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HOMOGENEOUS DARBOUX POLYNOMIALS AND GENERALISING INTEGRABLE ODE SYSTEMS

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Abstract. We show that any system of ODEs can be modified whilst preserving its homogeneous Darboux polynomials. We employ the result to generalise a hierarchy of integrable Lotka-Volterra systems.

1. **Introduction.**

We are concerned with systems of Ordinary Differential Equations (ODEs),

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\dot{\mathbf{x}}$ denotes the time derivative of a vector $\mathbf{x}$. A Darboux polynomial (or second integral) of (1) is a polynomial $P(\mathbf{x})$ such that $\dot{P} = C(\mathbf{x})P$ for some function C which is called the cofactor of P [6]. Darboux polynomials are important as the existence of sufficiently many Darboux polynomials implies the existence of a first integral, cf. Theorems 2.2 and 2.3 in [6]. Recently their use was extended to the discrete setting in [2].

In this paper, we propose the following generalisation of any ODE system of the form (1):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + b(\mathbf{x}, t)\mathbf{x}, \tag{2}$$

where b is a scalar function of $\mathbf{x}, t$. We will prove that if P is a homogeneous Darboux polynomial for (1), then P is also a Darboux polynomial for (2) with a modified cofactor.

We show that in several examples the above generalisation preserves the integrability of the ODE, e.g. this is the case for generalisations of: (i) the 2-dimensional system

$$\begin{aligned} \dot{x} &= x^2 + 2xy + 3y^2, \\ \dot{y} &= 2y(2x + y), \end{aligned} \tag{3}$$

found in [4, Appendix], (ii) the 4-dimensional Lotka-Volterra (LV) system

$$\begin{aligned} \dot{x}_1 &= x_1(+x_2 + x_3 + x_4) \\ \dot{x}_2 &= x_2(-x_1 + x_3 + x_4) \\ \dot{x}_3 &= x_3(-x_1 - x_2 + x_4) \\ \dot{x}_4 &= x_4(-x_1 - x_2 - x_3), \end{aligned} \tag{4}$$

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as well as (iii) higher dimensional LV systems found in [9]. For the LV systems we show that both Liouville integrability and superintegrability are preserved under certain generalisations given by (2).

2. **Darboux polynomials and integrals/integrability.** Note that if P_1 and P_2 are Darboux polynomials with cofactors C_1 and C_2 respectively, the product $P_1^a P_2^b$ is a Darboux polynomial with cofactor $aC_1 + bC_2$. This implies that linear relations between cofactors give rise to integrals.

For the 2-dimensional system (3) three Darboux polynomials

$$P_1 = x + y, \qquad P_2 = x - y, \qquad P_3 = y, \tag{5}$$

with cofactors given by

$$C_1 = x + 5y, \qquad C_2 = x - y, \qquad C_3 = 4x + 2y, \tag{6}$$

respectively, were given in [6, Example 2.21]. As these cofactors satisfy the linear relation $C_1 + 3C_2 - C_3 = 0$, an integral is given by

$$I = P_1 P_2^3 P_3^{-1} = \frac{(x+y)(x-y)^3}{y}.$$

The 4-dimensional LV system (4) admits linear Darboux polynomials of the form

$$P_{i,j} = \sum_{k=i}^{j} x_k, \text{ with } 1 \leq i \leq j \leq 4,$$

with corresponding cofactor

$$C_{i,j} = -\sum_{k=1}^{i-1} x_k + \sum_{k=j+1}^{n} x_k.$$

Because

$$C_{1,2} - C_{3,3} + C_{4,4} = (x_3 + x_4) - (-x_1 - x_2 + x_4) + (-x_1 - x_2 - x_3) = 0,$$

the rational function

$$F = P_{1,2} P_{3,3}^{-1} P_{4,4} = (x_1 + x_2)\frac{x_4}{x_3}$$

is an integral. And similarly,

$$C_{3,4} - C_{2,2} + C_{1,1} = (-x_1 - x_2) - (-x_1 + x_3 + x_4) + (x_2 + x_3 + x_4) = 0$$

yields the rational integral

$$G = P_{3,4} P_{2,2}^{-1} P_{1,1} = (x_3 + x_4)\frac{x_1}{x_2}.$$

As $C_{1,4} = 0$, the function

$$H = P_{1,4} = x_1 + x_2 + x_3 + x_4$$

provides a third integral. The functions F, G, H are functionally independent, as their gradients are linearly independent, and therefore the LV system (4) is superintegrable. The variables $u_i = P_{1,i}$ provide a separation of variables, i.e. each variable satisfies the same differential equation $\dot{u}_i = u_i(H - u_i)$ which can be explicitly integrated, cf. [1]

The system (4) is also a Hamiltonian system, with Hamiltonian H and quadratic Poisson bracket, of rank 4,

$$\{x_i, x_j\} = x_i x_j, \qquad i < j. \tag{7}$$

As both F and G Poisson commute with H, the systems F, H and G, H, and hence the vector field (4), are Liouville integrable, cf. [9].

3. **Generalising ODE systems.** The following result is quite general, it generalises any ODE system (1) whilst preserving all homogeneous Darboux polynomials.

Theorem 3.1. *Let $P(\mathbf{x})$ be a homogeneous Darboux polynomial of degree d with cofactor $C(\mathbf{x})$ for the system of ODEs $\dot{\mathbf{x}} = f(\mathbf{x})$. Then P is a Darboux polynomial for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + b(\mathbf{x}, t)\mathbf{x}$, with cofactor $C + db(\mathbf{x}, t)$, where b is a scalar function of $\mathbf{x}, t$.*

Proof. As P is homogeneous of degree d, we have $\mathbf{x} \cdot \nabla P = dP$. As P is a Darboux polynomial for $\dot{\mathbf{x}} = f(\mathbf{x})$, we have $\dot{P} = \nabla P \cdot \mathbf{f} = CP$. For the generalised system we then have

$$\dot{P} = \nabla P \cdot (\mathbf{f} + b\mathbf{x}) = CP + bdP = (C + db)P.$$

□

We first apply Theorem 3.1 to the 2-dimensional system (3). With $b = ax + cy$ we obtain a generalisation of (3),

$$\begin{aligned} \dot{x} &= x^2 + 2xy + 3y^2 + (ax + cy)x, \\ \dot{y} &= 2y(2x + y) + (ax + cy)y. \end{aligned} \tag{8}$$

Each P_i, $i = 1, 2, 3$, given by (5), is a linear Darboux polynomial for the system (8) with modified cofactor $C_i' = C_i + ax + cy$, where C_i is given by (6). As

$$(c - a - 2)C_1' - (a + c + 6)C_2' + 2(a + 1)C_3' = 0,$$

the function

$$K = P_1^{c-a-2} P_2^{-(a+c+6)} P_3^{2(a+1)} = \frac{(x + y)^{c-a-2} y^{2(a+1)}}{(x - y)^{a+c+6}}.$$

is a first integral of (8).

Applying Theorem 3.1 to the 4-dimensional system (4), taking b to be a constant, yields

$$\begin{aligned} \dot{x}_1 &= x_1(b + x_2 + x_3 + x_4) \\ \dot{x}_2 &= x_2(b - x_1 + x_3 + x_4) \\ \dot{x}_3 &= x_3(b - x_1 - x_2 + x_4) \\ \dot{x}_4 &= x_4(b - x_1 - x_2 - x_3), \end{aligned} \tag{9}$$

whose Darboux polynomials $P_{i,j}$ now have cofactors $C_{i,j}' = C_{i,j} + b$. In particular, $H = P_{1,4}$ is no longer an integral, and the linear combinations $C_{1,2}' - C_{3,3}' + C_{4,4}' = C_{3,4}' - C_{2,2}' + C_{1,1}' = b$ do not vanish. We have to subtract the cofactor $C_{1,4}' = b$, which corresponds to dividing by H. This yields two integrals

$$F' = \frac{(x_1 + x_2)x_4}{(x_1 + x_2 + x_3 + x_4)x_3},$$

and

$$G' = \frac{(x_3+x_4)x_2}{(x_1+x_2+x_3+x_4)x_1}.$$

The new system (9) is still Hamiltonian, with the same bracket (7). The new Hamiltonian

$$H' = H - b\ln\left(\frac{x_1x_3}{x_2x_4}\right)$$

is no longer rational. The integrals F', G', H' are functionally independent, and so the system (9) is superintegrable. Moreover, the functions F' and G' Poisson commute with H', hence the systems F', H' and G', H' are Liouville integrable. In the next section we generalise this example to arbitrary even dimensions.

4. **Integrability of a generalised n-dimensional LV system.** In [9] the system of ODEs

$$\dot{x}_i = x_i\left(\sum_{j>i} x_j - \sum_{j<i} x_j\right), \qquad i = 1,\dots,n, \tag{10}$$

arose as a subsystem of the quadratic vector fields associated with multi-sums of products, and it was shown to be superintegrable as well as Liouville integrable. Integrable generalisations of the system (10) have been obtained in [3, 5, 7]. The generalisation

$$\dot{x}_i = x_i\left(b + \sum_{j>i} x_j - \sum_{j<i} x_j\right), \qquad i = 1,\dots,n, \tag{11}$$

of which (9) is a special case, seems to be new. In [9] the LV system of ODEs (10), with $n = 2r$ even, was shown to admit the integrals, for $k = 1,\dots,r$,

$$\begin{aligned} F_k &= (x_1 + x_2 + \cdots + x_{2k})\frac{x_{2k+2}x_{2k+4}\cdots x_n}{x_{2k+1}x_{2k+3}\cdots x_{n-1}}, \\ G_k &= (x_{n-2k+1} + x_{n-2k+2} + \cdots + x_n)\frac{x_1x_3\cdots x_{n-2k-1}}{x_2x_4\cdots x_{n-2k}}, \end{aligned} \tag{12}$$

The $n-1$ integrals $F_1,\dots,F_{r-1}, G_1,\dots,G_{r-1}, F_r = G_r = H = P_{1,n}$ were proven to be independent, and the sets $\{F_1,\dots,F_{r-1},H\}$, $\{G_1,\dots,G_{r-1},H\}$ were proven to pairwise Poisson commute with respect to the bracket (7), which has rank n. Similar results were obtained for n odd (here the rank of (7) is $n-1$), establishing the superintegrability as well as Liouville integrability of the n-dimensional LV system (10) for all n. We consider a generalisation of the even-dimensional system.

Theorem 4.1. *The system*

$$\dot{x}_i = x_i(b + \sum_{j>i} x_j - \sum_{j<i} x_j), \qquad i = 1,\dots,n, \tag{13}$$

where $n = 2r$ is even, is both superintegrable and Liouville integrable.

Proof. The system is Hamiltonian with Hamiltonian

$$H' = H - bS, \text{ with } S = \ln\left(\frac{x_1x_3\cdots x_{n-1}}{x_2x_4\cdots x_n}\right).$$

According to Theorem 3.1 the functions (12) and H are Darboux functions (functions F such that $\dot{F} = C(\mathbf{x})F$ for some C) with cofactor b. Therefore, $n-2$ integrals are given by $F_i' = F_i/H$,$G_i' = G_i/H$, $i = 1, \ldots, r-1$. Together with H' they form a set of $n-1$ integrals,

$$\mathcal{S} = \{F_1', \ldots, F_{r-1}', G_1', \ldots, G_{r-1}', H'\},$$

for which we will prove functional independence, thereby showing the superintegrability of (13). The trick is to add a function, H, and show that the bigger set $\mathcal{S} \cup \{H\}$ is functionally independent, by showing the determinant of the Jacobian to be non-zero, which is done using LU-decomposition, cf. [8, Chapter 5]. We may perform row operations, which we do by taking linear combinations of the functions H and H' and ordering the functions in a particular way:

$$Z = \big(2(H - H'/2)/n^2, H/n^2, G_{n/2-1}', F_1', G_{n/2-2}', F_2', \ldots, G_1', F_{n/2-1}'\big).$$

We then consider the scaled Jacobian $J = n^2\mathrm{Jac}(Z)/2$ in the point $x_1 = x_2 = \cdots = x_n = b = 1$. The first two functions in Z are chosen so the first two rows in J are given by $J_{i,j} = i + j + 1 \mod 2$ $(i = 1, 2)$.

We conveniently introduce two sets of elementary functions

$$P_{i,j} = x_i + x_{i+1} + \cdots + x_j, \qquad Q_{i,j} = x_i^{-1} x_{i+1} x_{i+2}^{-1} \cdots x_j^{(-1)^{j-i+1}},$$

so that e.g. $F_k' = \frac{P_{1,2k}Q_{2k+1,n}}{P_{1,n}}$. As

$$\frac{\partial F_k'}{\partial x_i} = \begin{cases} \frac{Q_{2k+1,n}}{P_{1,n}} - \frac{P_{1,2k}Q_{2k+1,n}}{P_{1,n}^2} & i \le 2k \\ -(-1)^i \frac{P_{1,2k}Q_{2k+1,n}}{x_i P_{1,n}} - \frac{P_{1,2k}Q_{2k+1,n}}{P_{1,n}^2} & i > 2k, \end{cases}$$

we have

$$\frac{n^2}{2}\frac{\partial F_k'}{\partial x_i}\,|_{\mathbf{x}=\mathbf{1}} = \begin{cases} \frac{n}{2} - k & i \le 2k \\ -(-1)^i kn - k & i > 2k. \end{cases}$$

In the point $\mathbf{1}$ the gradient of G_k is the gradient of F_k read from right to left. This yields, for $i > 2$

$$J_{i,j} = \begin{cases} -((-1)^j n + 1)(n - i + 1)/2 & i \equiv 1,\ j < i \\ (i-1)/2 & i \equiv 1, j \ge i \\ (n - i + 2)/2 & i \equiv 0, j < i - 1 \\ ((-1)^j n - 1)(i-2)/2 & i \equiv 0,\ j \ge i - 1, \end{cases}$$

where (here and in the sequel) the equivalence is taken modulo 2. Explicitly, for $n = 10$ we have

$$J = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 36 & -44 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & -11 & 9 & -11 & 9 & -11 & 9 & -11 & 9 \\ 27 & -33 & 27 & -33 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & -22 & 18 & -22 & 18 & -22 & 18 \\ 18 & -22 & 18 & -22 & 18 & -22 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & -33 & 27 & -33 & 27 \\ 9 & -11 & 9 & -11 & 9 & -11 & 9 & -11 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -44 & 36 \end{pmatrix}.$$

We define lower and upper triangular matrices

$$L_{i,k} = \begin{cases} M_{i,k} & k = 1, 2 \\ 1 & k = i \\ k/(n-k) & 1 \equiv i = k+1, k > 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{k,j} = \begin{cases} M_{k,j} & k = 1, 2 \\ -n(n-k)/2 & k \equiv 1, j \equiv 1, j \geq k \\ n(n-k+2)/2 & k \equiv 1, j \equiv 0, j \geq k \\ -n^2/(n-k+1) & k \equiv 0, j \equiv 0, j \geq k \\ 0 & k \equiv 0, j \equiv 0, j \geq k \text{ or } k > j. \end{cases}$$

When $n = 10$ we have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 36 & -44 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & \frac{3}{7} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & -33 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 18 & -22 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & \frac{7}{3} & 1 & 0 & 0 \\ 9 & -11 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -35 & 45 & -35 & 45 & -35 & 45 & -35 & 45 \\ 0 & 0 & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} \\ 0 & 0 & 0 & 0 & -25 & 35 & -25 & 35 & -25 & 35 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 & -20 & 0 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 25 & -15 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{100}{3} & 0 & -\frac{100}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -100 \end{pmatrix}.$$

We now show that $J = LU$, making use of the Kronecker delta, $\delta_{i,k} = 1$ if $i = k$ and 0 otherwise, and using summation over repeated indices. There are three cases:

- $i = 1, 2$. We have $L_{i,k} = \delta_{i,k}$, so $L_{i,k}U_{k,j} = U_{i,j} = M_{i,j}$.
- $1 \equiv i > 2$. We have

$$\begin{aligned} L_{i,k}U_{k,j} &= (n-1)(n-i+1)U_{1,j}/2 - (n+1)(n-i+1)U_{2,j}/2 + U_{i,j} \\ &= \begin{cases} -((-1)^j n + 1)(n-i+1)/2 & i > j \\ (n-1)(n-i+1)/2 - n(n-i)/2 = (i-1)/2 & 1 \equiv j \geq i \\ -(n+1)(n-i+1)/2 + n(n-i+2)/2 = (i-1)/2 & 0 \equiv j \geq i. \end{cases} \end{aligned}$$

- $0 \equiv i > 2$. We have

$$L_{i,k}U_{k,j} = (n-i+2)(U_{1,j}+U_{2,j})/2 + (i-1)U_{i-1,j}/(n-i+1) + U_{i,j}$$
$$= \begin{cases} \frac{n-i+2}{2} & j < i-1 \\ \frac{n-i+2}{2} - \frac{(i-1)n(n-i+1)}{2(n-i+1)} = -\frac{(i-2)(n+1)}{2} & 1 \equiv j \geq i \\ \frac{n-i+2}{2} + \frac{(i-1)n(n-i+3)}{2(n-i+1)} - \frac{n^2}{n-i+1} = \frac{(i-2)(n-1)}{2} & 0 \equiv j \geq i. \end{cases}$$

As both L and U have non-zero diagonal elements, the determinant of J is non-zero. Hence the set S is functionally independent. This shows that (13) is superintegrable.

Next we prove that each pair of functions in the set $\{F'_1, \ldots, F'_{r-1}, H'\}$ Poisson commutes with respect to the bracket (7). Due to the Leibniz rule, the brackets $\{F'_i, F'_j\} = \{F_i/H, F_j/H\}$, with $1 \leq i, j < r$, can be expressed in terms of $\{F_i, F_j\}$, $\{F_i, H\}$, $\{H, F_j\}$, which all vanish. We also have $\{F'_i, H'\} = 0$ as the F'_i are integrals and H' is the Hamiltonian function of the system. Similarly, it follows that the functions in $\{G'_1, \ldots, G'_{r-1}, H'\}$ Poisson commute. This shows that (13) is Liouville integrable. □

Remark 1. Similar to the above, one can also show that the system

$$\dot{x}_i = x_i \left(b(S) + \sum_{j>i} x_j - \sum_{j<i} x_j \right), \qquad i = 1, \ldots, n = 2r, \tag{14}$$

where b is an arbitrary integrable function, is both superintegrable and Liouville integrable. The system (14) is a Hamiltonian system with Hamiltonian $H^* = H - B(S)$, where B is the anti-derivative of b.

Remark 2. In general, the b-generalisation (2) of a Hamiltonian system (1) will not be Hamiltonian. We hope to discuss some other cases in which the generalisation is Hamiltonian in a future publication. The reason that we have restricted the dimension of the Lotka-Volterra systems (13) to be even is that it seems unclear whether a Hamiltonian exists in the general odd-dimensional case.

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