qquad & (\mathrm{sub}[G'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & (\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{k-1}/N(t_{k+1}), x_k/N(t_k), \ldots, x_{n-1}/N(t_n)] \;\in\; T) \\ \iff & \neg R_1(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n) \iff \neg S(t_1, \ldots, t_n), \end{aligned}$$

so for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and
(2) $\neg S$ is represented by G' in T.

(GD7). Let $T \subseteq \mathrm{wff}$ be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Let $S := \{(t_1, \ldots, t_n) \in \mathrm{Tx}^n \mid \neg R_1(t_1, \ldots, t_n)\}$. Then for all $t_1, \ldots, t_n \in$

Tx,

$$\begin{aligned}
&(\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &\neg R_1(t_1, \ldots, t_n) \iff S(t_1, \ldots, t_n),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
\iff &R_1(t_1, \ldots, t_n) \iff \neg(\neg R_1(t_1, \ldots, t_n)) \iff \neg S(t_1, \ldots, t_n),
\end{aligned}$$

so for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by F_1' in T, and
(2) $\neg S$ is represented by F_1 in T.

(GD8). Let $T \subseteq \mathrm{wff}$ be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Suppose $R_1 \subseteq \mathrm{Tx}^n$ and $R_2 \subseteq \mathrm{Tx}^k$ for some $n \in \mathbb{N}$ and $k \in \{1, \ldots, n-1\}$.

Now the theories T and $\mathrm{Th}(\mathbf{Tx})$ both satisfy the properties of containing TC, being closed under logical operations and being contained by $\mathrm{Th}(\mathbf{Tx})$, so for all $t_1, \ldots, t_{n+k} \in \mathrm{Tx}$, we have

- $\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T$
  $\iff \neg R_1(t_1, \ldots, t_n)$
  $\iff \mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ \mathrm{Th}(\mathbf{Tx})$, and
- $\mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T$
  $\iff \neg R_2(t_{n+1}, \ldots, t_{n+k})$
  $\iff \mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ \mathrm{Th}(\mathbf{Tx})$ by assumption.

Hence

$$\begin{aligned}
&([\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \vee \mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})]] \ \in \ T) \\
\iff &([\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \vee \mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})]] \ \in \ \mathrm{Th}(\mathbf{Tx})) \\
\iff &(\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ \mathrm{Th}(\mathbf{Tx})) \\
&\text{ or } (\mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ \mathrm{Th}(\mathbf{Tx})) \ \text{ (as } \mathrm{Th}(\mathbf{Tx}) \text{ is complete)} \\
\iff &(\mathrm{sub}[F_1'; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
&\text{ or } (\mathrm{sub}[F_2'; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&([\mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \wedge \mathrm{sub}[F_2; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})]] \ \in \ T) \\
\iff &(\mathrm{sub}[F_1; x_0/N(t_1), \ldots, x_{n-1}/N(t_n)] \ \in \ T) \\
&\text{ and } (\mathrm{sub}[F_2; x_0/N(t_{n+1}), \ldots, x_{k-1}/N(t_{n+k})] \ \in \ T)
\end{aligned}$$

holds for all theories T of interest.

Now let

- $S := \{(t_1, \dots, t_{n+k}) \in \mathrm{Tx}^{n+k} \mid R_1(t_1, \dots, t_n) \text{ and } R_2(t_{n+1}, \dots, t_{n+k})\}$,
- $G := [F_1 \wedge \mathrm{sub}[F_2; x_0/x_n, \dots, x_{k-1}/x_{n+k-1}]]$ and
- $G' := [F_1' \vee \mathrm{sub}[F_2'; x_0/x_n, \dots, x_{k-1}/x_{n+k-1}]]$.

Then for all $t_1, \dots, t_{n+k} \in \mathrm{Tx}$,

$$\begin{aligned}
&(\mathrm{sub}[G; x_0/N(t_1), \dots, x_{n+k-1}/N(t_{n+k})] \;\in\; T) \\
\iff &([\mathrm{sub}[F_1; x_0/N(t_1), \dots, x_{n-1}/N(t_n)] \wedge \mathrm{sub}[F_2; x_0/N(t_{n+1}), \dots, x_{k-1}/N(t_{n+k})]] \;\in\; T) \\
\iff &(\mathrm{sub}[F_1; x_0/N(t_1), \dots, x_{n-1}/N(t_n)] \;\in\; T) \\
&\text{ and } (\mathrm{sub}[F_2; x_0/N(t_{n+1}), \dots, x_{k-1}/N(t_{n+k})] \;\in\; T) \\
\iff &(R_1(t_1, \dots, t_n) \text{ and } R_2(t_{n+1}, \dots, t_{n+k})) \iff S(t_1, \dots, t_{n+k}),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{sub}[G'; x_0/N(t_1), \dots, x_{n+k-1}/N(t_{n+k})] \;\in\; T) \\
\iff &([\mathrm{sub}[F_1'; x_0/N(t_1), \dots, x_{n-1}/N(t_n)] \vee \mathrm{sub}[F_2'; x_0/N(t_{n+1}), \dots, x_{k-1}/N(t_{n+k})]] \;\in\; T) \\
\iff &(\mathrm{sub}[F_1'; x_0/N(t_1), \dots, x_{n-1}/N(t_n)] \;\in\; T) \\
&\text{ or } (\mathrm{sub}[F_2'; x_0/N(t_{n+1}), \dots, x_{k-1}/N(t_{n+k})] \;\in\; T) \\
\iff &\neg R_1(t_1, \dots, t_n) \text{ or } \neg R_2(t_{n+1}, \dots, t_{n+k}) \\
\iff &\neg(R_1(t_1, \dots, t_n) \text{ and } R_2(t_{n+1}, \dots, t_{n+k})) \iff \neg S(t_1, \dots, t_{n+k}),
\end{aligned}$$

so for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and
(2) $\neg S$ is represented by G' in T.

(GD9). Let $T \subseteq \mathrm{wff}$ be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Suppose $R_1 \subseteq \mathrm{Tx}^n$ and $R_2 \subseteq \mathrm{Tx}^k$. Let

- $S := \{(y, t_2, \dots, t_n) \in \mathrm{Tx}^n \mid (\forall t_1 \in \mathrm{Tx})\, t_1 \sqsubset^{\mathbf{Tx}} y \implies R_1(t_1, t_2, \dots, t_n)\}$,
- $G := [x_n][x_n \sqsubset x_0 \to \mathrm{sub}[F_1; x_0/x_n, x_1/x_1, \dots, x_{n-1}/x_{n-1}]]$ and
- $G' := [Ex_n][x_n \sqsubset x_0 \,\wedge\, \mathrm{sub}[F_1'; x_0/x_n, x_1/x_1, \dots, x_{n-1}/x_{n-1}]]$.

Let $y, t_2, \dots, t_n \in \mathrm{Tx}$ and suppose $S(y, t_2, \dots, t_n)$. Then $(\forall t_1 \in \mathrm{Tx})\, (t_1 \sqsubset^{\mathbf{Tx}} y \implies R_1(t_1, t_2, \dots, t_n))$. Then $(\forall t_1 \in \mathrm{Tx})\, (t_1 \sqsubset^{\mathbf{Tx}} y \implies (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \dots, x_{n-1}/N(t_n)] \;\in\; T))$.

Then

$$\begin{aligned} X_y &:= \{s \in \mathrm{Tx} \,|\, s \sqsubset^{\mathbf{Tx}} y\} \\ &\subseteq \{s \in \mathrm{Tx} \,|\, \mathrm{sub}[F_1; x_0/N(s), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\}. \end{aligned}$$

Then

$$\begin{aligned} N(X_y) &\subseteq N(\{s \in \mathrm{Tx} \,|\, \mathrm{sub}[F_1; x_0/N(s), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\}) \\ &= \{c \in \mathrm{Cterm} \,|\, \mathrm{sub}[F_1; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T\} \end{aligned}$$

by Lemma 3.1.

In particular, for every $c \in N(X_y)$, the sentence $\mathrm{sub}[F_1; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]$ is in T.

Now T is closed under logical operations, so for every $c \in N(X_y)$, the sentence $[x_n][x_n \approx c \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)]$ is in T.

Then the (finite) conjunction of all such sentences is in T, which implies

$$[x_n][\bigvee\{x_n \approx c \,|\, c \in N(X_y)\} \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T.$$

But $\{x_n \approx c \,|\, c \in N(X_y)\} = \{x_n \approx N(t_1) \,|\, t_1 \in X_y\}$, so

$$[x_n][\bigvee\{x_n \approx N(t_1) \,|\, t_1 \in X_y\} \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T,$$

so $[x_n][x_n \sqsubset N(y) \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$ as

$$[x_n][x_n \sqsubset N(y) \leftrightarrow \bigvee\{x_n \approx N(t_1) \,|\, t_1 \in X_y\}] \ \in \ T$$

by Lemma 3.3.

But

$$\begin{aligned} &[x_n][x_n \sqsubset N(y) \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \\ = &\,\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)], \end{aligned}$$

so $\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$.

Conversely, suppose $\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$. Then

$$[x_n][x_n \sqsubset N(y) \to \mathrm{sub}[F_1; x_0/x_n, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T.$$

By quantifier rules in T, this implies

$$[c \sqsubset N(y) \to \mathrm{sub}[F_1; x_0/c, x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T.$$

for all $c \in$ Cterm. By Lemma 3.1, $N(t_1) \in$ Cterm for all $t_1 \in X_y$, so in particular,

$$[N(t_1) \sqsubset N(y) \to \mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T. \qquad (*)$$

for all $t_1 \in X_y$.

Furthermore, for all $t_1 \in X_y$, we have $[N(t_1) \approx N(t_1)] \ \in \ T$, as $[N(t_1) \approx N(t_1)]$ is a tautology. Then $\bigvee\{[N(t_1) \approx N(t)] \,|\, t \in X_y\} \ \in \ T$,
since $[N(t_1) \approx N(t_1)] \in \{N(t_1) \approx N(t) \,|\, t \in X_y\}$.

Now $[x_n][x_n \sqsubset N(y) \leftrightarrow \bigvee\{[x_n \approx N(t)] \,|\, t \in X_y\}] \ \in \ T$ by Lemma 3.3, so

$$N(t_1) \sqsubset N(y) \ \in \ T \qquad (**)$$

By $(*)$ and $(**)$,

$$\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T$$

for all $t_1 \in X_y$, by modus pollens in T. Hence for all $t_1 \in$ Tx,

$$t_1 \in X_y \implies (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T).$$

Now for all $t_1 \in$ Tx, we have $t_1 \in X_y \iff t_1 \sqsubset^{\mathbf{Tx}} y$ by definition of X_y. Hence

$$(\forall t_1 \in \mathrm{Tx})\, t_1 \sqsubset^{\mathbf{Tx}} y \implies (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T),$$

and so $S(y, t_2, \ldots, t_n)$ by definition of S.

Hence $S(y, t_2, \ldots, t_n) \iff (\mathrm{sub}[G; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T)$ for all $y, t_2, \ldots, t_n \in$ Tx, so S is represented in T by G.

Now suppose $\neg S(y, t_2, \ldots, t_n)$. Then there exists $t_1 \in$ Tx such that $t_1 \sqsubset^{\mathbf{Tx}} y$ and $\neg R_1(t_1, t_2, \ldots, t_n)$.
We have $t_1 \sqsubset^{\mathbf{Tx}} y \iff (t_1 = y$ or $(\exists z \in \mathrm{Tx})\,(y = t_1 z$ or $y = z t_1)$ or
$(\exists z, w \in \mathrm{Tx})\,(y = z t_1 w))$, by definition of $\sqsubset^{\mathbf{Tx}}$.

We have

$$\begin{aligned} t_1 = y &\iff R_2(t_1, y) && \text{(by definition of } R_2\text{))} \\ &\iff ([N(t_1) \approx N(y)] \;\in\; T) \end{aligned}$$

since R_2 is strongly represented in T by $[x_0 \approx x_1]$.

We have

$$\begin{aligned} &(\exists z \in \mathrm{Tx})\,(y = t_1 z \text{ or } y = z t_1) \\ \iff &(\exists z \in \mathrm{Tx})\,(R_3(y, t_1, z) \text{ or } R_3(y, z, t_1)) && \text{(by definition of } R_3\text{))} \\ \iff &(\exists z \in \mathrm{Tx})\,([N(y) \approx N(t_1) * N(z)] \;\in\; T \text{ or } [N(y) \approx N(z) * N(t_1)] \;\in\; T) \\ & && \text{(since } R_3 \text{ is strongly represented in } T \text{ by } [x_0 \approx [x_1 * x_2]]\text{)} \\ \implies &(\exists z \in \mathrm{Tx})\,([N(y) \approx N(t_1) * N(z)] \vee [N(y) \approx N(z) * N(t_1)] \;\in\; T) \\ & && \text{(since } T \text{ is closed under logical operations)} \\ \implies &([Ex_0][N(y) \approx N(t_1) * x_0] \vee [N(y) \approx x_0 * N(t_1)] \;\in\; T), \end{aligned}$$

by quantifier rules in T, since there exists $c = N(z) \in$ Cterm such that $[N(y) \approx N(t_1) * c] \vee [N(y) \approx c * N(t_1)] \;\in\; T$.

We have

$$\begin{aligned} &(\exists z, w \in \mathrm{Tx})\,(y = z t_1 w) \\ \iff &(\exists z, w \in \mathrm{Tx})\,R_3(y, z, t_1 w) && \text{(by definition of } R_3\text{))} \\ \iff &(\exists z, w \in \mathrm{Tx})\,([N(y) \approx N(z) * N(t_1 w)] \;\in\; T) \\ & && \text{(since } R_3 \text{ is strongly represented in } T \text{ by } [x_0 \approx [x_1 * x_2]]\text{)} \\ \iff &(\exists z, w \in \mathrm{Tx})\,([N(y) \approx N(z) * [N(t_1) * N(w)]] \;\in\; T) \\ & && \text{(since } N(t_1 w) = [N(t_1) * N(w)]\text{)} \\ \implies &([Ex_0][Ex_1][N(y) \approx x_0 * [N(t_1) * x_1]] \;\in\; T), \end{aligned}$$

by quantifier rules in T, since there exist $c, d \in$ Cterm (with $c = N(z)$ and $d = N(w)$) such that $[N(y) \approx c * [N(t_1) * d]] \;\in\; T$.

Thus $[N(t_1) \approx N(y)] \;\in\; T$, or
$[Ex_0][N(y) \approx N(t_1) * x_0] \vee [N(y) \approx x_0 * N(t_1)] \;\in\; T$, or
$[Ex_0][Ex_1][N(y) \approx x_0 * [N(t_1) * x_1]] \;\in\; T$, so their disjunction is in T, as T is closed under

logical operations. Hence by (A6), $N(t_1) \sqsubset N(y) \;\in\; T$.

Furthermore, since $\neg R_1(t_1, t_2, \dots, t_n)$, we have $\mathrm{sub}[F_1'; x_0/N(t_1), x_1/N(t_2), \dots, x_{n-1}/N(t_n)] \;\in\; T$ as $\neg R$ is represented by F_1' in T. Hence $[N(t_1) \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/N(t_1), x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; T$, as T is closed under logical operations.

Then

$$[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/x_n, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; T,$$

by quantifier rules in T, since there exists $c = N(t_1) \in \mathrm{Cterm}$ such that $[c \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/c, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; T$.

But

$$\begin{aligned}&[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/x_n, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]]\\ =\,&\mathrm{sub}[G'; x_0/N(y), x_1/N(t_2), \dots, x_{n-1}/N(t_n)],\end{aligned}$$

so $\mathrm{sub}[G'; x_0/N(y), x_1/N(t_2), \dots, x_{n-1}/N(t_n)] \;\in\; T$.

Conversely, suppose $\mathrm{sub}[G'; x_0/N(y), x_1/N(t_2), \dots, x_{n-1}/N(t_n)] \;\in\; T$. Then

$$[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/x_n, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; T,$$

so

$$[Ex_n][x_n \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/x_n, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; \mathrm{Th}(\mathbf{Tx}),$$

since $T \subseteq \mathrm{Th}(\mathbf{Tx})$. Then

$$(\exists c \in \mathrm{Cterm})\,([c \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/c, x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; \mathrm{Th}(\mathbf{Tx}))$$

by Lemma 7.4. Now $c \approx N(\mathrm{De}(c)) \;\in\; \mathrm{TC}$ by Lemma 7.2(2), so $c \approx N(\mathrm{De}(c)) \;\in \mathrm{Th}(\mathbf{Tx})$ as $\mathrm{TC} \subseteq \mathrm{Th}(\mathbf{Tx})$. Then

$$\begin{aligned}&(\exists c \in \mathrm{Cterm})\\ &([N(\mathrm{De}(c)) \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/N(\mathrm{De}(c)), x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; \mathrm{Th}(\mathbf{Tx})),\end{aligned}$$

by extensionality in $\mathrm{Th}(\mathbf{Tx})$. Then

$$\begin{aligned}&(\exists t \in \mathrm{Tx})\\ &([N(t) \sqsubset N(y) \wedge \mathrm{sub}[F_1'; x_0/N(t), x_1/N(t_2), \dots, x_{n-1}/N(t_n)]] \;\in\; \mathrm{Th}(\mathbf{Tx})).\end{aligned}$$

Then

$$(\exists t \in \mathrm{Tx})\,(N(t) \sqsubset N(y) \ \in \ \mathrm{Th}(\mathbf{Tx}))$$
$$\text{and } (\mathrm{sub}[F_1'; x_0/N(t), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ \mathrm{Th}(\mathbf{Tx})),$$

as $\mathrm{Th}(\mathbf{Tx})$ is complete. Then

$$(\exists t \in \mathrm{Tx})\, t \sqsubset^{\mathbf{Tx}} y \text{ and } (\mathrm{sub}[F_1'; x_0/N(t), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ \mathrm{Th}(\mathbf{Tx}))$$

by Lemma 6.3. Then

$$(\exists t \in \mathrm{Tx})\, t \sqsubset^{\mathbf{Tx}} y \text{ and } \neg R_1(t_1, t_2, \ldots, t_n),$$

since $\mathrm{Th}(\mathbf{Tx}) \subseteq$ wff is a theory closed under logical operations and
$\mathrm{TC} \subseteq \mathrm{Th}(\mathbf{Tx}) \subseteq \mathrm{Th}(\mathbf{Tx})$, so $\neg R_1$ is represented by F_1' in $\mathrm{Th}(\mathbf{Tx})$. Then

$$\neg S(y, t_2, \ldots, t_n),$$

by definition of S.

Hence $\neg S(y, t_2, \ldots, t_n) \iff (\mathrm{sub}[G'; x_0/N(y), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \ \in \ T)$ for all $y, t_2, \ldots, t_n \in \mathrm{Tx}$, so $\neg S$ is represented in T by G'.

Hence for all theories $T \subseteq$ wff where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and
(2) $\neg S$ is represented by G' in T.

(GD10). Let $T \subseteq$ wff be a theory closed under logical operations with

$$\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx}).$$

Suppose $R_1 \subseteq \mathrm{Tx}^{n+k}$ and $R_2 \subseteq \mathrm{Tx}^{n+l}$. Let $S \subseteq \mathrm{Tx}^n$ and suppose that:

$$S(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+k} \in \mathrm{Tx})\, R_1(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}) \text{ and}$$
$$\neg S(t_1, \ldots, t_n) \iff (\exists t_{n+1}, \ldots, t_{n+l} \in \mathrm{Tx})\, R_2(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+l}).$$

Let $G := [Ex_n]\dots[Ex_{n+k-1}]F_1$ and $G' := [Ex_n]\dots[Ex_{n+l-1}]F_2$. Then for all $t_1,\dots,t_n \in \mathrm{Tx}$,

$$
\begin{aligned}
& S(t_1,\dots,t_n) \\
\Longleftrightarrow\ & (\exists t_{n+1},\dots,t_{n+k} \in \mathrm{Tx})\, R_1(t_1,\dots,t_n,t_{n+1},\dots,t_{n+k}) && \text{(by definition of } S\text{)} \\
\Longleftrightarrow\ & (\exists t_{n+1},\dots,t_{n+k} \in \mathrm{Tx}) \\
& (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n), x_n/N(t_{n+1}),\dots,x_{n+k-1}/N(t_{n+k})] \ \in\ T) \\
& && \text{(as } R_1 \text{ is represented in } T \text{ by } F_1\text{)} \\
\Longrightarrow\ & (\ [Ex_n]\dots[Ex_{n+k-1}] \\
& \quad \mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n), x_n/x_n,\dots,x_{n+k-1}/x_{n+k-1}] \ \in\ T) \\
& && \text{(by quantifier rules in } T\text{)} \\
\Longleftrightarrow\ & (\mathrm{sub}[G; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n)] \ \in\ T) && \text{(by definition of } G\text{).}
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
& (\mathrm{sub}[G; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n)] \ \in\ T) \\
\Longleftrightarrow\ & (\ [Ex_n]\dots[Ex_{n+k-1}] \\
& \quad \mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n), x_n/x_n,\dots,x_{n+k-1}/x_{n+k-1}] \ \in\ T) \\
& && \text{(by definition of } G\text{)} \\
\Longrightarrow\ & (\ [Ex_n]\dots[Ex_{n+k-1}] \\
& \quad \mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n), x_n/x_n,\dots,x_{n+k-1}/x_{n+k-1}] \ \in\ \mathrm{Th}(\mathbf{Tx})) \\
& && \text{(as } T \subseteq \mathrm{Th}(\mathbf{Tx})\text{)} \\
\Longrightarrow\ & (\exists c_1,\dots,c_k \in \mathrm{Cterm}) \\
& (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2),\dots,x_{n-1}/N(t_n), x_n/c_1,\dots,x_{n+k-1}/c_k] \ \in\ \mathrm{Th}(\mathbf{Tx})) \\
& && \text{(by Lemma 7.4).}
\end{aligned}
$$

Now $c_i \approx N(\mathrm{De}(c_i)) \in \mathrm{TC}$ for all $i \in \{1, \ldots, k\}$ by Lemma 7.2(2), so $c_i \approx N(\mathrm{De}(c_i)) \in \mathrm{Th}(\mathbf{Tx})$ for all $i \in \{1, \ldots, k\}$ as $\mathrm{TC} \subseteq \mathrm{Th}(\mathbf{Tx})$. Then

$$
\begin{aligned}
& (\exists c_1, \ldots, c_k \in \mathrm{Cterm}) \\
& (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n), x_n/c_1, \ldots, x_{n+k-1}/c_k] \in \mathrm{Th}(\mathbf{Tx})) \\
\iff & (\exists c_1, \ldots, c_k \in \mathrm{Cterm}) \\
& (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n), x_n/N(\mathrm{De}(c_1), \ldots, x_{n+k-1}/N(\mathrm{De}(c_k)] \\
& \qquad \in \mathrm{Th}(\mathbf{Tx}) \;) \\
& \qquad \text{(by extensionality in } T) \\
\implies & (\exists t_{n+1}, \ldots, t_{n+k} \in \mathrm{Tx}) \\
& (\mathrm{sub}[F_1; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n), x_n/t_{n+1}, \ldots, x_{n+k-1}/t_{n+k}] \in \mathrm{Th}(\mathbf{Tx})) \\
\iff & (\exists t_{n+1}, \ldots, t_{n+k} \in \mathrm{Tx})\, R_1(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}) \text{ (as } R_1 \text{ is represented in } T \text{ by } F_1) \\
\iff & S(t_1, \ldots, t_n) \qquad \text{(by definition of } S).
\end{aligned}
$$

Hence $S(t_1, t_2, \ldots, t_n) \iff (\mathrm{sub}[G; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \in T)$ for all $t_1, t_2, \ldots, t_n \in \mathrm{Tx}$, so S is represented in T by G. By a similar argument, $\neg S(t_1, t_2, \ldots, t_n) \iff (\mathrm{sub}[G'; x_0/N(t_1), x_1/N(t_2), \ldots, x_{n-1}/N(t_n)] \in T)$ for all $t_1, t_2, \ldots, t_n \in \mathrm{Tx}$, so $\neg S$ is represented in T by G'.

Hence for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) S is represented by G in T, and

(2) $\neg S$ is represented by G' in T.

Thus for all $R \in \mathbf{GD}$, there exist $F, G \in \mathrm{wff}$ such that for all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations,

(1) R is represented by F in T, and

(2) $\neg R$ is represented by G in T.

In particular, for all $R \in \mathbf{GD}$, there exists $F \in \mathrm{wff}$ such that R is represented by F in all theories $T \subseteq \mathrm{wff}$ where $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$ and T is closed under logical operations. $\Box$

CHAPTER 8

Discernibility of encoded classes

In this chapter, we introduce an encoding map (distinct from the one in Chapter 5), which lets us speak of conceptions in the language of TC with only elements of Tx. This involves encoding strings on the alphabet A (defined in Section 1) plus a few additional symbols into elements of Tx. In particular, this lets us represent theories such as TC and Th(**Tx**) as subsets of Tx, which are also unary relations on Tx. We want to do this so that it makes sense to talk about whether or not a theory is GD. This chapter is based on Section 12 under Part Four of 'Undecidability without Arithmetization' [**8**], and is referenced by part (iv) of the overview.

1. Definitions

DEFINITION 8.1. Let S be the alphabet containing precisely the following symbols:

$$, \quad \langle \quad ; \quad \rangle \quad \backslash$$

The map $\langle\!\langle \cdot \rangle\!\rangle : (A \cup S)^+ \to \text{Tx}$ is defined by

$$\langle\!\langle \lambda \rangle\!\rangle = b\underbrace{a \ldots a}_{n \text{ times}}b$$

where λ is the n-th element of $(A \cup S)$ when written in this order:

$$\alpha \quad \beta \quad [\quad] \quad * \quad x \quad / \quad \approx \quad E \quad \sqsubset \quad \to \quad \wedge \quad \vee \quad \neg \quad , \quad \langle \quad ; \quad \rangle \quad \backslash$$

For instance, $\langle\!\langle * \rangle\!\rangle = baaaaab$, since $*$ was the 5^{th} symbol on that list. Furthermore, if $\eta, \rho \in (A \cup S)^+$, then $\langle\!\langle \eta\rho \rangle\!\rangle = \langle\!\langle \eta \rangle\!\rangle \langle\!\langle \rho \rangle\!\rangle$.

For the sake of readability, we shall denote $\langle\!\langle \eta \rangle\!\rangle$ by $\underline{\eta}$ for all $\eta \in (A \cup S)^+$. Then for all $\eta, \rho \in (A \cup S)^+$, we have $\langle\!\langle \eta\rho \rangle\!\rangle = \underline{\eta\rho} = \langle\!\langle \eta \rangle\!\rangle \langle\!\langle \rho \rangle\!\rangle$, but since $\langle\!\langle \eta\rho \rangle\!\rangle = \langle\!\langle \eta \rangle\!\rangle \langle\!\langle \rho \rangle\!\rangle$, this notation is unambiguous. It should be straightforward to show that $\langle\!\langle \cdot \rangle\!\rangle$ is injective. We denote the image of a set X under $\langle\!\langle \cdot \rangle\!\rangle$ by $\langle\!\langle X \rangle\!\rangle$.

DEFINITION 8.2. It may be helpful to define the following relations in Tx. Roughly speaking,

- $\text{Symb}(t) \iff t$ is the encoding of some symbol in $A \cup S$
- $\text{Form}(t) \iff t$ is the encoding of some string in $(A \cup S)^+$ (i.e. a (not necessarily well-formed) formula)

- $\mathrm{Var}(t) \iff t$ is the encoding of a variable x_n for some $n \in \mathbf{N} \cup \{0\}$
- $\mathrm{Seq}(t) \iff t$ is the encoding of some sequence of strings in $(A \cup S\backslash\{,\})^+$
- $\mathrm{Mb}(s,t) \iff s$ is the encoding of some sequence and t is the encoding of a member of that sequence
- $\mathrm{Dgr}(s,t) \iff t = t_1 \ldots t_n$ for some $t_1, \ldots, t_n \in \{a,b\}$ and s is the chain

$$\underline{\langle} t_1 \underline{;} \underline{N(t_1)\rangle,}\underline{\langle} t_1t_2\underline{;}\underline{N(t_1t_2)\rangle,} \ldots, \underline{\langle} t_1 \ldots t_n\underline{;}\underline{N(t_1 \ldots t_n)\rangle}.$$

(1) Unary relations: For all $t \in \mathrm{Tx}$,

(a) $\mathrm{Symb}(t) \iff (\exists s \in \mathrm{Tx})\,(\, t = bsb$ and $((\forall u \in \mathrm{Tx})\, u \sqsubset^{\mathrm{Tx}} s \implies a \sqsubset^{\mathrm{Tx}} u)$ and $\neg(\underbrace{a \ldots a}_{20\text{ times}} \sqsubset^{\mathrm{Tx}} s)\quad)$

(b) $\mathrm{Form}(t) \iff$

- $\mathrm{Symb}(t)$ or
- $(\ \ ((\exists z \in \mathrm{Tx})\, t = bazab)$ and
  $(\forall u, v \in \mathrm{Tx})$
  $(\, t = ubv \implies (\exists w, s \in \mathrm{Tx})$
  $((u = wab$ and $v = as)$ or $(u = wa$ and $v = bas))\,)\ \ )$

(c) $\mathrm{Var}(t) \iff$

- $(t = \underline{x})$ or
- $(\exists z \in \mathrm{Tx})\,(\ \mathrm{Form}(z)$ and $t = \underline{x}z$ and
  $(\forall u \in \mathrm{Tx})\,((u \sqsubset^{\mathrm{Tx}} z$ and $\mathrm{Symb}(u)) \implies u = \underline{/})\ \ )$

(d) $\mathrm{Seq}(t) \iff (\forall s \in \mathrm{Tx})\,(t \neq \underline{,}s$ and $t \neq s\underline{,})$ and
$(\forall u, w, z \in \mathrm{Tx})\,(t = u\underline{,}w \implies (u \neq z\underline{,}$ and $w \neq \underline{,}z))$

(2) Binary relations: For all $s, t \in \mathrm{Tx}$,

(a) $\mathrm{Mb}(s,t) \iff$

- $\mathrm{Seq}(t)$ and
- $\neg(\underline{,} \sqsubset^{\mathrm{Tx}} s)$ and
- $f = s$ or $(\exists u, w \in \mathrm{Tx})\,(t = s\underline{,}u$ or $t = w\underline{,}s$ or $t = w\underline{,}s\underline{,}u)$

(b) $\mathrm{Dgr}(s,t) \iff$

(i) $\mathrm{Seq}(s)$ and

(ii) $(\forall k \in \mathrm{Tx})\,(\ \mathrm{Mb}(k,s) \implies$
$(\exists u, v \in \mathrm{Tx})$
$(\ \ k = \underline{\langle} u\underline{;}v\underline{\rangle}$ and $(u = t$ or $((\exists q \in \mathrm{Tx})\, uq = t))\ \ )\ \ )$ and

(iii) $(t = a \iff s = \underline{\langle} a\underline{;}\,\underline{\alpha\rangle})$ and $(t = b \iff s = \underline{\langle} b\underline{;}\,\underline{\beta\rangle})$ and

(iv) $((\exists q \in \mathrm{Tx})\, t = aq) \implies ((\exists r \in \mathrm{Tx})\, s = \underline{\langle} a\underline{;}\,\underline{\alpha\rangle}\underline{,}r)$ and

(v) $((\exists q \in \mathrm{Tx})\, t = bq) \implies ((\exists r \in \mathrm{Tx})\, s = \underline{\langle} b\underline{;}\,\underline{\beta\rangle}\underline{,}r)$ and

(vi) $(\forall m, n, r, r' \in \mathrm{Tx})$
$(\ (\ \mathrm{Mb}(m,s)$ and $\mathrm{Mb}(n,s)$ and

$$
\begin{aligned}
&\quad (s = m\underline{,}n \text{ or } s = rm\underline{,}n \text{ or } s = m\underline{,}nr' \text{ or } s = rm\underline{,}nr')\ )\\
&\implies (\exists u, w \in \mathrm{Tx})\\
&\qquad (\ (m = \underline{\langle} u\underline{;}w\underline{\rangle} \text{ and } n = \underline{\langle} ua\underline{;}\,\underline{[}w\underline{*}\underline{\alpha}\underline{]}\underline{\rangle}) \text{ or}\\
&\qquad\quad (m = \underline{\langle} u\underline{;}w\underline{\rangle} \text{ and } n = \underline{\langle} ub\underline{;}\,\underline{[}w\underline{*}\underline{\beta}\underline{]}\underline{\rangle})\ )\qquad)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(vii)}\ &(\forall m, n, r, u, w, v \in \mathrm{Tx})\\
&(\ (\ \mathrm{Mb}(m, s) \text{ and } \mathrm{Mb}(n, s)\\
&\quad \text{and } (s = m\underline{,}n \text{ or } s = rm\underline{,}n) \text{ and } t = ua \text{ and } m = \underline{\langle} u\underline{;}w\underline{\rangle}\ )\\
&\qquad\qquad\qquad\qquad \implies n = \underline{\langle} t\underline{;}\,\underline{[}w\underline{*}\underline{\alpha}\underline{]}\underline{\rangle}\ )
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(viii)}\ &(\forall m, n, r, u, w, v \in \mathrm{Tx})\\
&(\ (\ \mathrm{Mb}(m, s) \text{ and } \mathrm{Mb}(n, s)\\
&\quad \text{and } (s = m\underline{,}n \text{ or } s = rm\underline{,}n) \text{ and } t = ub \text{ and } m = \underline{\langle} u\underline{;}w\underline{\rangle}\ )\\
&\qquad\qquad\qquad\qquad \implies n = \underline{\langle} t\underline{;}\,\underline{[}w\underline{*}\underline{\beta}\underline{]}\underline{\rangle}\ ).
\end{aligned}
$$

It should be straightforward, if tedious, to prove by construction that all of the above relations are **ED** from the explicit definitions given in Definition 7.2.

Lemma 8.1. For all $s, t \in \mathrm{Tx}$,

$$(\mathrm{Form}(s) \text{ and } \mathrm{Form}(t)) \iff \mathrm{Form}(st) \text{ and}$$

$$(\mathrm{Seq}(s) \text{ and } \mathrm{Seq}(t)) \iff \mathrm{Seq}(s\underline{,}t)$$

It should be straightforward to check this from the explicit definitions given in Definition 7.2, and we shall not do it here.

Definition 8.3. The map $\mathrm{Deco} : \mathrm{Form} \to A \cup S$ is defined by

$$\mathrm{Deco}(b\underbrace{a \dots a}_{n \text{ times}}b) = \lambda$$

where λ is the n-th element of $(A \cup S)$ when written in this order:

$$\alpha \quad \beta \quad [\quad] \quad * \quad x \quad / \quad \approx \quad E \quad \sqsubset \quad \to \quad \wedge \quad \vee \quad \neg \quad , \quad \langle \quad ; \quad \rangle \quad \backslash$$

For instance, $\mathrm{Deco}(b\underbrace{a \dots a}_{17 \text{ times}}b) =\ ;$, since ; was the 17^{th} symbol on that list. Furthermore, if $\eta, \rho \in \mathrm{Form}$, then $\mathrm{Deco}(\eta\rho) = \mathrm{Deco}(\eta)\mathrm{Deco}(\rho)$.

It should be straightforward to show that $\mathrm{Deco} = \langle\!\langle \cdot \rangle\!\rangle^{-1}$ (technically f^{-1} where f is $\langle\!\langle \cdot \rangle\!\rangle$ with codomain restricted to $\langle\!\langle (A \cup S)^+ \rangle\!\rangle$).

2. Some lemmas

LEMMA 8.2. For all $t \in \mathrm{Tx}$, there exists $s \in \mathrm{Tx}$ such that $\mathrm{Dgr}(s,t)$ and

$$s = \underline{\langle} t\underline{; N(t)\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{, \langle} t\underline{; N(t)\rangle}.$$

PROOF. Let
$X := \{t \in \mathrm{Tx} \mid (\exists s \in \mathrm{Tx})\,(\mathrm{Dgr}(s,t) \text{ and } (\underline{\langle} t\underline{; N(t)\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{, \langle} t\underline{; N(t)\rangle})\,)\}.$

Let $s = \underline{\langle} t\underline{; \alpha\rangle}$. Now $N(a) = \alpha$, so $s = \underline{\langle} t\underline{; N(a)\rangle}$. Furthermore, we can see that $\mathrm{Dgr}(s,a)$; conditions (i–iii) are satisfied and the conditions in front of the implications for (iv–viii) are not satisfied, and thus (iv–viii) are satisfied vacuously. Hence $a \in X$. By a similar argument, $b \in X$.

Now suppose $t \in X$. Then there exists $s \in \mathrm{Tx}$ such that $\mathrm{Dgr}(s,t)$ and

$$s = \underline{\langle} t\underline{; N(t)\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{, \langle} t\underline{; N(t)\rangle}.$$

Let $t' = ta$ and $s' = s\underline{, \langle} ta\underline{; [N(t) * \alpha]\rangle}$. Since $N(ta) = [N(t) * N(a)] = [N(t) * \alpha]$, there exists $r = v \in \mathrm{Tx}$ such that $s' = r\underline{, \langle} ta\underline{; N(ta)\rangle} = r\underline{, \langle} t'\underline{; N(t')\rangle}$. We now show that $\mathrm{Dgr}(s', t')$.

i. Since $\mathrm{Dgr}(s,t)$, by (i), we have $\mathrm{Seq}(s)$, and we can see from the definition of Seq that $\mathrm{Seq}(\underline{\langle} ta\underline{; [N(t) * \alpha]\rangle})$. Hence by Lemma 8.1, $\mathrm{Seq}(s\underline{, \langle} ta\underline{; [N(t) * \alpha]\rangle})$, so $\mathrm{Seq}(s')$.

ii. For all $k \in \mathrm{Tx}$, if $\mathrm{Mb}(k, s')$, then either

- $\mathrm{Mb}(k,s)$, in which case

$(\exists u, v \in \mathrm{Tx})$

$(k = \underline{\langle} u\underline{;}v\underline{\rangle} \text{ and } (u = t \text{ or } (\exists q \in \mathrm{Tx})\, uq = t))$ (by (ii), since $\mathrm{Dgr}(s,t)$),

so $(\exists u, v \in \mathrm{Tx})$

$(k = \underline{\langle} u\underline{;}v\underline{\rangle} \text{ and } (ua = ta = t' \text{ or } (\exists q \in \mathrm{Tx})\, uqa = ta = t'))$ (since $t = ua$),

so $(\exists u, v \in \mathrm{Tx})\,(k = \underline{\langle} u\underline{;}v\underline{\rangle} \text{ and } (\exists q \in \mathrm{Tx})\, uq = t')$, or

- $k = \underline{\langle}ta\underline{;\,[N(t) * \alpha]}\rangle$, in which case there exist $u = ta = t' \in \mathrm{Tx}$ and $v = \underline{[N(t) * \alpha]} \in \mathrm{Tx}$ such that

$$k = \underline{\langle}u\underline{;}v\underline{\rangle} \text{ and } u = t'.$$

iii. We have $b \neq ta = t' = ta \neq a$ and $\underline{\langle}a\underline{;\,\alpha\rangle} \neq s' \neq \underline{\langle}b\underline{;\,\beta\rangle}$, so

$$(t' = a \iff s' = \underline{\langle}a\underline{;\,\alpha\rangle}) \text{ and } (t' = b \iff s' = \underline{\langle}b\underline{;\,\beta\rangle}).$$

iv and v. Suppose there exists $q \in \mathrm{Tx}$ such that $t' = ta = aq$. Then either

- $t = a$, in which case

$$s = \underline{\langle}a\underline{;\,\alpha\rangle} \qquad \text{(by (iii), since } \mathrm{Dgr}(s,t)),$$

so there exists $r = \underline{\langle}ta\underline{;\,[N(t) * \alpha]\rangle} \in \mathrm{Tx}$ such that $s' = \underline{\langle}a\underline{;\,\alpha\rangle}\underline{,}r$, or

- there exists $p \in \mathrm{Tx}$ such that $t = ap$, in which case there exists $r \in \mathrm{Tx}$ such that

$$s = \underline{\langle}a\underline{;\,\alpha\rangle}\underline{,}r \qquad \text{(by (iv), since } \mathrm{Dgr}(s,t)),$$

so there exists $r' = r\underline{,\,\langle}ta\underline{;\,[N(t) * \alpha]\rangle} \in \mathrm{Tx}$ such that $s' = \underline{\langle}a\underline{;\,\alpha\rangle}\underline{,}r'$.

By a similar argument, if $t' = ta = bq$, then there exists $r' \in \mathrm{Tx}$ such that $s' = \underline{\langle}b\underline{;\,\beta\rangle}\underline{,}r'$ (where $r' = \underline{\langle}ta\underline{;\,[N(t) * \alpha]\rangle}$ or $r' = r\underline{,\,\langle}ta\underline{;\,[N(t) * \alpha]\rangle}$ for some $r \in \mathrm{Tx}$ such that $s = \underline{\langle}b\underline{;\,\beta\rangle}\underline{,}r$).

vi. Let $m, n, r, r' \in \mathrm{Tx}$ and suppose $\mathrm{Mb}(m, s')$ and $\mathrm{Mb}(n, s')$.

- Suppose $s' = m\underline{,}n$. Then $s = m$ and $n = \underline{\langle}ta\underline{;\,[N(t) * \alpha]\rangle}$.

Suppose $a \neq t \neq b$. Since $\mathrm{Dgr}(s,t)$, by (iv) and (v), we have $(\underline{,} \sqsubset^{\mathrm{Tx}} s)$ and thus $(\underline{,} \sqsubset^{\mathrm{Tx}} m)$, which contradicts $\mathrm{Mb}(m, s')$. Hence $t = a$ (in which case $s = \underline{\langle}a\underline{;\,\alpha\rangle} = \underline{\langle}t\underline{;\,N(t)\rangle}$ by (iii)) or $t = b$ (in which case $s = \underline{\langle}b\underline{;\,\beta\rangle} = \underline{\langle}t\underline{;\,N(t)\rangle}$ by (iii)).

Hence $m = \underline{\langle}t\underline{;\,N(t)\rangle}$ and $n = \underline{\langle}ta\underline{;\,[N(t) * \alpha]\rangle}$, so there exist $u = t \in \mathrm{Tx}$ and $w = \underline{N(t)} \in \mathrm{Tx}$ such that $m = \underline{\langle}u\underline{;}w\underline{\rangle}$ and $n = \underline{\langle}ua\underline{;\,[}w{*}\underline{\alpha]\rangle}$.

- Suppose $s' = rm\underline{,}n$. Then $s = rm$ and $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle$. Since $\mathrm{Mb}(m, s')$, we have $(\underline{,} \sqsubset^{\mathrm{Tx}} r)$, so $(\underline{,} \sqsubset^{\mathrm{Tx}} s)$. Thus $s \neq \langle \underline{t}; \underline{N(t)}\rangle$, so $(\exists q \in \mathrm{Tx})\, s = q\underline{,}\langle \underline{t}; \underline{N(t)}\rangle$ by our inductive assumption. Then $m = \langle \underline{t}; \underline{N(t)}\rangle$.

  Hence $m = \langle \underline{t}; \underline{N(t)}\rangle$ and $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle$, so there exist $u = t \in \mathrm{Tx}$ and $w = \underline{N(t)} \in \mathrm{Tx}$ such that $m = \langle \underline{u}; \underline{w}\rangle$ and $n = \langle \underline{ua}; \underline{[w{*}\alpha]}\rangle$.

- Suppose $s' = m\underline{,}nr'$ or $s' = rm\underline{,}nr'$. As $\mathrm{Mb}(n, s')$, we have $(\underline{,} \sqsubset^{\mathrm{Tx}} r')$, so $\mathrm{Mb}(m, s)$ and $\mathrm{Mb}(n, s)$ and ($s = m\underline{,}n$ or $s = rm\underline{,}n$ or $s = m\underline{,}nr''$ or $s = rm\underline{,}nr''$ for some $r'' \sqsubset^{\mathrm{Tx}} r$). Then there exist $u, w \in \mathrm{Tx}$ such that $m = \langle \underline{u}; \underline{w}\rangle$ and $n = \langle \underline{ua}; \underline{[w{*}\alpha]}\rangle$ by (vi), since $\mathrm{Dgr}(s, t)$.

vii. Let $m, n, r, u, w, v \in \mathrm{Tx}$ and suppose $\mathrm{Mb}(m, s')$, $\mathrm{Mb}(n, s')$, $t' = ua$ and $m = \langle \underline{u}; \underline{w}\rangle$.

If $s' = m\underline{,}n$, then $s = m$ and $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle$. Then $\neg(\underline{,} \sqsubset^{\mathrm{Tx}} s)$, so $s \neq q\underline{,}\langle \underline{t}; \underline{N(t)}\rangle$ for all $q \in \mathrm{Tx}$, so $s = \langle \underline{t}; \underline{N(t)}\rangle$ by our inductive assumption, so $m = \langle \underline{t}; \underline{N(t)}\rangle$. On the other hand, if $s' = rm\underline{,}n$, then $s = rm$ and $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle$. Since $\mathrm{Mb}(m, s')$, we have $(\underline{,} \sqsubset^{\mathrm{Tx}} r)$, so $(\underline{,} \sqsubset^{\mathrm{Tx}} s)$. Thus $s \neq \langle \underline{t}; \underline{N(t)}\rangle$, so $(\exists q \in \mathrm{Tx})\, s = q\underline{,}\langle \underline{t}; \underline{N(t)}\rangle$ by our inductive assumption. Then $m = \langle \underline{t}; \underline{N(t)}\rangle$. In either case, we have $m = \langle \underline{t}; \underline{N(t)}\rangle$ and $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle$.

Then $u = t$ and $w = \underline{N(t)}$, so $n = \langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle = \langle \underline{ua}; \underline{[w{*}\alpha]}\rangle = \langle \underline{t'}; \underline{[w{*}\alpha]}\rangle$.

viii. Let $m, n, r, u, w, v \in \mathrm{Tx}$. Then $t' \neq ub$, since $t' = ta$. Hence (viii) is vacuously satisfied.

Thus $\mathrm{Dgr}(s', t')$, so there exists $s' = s\underline{,}\langle \underline{ta}; \underline{[N(t) * \alpha]}\rangle \in \mathrm{Tx}$ such that $\mathrm{Dgr}(s', t')$ and

$$s' = \langle \underline{t'}; \underline{N(t')}\rangle \text{ or } (\exists r \in \mathrm{Tx})\, s' = r\underline{,}\langle \underline{t'}; \underline{N(t')}\rangle.$$

By similar arguments, if $t' = tb$, then there exists $s' = s\underline{,}\langle \underline{tb}; \underline{[N(t) * \beta]}\rangle \in \mathrm{Tx}$ such that $\mathrm{Dgr}(s', t')$ and

$$s' = \langle \underline{t'}; \underline{N(t')}\rangle \text{ or } (\exists r \in \mathrm{Tx})\, s' = r\underline{,}\langle \underline{t'}; \underline{N(t')}\rangle.$$

Hence $ta, tb \in X$, so $X \in \mathcal{B} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have $\mathrm{Tx} = \bigcap \mathcal{B} \subseteq X$, so

$$\begin{aligned} t \in \mathrm{Tx} &\implies t \in X \\ &\iff (\exists s \in \mathrm{Tx})\,(\mathrm{Dgr}(s, t) \text{ and } (\langle \underline{t}; \underline{N(t)}\rangle \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{,}\langle \underline{t}; \underline{N(t)}\rangle)). \end{aligned}$$

Hence $(\exists s \in \mathrm{Tx})\,(\mathrm{Dgr}(s,t)$ and $(\underline{\langle} t; \underline{N(t)\rangle}$ or $(\exists r \in \mathrm{Tx})\, s = r \underline{,} \underline{\langle} t; \underline{N(t)\rangle})$ for all $t \in \mathrm{Tx}$, as required. $\square$

Lemma 8.3. For all $s', t' \in \mathrm{Tx}$, if

- $s' = s\underline{,}n$ for some $s, n \in \mathrm{Tx}$ with $\mathrm{Mb}(n, s)$,
- $t' = tc$ for some $t, c \in \mathrm{Tx}$ with $c \in \{a, b\}$ and
- $\mathrm{Dgr}(s', t')$,

then $\mathrm{Dgr}(s, t)$.

This result may be slightly less trivial than Lemma 8.1, but we shall skip the proof for now regardless. We nonetheless mention it since it is used in the proof of the following result; a fact that appears to be overlooked in Grzegorcyzk's paper.

Lemma 8.4. For all $s, s', t \in \mathrm{Tx}$, if $\mathrm{Dgr}(s,t)$ and $\mathrm{Dgr}(s',t)$, then $s = s'$.

Proof. (Sketch)
Let $X := \{t \in \mathrm{Tx} \,|\, (\forall s, s' \in \mathrm{Tx})\,(\mathrm{Dgr}(s,t) \text{ and } \mathrm{Dgr}(s',t)) \implies s = s'\}$.

Let $s, s' \in \mathrm{Tx}$ and suppose $\mathrm{Dgr}(s,a)$ and $\mathrm{Dgr}(s',a)$. Then by Definition 7.2(2)(b)(iii), we have $s = \underline{\langle} a; \underline{\alpha\rangle} = s'$. Similarly, if $\mathrm{Dgr}(s,b)$ and $\mathrm{Dgr}(s',b)$, then $s = \underline{\langle} b; \underline{\beta\rangle} = s'$. Hence $a, b \in X$.

Now let $t \in \mathrm{Tx}$ and suppose for all $s, s' \in \mathrm{Tx}$, if $\mathrm{Dgr}(s,t)$ and $\mathrm{Dgr}(s',t)$, then $s = s'$.

Let $s, s' \in \mathrm{Tx}$ and suppose $\mathrm{Dgr}(s, ta)$ and $\mathrm{Dgr}(s', ta)$. Now $a \neq ta \neq b$, so there exists $q \in \mathrm{Tx}$ such that $ta = aq$ or $ta = bq$. Then by (iv) and (v), we have $(\underline{,} \sqsubset^{\mathrm{Tx}} s)$ and $(\underline{,} \sqsubset^{\mathrm{Tx}} s')$.

Then there exist some $r, r', n, n' \in \mathrm{Tx}$ such that $s = r\underline{,}n$ and $\mathrm{Mb}(n, s)$ and $s' = r'\underline{,}n'$ and $\mathrm{Mb}(n', s')$. Then by Lemma 8.3, we have $\mathrm{Dgr}(r, t)$ and $\mathrm{Dgr}(r', t)$. Hence $r = r'$ by our inductive assumption.

Now $\mathrm{Dgr}(s, ta)$ and $\mathrm{Dgr}(s', ta)$, so n and n' are determined by the last member of r and r' respectively. But $r = r'$, so in particular, their last members are equal. Thus $n = n'$, and so $s = r\underline{,}n = r'\underline{,}n' = s'$.

By a similar argument, if $\mathrm{Dgr}(s, tb)$ and $\mathrm{Dgr}(s', tb)$, then $s = s'$.

Hence $t \in X \implies ta, tb \in X$, so
$X \in \mathcal{B} = \{X \mid a, b \in X \text{ and } (\forall s \in X)\, sa, sb \in X\}$, so by Lemma 2.1, we have
$\mathrm{Tx} = \bigcap \mathcal{B} \subseteq X$, so
$t \in \mathrm{Tx} \implies t \in X \iff (\forall s, s' \in \mathrm{Tx})\,(\mathrm{Dgr}(s,t) \text{ and } \mathrm{Dgr}(s',t)) \implies s = s'$. Hence for all $t \in \mathrm{Tx}$, we have $(\mathrm{Dgr}(s,t) \text{ and } \mathrm{Dgr}(s',t)) \implies s = s'$ for all $s, s' \in \mathrm{Tx}$, so $(\mathrm{Dgr}(s,t) \text{ and } \mathrm{Dgr}(s',t)) \implies s = s'$ for all $s, s', t \in \mathrm{Tx}$, as required. $\square$

LEMMA 8.5. The function $N' : \mathrm{Tx} \to \mathrm{Tx}$ defined by $N'(t) = \underline{N(t)}$ is GD.

PROOF. Let $S \in \mathrm{Tx}^2$ be defined by $S(t,u) \iff N'(t) := \underline{N(t)} = u$. Then S satisfies condition 3 of Definition 4.2 by construction, and S satisfies conditions 1 and 2 due to N' being a well-defined function. Hence it suffices to show that $S \in \mathbf{GD}$.

Let $R_1 \in \mathrm{Tx}^3$ and $R_2 \in \mathrm{Tx}^4$ be defined by

- $R_1(u,t,k) \iff \mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle t;u \rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\,\underline{\langle t;u\rangle})$
- $R_1(u,t,v,k) \iff$
  $$\mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle t;v\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\,\underline{\langle t;v\rangle}) \text{ and } u \neq v$$

It should be straightforward, if tedious, to prove by construction that
$R_1, R_2 \in \mathbf{ED}$.

Let $t, u \in \mathrm{Tx}$. By Lemma 8.2, there exists $s \in \mathrm{Tx}$ such that $\mathrm{Dgr}(s,t)$ and

$$s = \underline{\langle t; N(t)\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{,}\,\underline{\langle t; N(t)\rangle}$$

Suppose $S(t,u)$. Then $u = \underline{N(t)}$. Then $\mathrm{Dgr}(s,t)$ and
$(s = \underline{\langle t;u\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, s = r\underline{,}\,\underline{\langle t;u\rangle})$, so there exists $k = s \in \mathrm{Tx}$ such that $\mathrm{Dgr}(k,t)$ and $(k = \underline{\langle t;u\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\,\underline{\langle t;u\rangle})$. Hence $(\exists k \in \mathrm{Tx})\, R_1(u,t,k)$, by definition of R_1.

Conversely, suppose $(\exists k \in \mathrm{Tx})\, R_1(u,t,k)$. Then there exists $k \in \mathrm{Tx}$ such that $\mathrm{Dgr}(k,t)$ and $(k = \underline{\langle t;u\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\,\underline{\langle t;u\rangle})$. Then $k = s$ by Lemma 8.4, so we have

(1) $\underline{\langle t; N(t)\rangle} = \underline{\langle t;u\rangle}$, or
(2) $(\exists r \in \mathrm{Tx})\, \underline{\langle t; N(t)\rangle} = r\underline{,}\,\underline{\langle t;u\rangle})$, or
(3) $(\exists r \in \mathrm{Tx})\, r\underline{,}\,\underline{\langle t; N(t)\rangle} = \underline{\langle t;u\rangle}$, or
(4) $(\exists r, r' \in \mathrm{Tx}) = r\underline{,}\,\underline{\langle t; N(t)\rangle} = r'\underline{,}\,\underline{\langle t;u\rangle})$.

Note that (2) and (3) are impossible, while (1) and (4) imply that $u = \underline{N(t)}$. Thus $S(t,u)$, by definition of S.

Hence $S(t,u) \iff (\exists k \in \mathrm{Tx})\, R_1(u,t,k)$.

Now suppose $\neg S(t,u)$. Then $u \neq \underline{N(t)}$. By definition of S, we have $S(\underline{N(t)}, t)$, so

$$(\exists k \in \mathrm{Tx})\, \mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle} t\underline{;}\, \underline{N(t)}\underline{\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\, \underline{\langle} t\underline{;}\, \underline{N(t)}\underline{\rangle}),$$

as $S(\underline{N(t)}, t) \iff (\exists k \in \mathrm{Tx})\, R_1(\underline{N(t)}, t, k)$. Then

$$(\exists k \in \mathrm{Tx})\, \mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle} t\underline{;}\, \underline{N(t)}\underline{\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\, \underline{\langle} t\underline{;}\, \underline{N(t)}\underline{\rangle}) \text{ and } u \neq \underline{N(t)},$$

as $u \neq \underline{N(t)}$. Then there exists $v = \underline{N(t)} \in \mathrm{Tx}$ such that

$$(\exists k \in \mathrm{Tx})\, \mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle} t\underline{;} v\underline{\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\, \underline{\langle} t\underline{;} v\underline{\rangle}) \text{ and } u \neq v.$$

Hence $(\exists v, k \in \mathrm{Tx})\, R_2(u,t,v,k)$, by definition of R_2.

Conversely, suppose $(\exists v, k \in \mathrm{Tx})\, R_2(u,t,v,k)$. Then there exist $v, k \in \mathrm{Tx}$ such that

$$\mathrm{Dgr}(k,t) \text{ and } (k = \underline{\langle} t\underline{;} v\underline{\rangle} \text{ or } (\exists r \in \mathrm{Tx})\, k = r\underline{,}\, \underline{\langle} t\underline{;} v\underline{\rangle}) \text{ and } u \neq v.$$

In particular, $(\exists k \in \mathrm{Tx})\, \mathrm{Dgr}(k,t)$ and $(k = \underline{\langle} t\underline{;} v\underline{\rangle}$ or $(\exists r \in \mathrm{Tx})\, k = r\underline{,}\, \underline{\langle} t\underline{;} v\underline{\rangle})$, so $(\exists k \in \mathrm{Tx}) R_1(v,t,k)$, so $S(v,t)$, since $S(t,u) \iff (\exists k \in \mathrm{Tx})\, R_1(u,t,k)$.

Thus $v = \underline{N(t)}$, by definition of S. But $u \neq v$, so $u \neq \underline{N(t)}$, so $\neg S(t,u)$, by definition of S.

Hence $\neg S(t,u) \iff (\exists v, k \in \mathrm{Tx})\, R_2(u,t,v,k)$.

Hence there exist $R_1, R_2 \in \mathbf{ED} \subseteq \mathbf{GD}$ such that

$$S(t,u) \iff (\exists k \in \mathrm{Tx})\, R_1(u,t,k) \text{ and}$$
$$\neg S(t,u) \iff (\exists v, k \in \mathrm{Tx})\, R_2(u,t,v,k),$$

so by inductive condition 10 of Definition 4.1, $S \in \mathbf{GD}$. Hence N' is a GD function. $\square$

CHAPTER 9

Undecidability of TC, Th(Tx) and everything in between

In this chapter, we prove the undecidability of TC and Th(**Tx**). This chapter is based on Section 13 under Part Four of 'Undecidability without Arithmetization' [**8**], and is referenced by part (vii) of the overview.

1. A few more results

LEMMA 9.1. Let $G \in$ wff such that G contains exactly one free variable x_n. Then there exists some $F \in$ wff which contains precisely one free variable x_0, which occurs in F once only such that for all $c \in$ Cterm,

$$\mathrm{sub}[F; x_0/c] \leftrightarrow \mathrm{sub}[G; x_n/c] \;\; \in \;\; \mathrm{TC}.$$

PROOF. Let $F = [Ex_1]\,[x_0 \approx x_1 \wedge \mathrm{sub}[G; x_n/x_1]]$. Then the required result holds since TC is closed under logical operations. □

LEMMA 9.2. For all unary $X \in \mathbf{GD}$, there exists $F \in$ wff such that

(1) X is represented by F in all theories $T \subseteq$ wff which are closed under logical operations, contains TC and is contained in Th(**Tx**).
(2) F contains precisely one free variable x_0, which occurs in F once only.

PROOF. Let $X \in \mathbf{GD}$ be unary. By Lemma 7.5, there exists $G \in$ wff which represents X in all theories $T \subseteq$ wff which are closed under logical operations, contains TC and is contained in Th(**Tx**).

Let $T \subseteq$ wff be a theory closed under logical operations with $\mathrm{TC} \subseteq T \subseteq \mathrm{Th}(\mathbf{Tx})$. Then for all $t \in$ Tx,

$$X(t) \iff (\mathrm{sub}[G; x_0/N(t)] \;\; \in \;\; T).$$

As T is a theory, $\mathrm{sub}[G; x_0/N(t)]$ is a sentence so G contains exactly one free variable x_0. But by Lemma 9.1, there exists $F \in$ wff which contains precisely one free variable x_0, which occurs in F once only such that for all $c \in$ Cterm,

$$\mathrm{sub}[F; x_0/c] \leftrightarrow \mathrm{sub}[G; x_0/c] \;\; \in \;\; \mathrm{TC}.$$

Then $\mathrm{sub}[F; x_0/N(t)] \leftrightarrow \mathrm{sub}[G; x_0/N(t)] \ \in\ \mathrm{TC}$, as $N(t) \in \mathrm{Cterm}$.

Then $\mathrm{sub}[F; x_0/N(t)] \leftrightarrow \mathrm{sub}[G; x_0/N(t)] \ \in\ T$, as $\mathrm{TC} \subseteq T$.

Then $(\mathrm{sub}[F; x_0/N(t)] \ \in\ T) \iff (\mathrm{sub}[G; x_0/N(t)] \ \in\ T)$, as T is closed under logical operations.

Thus $X(t) \iff (\mathrm{sub}[G; x_0/N(t)] \ \in\ T)$, so X is represented by F in T. $\square$

LEMMA 9.3. The function $\mathrm{Sub} : \mathbf{Tx}^2 \to \mathbf{Tx}$ defined by

$$\mathrm{Sub}(s,t) = \begin{cases} utv & \text{when } (\exists u, v \in \mathrm{Tx}) \\ & \quad (s = u\underline{x_0}v \text{ and } (\forall w, z \in \mathrm{Tx})(s = w\underline{x_0}z \implies (u = w \text{ and } v = z))) \\ a & \text{otherwise} \end{cases}$$

is GD, and for all

- $F \in \mathrm{wff}$ such that F contains precisely one free variable x_0 which occurs in F once only,
- $c \in \mathrm{Cterm}$, and
- $T \subseteq \mathrm{wff}$, we have

$$\mathrm{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle \iff \mathrm{sub}[F, x_0/c] \in T.$$

PROOF. Let $R \subseteq \mathrm{Tx}^3$ be defined as follows:

$R(s,t,q) \iff$

- $(\exists u, v \in \mathrm{Tx})\,(\ s = u\underline{x_0}v$ and
  $(\forall w, z \in \mathrm{Tx})(s = w\underline{x_0}z \implies (u = w \text{ and } v = z)\ )$
  and $q = utv$, or
- $(\ \neg((\exists u, v \in \mathrm{Tx})\,(\ s = u\underline{x_0}v$ and
  $(\forall w, z \in \mathrm{Tx})(s = w\underline{x_0}z \implies (u = w \text{ and } v = z)\ )\ )$
  and $q = a$.

It should be straightforward, if tedious, to prove by construction that
$R \in \mathbf{ED} \subseteq \mathbf{GD}$. Furthermore,

(1) $(\forall s, t \in \mathrm{Tx})\,(R(s,t,q) \text{ and } R(s,t,r)) \implies q = r$,
(2) $(\forall s, t \in \mathrm{Tx})\,\exists q \in \mathrm{Tx})\,(R(s,t,q)$ and
(3) $(\forall s, t, q \in \mathrm{Tx})\,\mathrm{Sub}(s,t) = q \iff (R(s,t,q)$

by construction. Hence by Definition 4.2, Sub is a GD function.

Now let

- $F \in \text{wff}$ such that F contains precisely one free variable x_0 which occurs in F once only,
- $c \in \text{Cterm}$, and
- $T \subseteq \text{wff}$.

Since F contains precisely one free variable x_0 which occurs in F once only, there exist some $u, v \in \text{Tx}$ such that $\underline{F} = u\underline{x_0}v$ and for all $w, z \in \text{Tx}$, we have $s = w\underline{x_0}z \implies (u = w \text{ and } v = z)$. Then $\text{Sub}(\underline{F}, \underline{c}) = u\underline{c}v$, by definition of Sub.

Now $\text{Form}(\underline{F})$, so by Lemma 8.1, we have $\text{Form}(u)$ and $\text{Form}(v)$. Then $\text{Deco}(u)$ and $\text{Deco}(v)$ are well-defined. Then
$\text{Deco}(\text{Sub}(\underline{F}, \underline{c})) = \text{Deco}(u\underline{c}v) = \text{Deco}(u)\text{Deco}(\underline{c})\text{Deco}(v) = \text{Deco}(u)c\text{Deco}(v)$. But
$F = \text{Deco}(\underline{F}) = \text{Deco}(u\underline{x_0}v) = \text{Deco}(u)\text{Deco}(\underline{x_0})\text{Deco}(v) = \text{Deco}(u)x_0\text{Deco}(v)$, so

$$\text{Deco}(\text{Sub}(\underline{F}, \underline{c})) = \text{sub}[F; x_0/c].$$

Suppose $\text{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle$. Since $\text{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle \iff \text{Deco}(\text{Sub}(\underline{F}, \underline{c})) \in T$, we have $\text{sub}[F; x_0/c] \in T$.

Conversely, suppose $\text{sub}[F; x_0/c] \in T$. Then $\text{Deco}(\text{Sub}(\underline{F}, \underline{c})) \in T$.
Since $\text{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle \iff \text{Deco}(\text{Sub}(\underline{F}, \underline{c})) \in T$, we have $\text{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle$.

Hence $\text{Sub}(\underline{F}, \underline{c}) \in \langle\!\langle T \rangle\!\rangle \iff \text{sub}[F, x_0/c] \in T$. □

2. The proof of undecidability

THEOREM 9.1. Let $T \subseteq \text{wff}$ be a theory closed under logical operations with $\text{TC} \subseteq T \subseteq \text{Th}(\mathbf{Tx})$. Then $\langle\!\langle T \rangle\!\rangle \notin \mathbf{GD}$.

PROOF. Let $T \subseteq \text{wff}$ be a theory closed under logical operations with $\text{TC} \subseteq T \subseteq \text{Th}(\mathbf{Tx})$ and let $X \subseteq \text{Tx}$ be defined by

$$X(t) \iff \text{Sub}(t, N'(t)) \notin \langle\!\langle T \rangle\!\rangle \qquad (*)$$

Now Sub is GD by Lemma 9.3 and N' is GD by Lemma 8.5, so by Definition 4.2, Lemma 4.2 and inductive conditions 6 and 5 of Definiton 4.1, the function $\varphi : \text{Tx} \to \text{Tx}$ defined by $\varphi(t) = \text{Sub}(t, N'(t))$ is GD. Then

$$X = \varphi^{-1}(\neg\langle\!\langle T \rangle\!\rangle),$$

where $\neg\langle\!\langle T\rangle\!\rangle := (\mathrm{Tx} \setminus \langle\!\langle T\rangle\!\rangle)$.

Suppose $\langle\!\langle T\rangle\!\rangle \in \mathbf{GD}$. Then $\neg\langle\!\langle T\rangle\!\rangle \in \mathbf{GD}$, by inductive property 10 of Definition 4.1. Then by Lemma 4.1, we have $X \in \mathbf{GD}$.

Then by Lemma 9.2, there exists $F \in \mathrm{wff}$ such that

(1) X is represented by F in T and

(2) F contains precisely one free variable x_0, which occurs in F once only.

Then for all $t \in \mathrm{Tx}$,

$$\begin{aligned} t \in X \iff X(t) &\iff (\mathrm{sub}[F, x_0/N(t)] \;\in\; T) && \text{(by definition of representability)} \\ &\iff \mathrm{Sub}(\underline{F}, \underline{N(t)}) \in \langle\!\langle T\rangle\!\rangle && \text{(by Lemma 9.3)} \\ &\iff \mathrm{Sub}(\underline{F}, N'(t)) \in \langle\!\langle T\rangle\!\rangle && \text{(by definition of } N'\text{)}. \end{aligned}$$

In particular, $\underline{F} \in X \iff \mathrm{Sub}(\underline{F}, N'(\underline{F})) \in \langle\!\langle T\rangle\!\rangle$, which contradicts $\underline{F} \in X \iff \mathrm{Sub}(\underline{F}, N'(\underline{F})) \notin \langle\!\langle T\rangle\!\rangle$ from definition $(*)$ of X. Hence $\langle\!\langle T\rangle\!\rangle \notin \mathbf{GD}$. $\square$

Future Directions

Going forward, it may be helpful to investigate the converse to Lemma 6.2; that if a relation R is is strongly represented in all consistent extensions of TC, then $R \in \mathbf{ED}$. If this is true, then we know for sure that relations that can be constructed from the elementary operations and relations in any way are **ED** without performing a tedious construction or handwaving it by saying such a construction should be "straightforward". It may also be worthwhile to generalize this to arbitrary theories with finite axioms in arbitrary languages, which would most likely necessitate a general definition of a "standard" model for a given set of axioms.

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