Single variable shear deformation theory for nonlocal beams

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Abstract

In this paper, a simple beam theory accounting for shear deformation effects with one unknown is proposed for static bending and free vibration analysis of isotropic nanobeams. The size-dependent behaviour is captured by using the nonlocal differential constitutive relations of Eringen. The governing equation of the present beam theory is obtained by using equilibrium equations of elasticity theory. The present theory has strong similarities with nonlocal Euler-Bernoulli beam theory in terms of the governing equation and boundary conditions. Analytical solutions for static bending and free vibration are derived for nonlocal beams with various types of boundary conditions. Verification studies indicate that the present theory is not only more accurate than Euler-Bernoulli beam theory, but also comparable with Timoshenko beam theory.
Keywords: Nanobeam, Nonlocal elasticity theory, Shear deformation beam theory, Bending, Vibration

1. Introduction

In recent years, the emergence of nanotechnology in high-tech devices requires a proper understanding of the mechanical behaviours of small-scale structures, which are considerably influenced by the size effects [1]. In general, the classical continuum theories failed to accurately predict the responses of such structures as they dismiss the size effects. In order to capture these effects, two different approaches have been proposed including atomistic and high-order continuum mechanics. Comparing to the former, the later approach is more popular in practice due to its simplicity and computational efficiency. Currently, there are three well-known high-order theories in continuum mechanics: strain gradient theory [2], modified couple stress theory [3], and nonlocal elasticity theory [4]. Among them, the nonlocal elasticity theory is widely used in the literature. This theory states that the stress at a point in continuum

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depends on the strains of all points in the body instead of being only determined by the strains at that point.

By using the nonlocal elasticity theory, a large number of papers published in the literature attempted to investigate the size effects on the behaviour of nanobeams in accordance with various beam theories such as Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT) and high-order shear deformation beam theories. Those investigations were carried out for bending ([5–11]), buckling ([12–16]) and vibration analyses ([17–23]). The nonlocal elasticity theory was also employed to study the size-dependant behaviours of functional graded nanobeams ([24–37]). It is well-known that the EBT underestimates deflection and overestimates buckling load and natural frequencies of thick beam due to neglecting the shear deformation effects. The TBT is capable of accurately predicting the response of thick beam, but it requires a shear correction factor, which is not always easy to be determined. To overcome this drawback, a number of high-order shear deformation beam theories were introduced, and only few of them are cited here ([5, 20, 33, 34, 39–41]). These high-order shear deformation theories, however, require at least two variables and consequently two governing equations to describe the behaviours of beams, and thus it is not convenient to use.

In this study, a simple high-order shear deformation beam theory involving only one variable is proposed to study static bending and vibration analyses of nanobeams. The displacement field of the present theory is based on the simple single variable shear deformation theory of Shimpi et al. [42], in which the equilibrium equations of elasticity theory is exploited to derive the governing equation. The nonlocal elasticity theory of Eringen [4] is employed to consider the size effects. Analytical solutions for the transverse deflections and natural frequencies with various boundary conditions are developed. Numerical results are also computed and compared with those predicted by the EBT and TBT to illustrate the accuracy and efficiency of the present study.

2. Single variable local beam theory

The displacement field of Shimpi et al. [42] for isotropic beams is given as follow

$$u_1 = -z\frac{dw_b}{dx} + \left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\frac{dw_s}{dx} \tag{1a}$$

$$u_3 = w_b + w_s \tag{1b}$$

in which w_b and w_s denote the bending and shear components of transverse deflections, and h is the thickness of the beam. For the given displacement field, the quadratic distribution of transverse shear stress across the thickness of the beam is observed. The non-zero stresses are expressed by constitutive relations below

$$\sigma_x = E\varepsilon_x; \;\; \sigma_{xz} = G\gamma_{xz} \tag{2}$$

where E and G are Young's modulus and shear modulus, respectively. The linear strains are given by

$$\varepsilon_x = -z\frac{d^2w_b}{dx^2} + \left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\frac{d^2w_s}{dx^2} \tag{3a}$$

$$\gamma_{xz} = \left(\frac{5}{4} - \frac{5z^2}{h^2}\right)\frac{dw_s}{dx} \tag{3b}$$

For two-dimensional problem of elasticity theory, the equilibrium equations without body forces are stated as

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_{xz}}{dz} = \rho\ddot{u}_1 \tag{4a}$$

$$\frac{d\sigma_{xz}}{dx} + \frac{d\sigma_z}{dz} = \rho\ddot{u}_3 \tag{4b}$$

where the dot-superscript convention denotes the differentiation with respect to time, and ρ is the mass density. The equilibrium equations can be rewritten in terms of stress resultants by multiplying Eq. (4a) with z and then integrating the result and Eq. (4b) over the cross-section along with applying boundary conditions $\sigma_{xz} = 0$ at $z = \pm h/2$ and $\sigma_z = -q\left(x\right)$ at $z = h/2$. The resulting equations are given by

$$\frac{dM}{dx} - V - \rho I\frac{d\ddot{w}_b}{dx} = 0 \tag{5a}$$

$$\frac{dV}{dx} + q\left(x\right) - \rho A\left(\ddot{w}_b + \ddot{w}_s\right) = 0 \tag{5b}$$

where the bending moment M and shear force V are defined as follow

$$M = \int\limits_{-h/2}^{h/2} \sigma_x z b dz = -EI\frac{d^2w_b}{dx^2} \tag{6a}$$

$$V = \int\limits_{-h/2}^{h/2} \sigma_{xz} b dz = \frac{5AG}{6}\frac{dw_s}{dx} \tag{6b}$$

in which

$$\left(A, I\right) = b\int\limits_{-h/2}^{h/2} \left(1, z^2\right) dz \tag{7}$$

where b is the width of the beam. Substituting Eq. (6) into Eq. (5), the shear component w_s can be expressed with respect to the bending component w_b as follow

$$w_s = \frac{6}{5GA}\left(-EI\frac{d^2 w_b}{dx^2} + I\rho\ddot{w}_b\right) \tag{8}$$

Therefore, the displacement field can be rewritten in terms of w_b as

$$u_1 = z\frac{dw_b}{dx} + \frac{6}{5GA}\left(\frac{z}{4} - \frac{5z^3}{3h^2}\right)\left(-EI\frac{d^3 w_b}{dx^3} + I\rho\frac{d\ddot{w}_b}{dx}\right) \tag{9a}$$

$$u_3 = w_b + \frac{6}{5GA}\left(-EI\frac{d^2 w_b}{dx^2} + I\rho\ddot{w}_b\right) \tag{9b}$$

It can be seen that there is only one unknown variable in the displacement field, and consequently only one governing equation is required to describe the behaviour of the beam. The governing equation is derived by using Eqs. (5), (6) and (8) as presented below

$$EI\frac{d^4 w_b}{dx^4} + \rho A\ddot{w}_b + \frac{6\rho^2 I}{5G}\ddddot{w}_b - \rho I\left(1 + \frac{6E}{5G}\right)\frac{d^2\ddot{w}_b}{dx^2} = q\left(x\right) \tag{10}$$

3. Single variable nonlocal beam theory

The well-known nonlocal elasticity theory proposed by Eringen [4] states that the stress at a point in an elasticity continuum not only depends on the strains at that point but also on the strains at all other points in the body. The constitutive relations for the nonlocal beams can be expressed as follow:

$$\sigma_x - \mu\frac{d^2\sigma_x}{dx^2} = E\varepsilon_x \tag{11a}$$

$$\sigma_{xz} - \mu\frac{d^2\sigma_{xz}}{dx^2} = G\gamma_{xz} \tag{11b}$$

where $\mu = \left(e_0 a\right)^2$ is the nonlocal parameter with e_0 being a material-defined constant, and a being the internal length. Generally, the nonlocal parameter depends on various properties such as boundary conditions, chirality, mode shape, number of walls and type of motions [43]. However, there is no comprehensive study made on investigating the values of the nonlocal parameters so far. In this study, the nonlocal parameter $e_0 a \leqslant 2\,nm$ is utilized as a conservative estimate for single-walled carbon nanotubes [44]. The constitutive equations of the nonlocal beam can be rewritten in terms of the stress resultants by multiplying Eq. (11a) with z and then integrating the result and Eq. (11b) over the cross-section.

$$M - \mu\frac{d^2 M}{dx^2} = -EI\frac{d^2 w_b}{dx^2} \tag{12a}$$

$$V-\mu\frac{d^2V}{dx^2}=\frac{5GA}{6}\frac{dw_s}{dx} \tag{12b}$$

Using a similar approach presented in the previous section for the local beams, the nonlocal constitutive equations are also substituted into the equilibrium equations in Eq. (5) to derive the relationship between the bending and shear components of transverse deflection, which yields

$$w_s=\frac{6}{5GA}\left[-EI\frac{d^2w_b}{dx^2}+\rho I\left(\ddot{w}_b-\mu\frac{d^2\ddot{w}_b}{dx^2}\right)\right] \tag{13}$$

Then, the displacement field in Eq. (1) can be rewritten in terms of only w_b for nonlocal beams as follow

$$u_1=z\frac{dw_b}{dx}+\frac{6}{5GA}\left(\frac{z}{4}-\frac{5z^3}{3h^2}\right)\left[-EI\frac{d^3w_b}{dx^3}+\rho I\left(\frac{d\ddot{w}_b}{dx}-\mu\frac{d^3\ddot{w}_b}{dx^3}\right)\right] \tag{14a}$$

$$u_3=w_b+\frac{6}{5GA}\left[-EI\frac{d^2w_b}{dx^2}+\rho I\left(\ddot{w}_b-\mu\frac{d^2\ddot{w}_b}{dx^2}\right)\right] \tag{14b}$$

In order to obtain the governing equation for nonlocal beams, using Eqs. (5), (12), and (13) to obtain

$$\begin{aligned}EI\frac{d^4w_b}{dx^4}+\rho A\ddot{w}_b-\rho I\frac{d^2\ddot{w}_b}{dx^2}-\mu\left(\rho A\frac{d^2\ddot{w}_b}{dx^2}-\rho I\frac{d^4\ddot{w}_b}{dx^4}\right)\\+\underline{\frac{6\rho^2I}{5G}\ddddot{w}_b-\rho\frac{6EI}{5G}\frac{d^2\ddot{w}_b}{dx^2}-\mu\left(-\rho I\frac{6E}{5G}\frac{d^4\ddot{w}_b}{dx^4}+\frac{12\rho^2I}{5G}\frac{d^2\ddddot{w}_b}{dx^2}-\frac{6\mu\rho^2I}{5G}\frac{d^4\ddddot{w}_b}{dx^4}\right)}\\=-\mu\frac{d^2q(x)}{dx^2}+q(x)\end{aligned} \tag{15}$$

Then, the shear force V and bending moment M are given as follow

$$\begin{aligned}V=&-EI\frac{d^3w_b}{dx^3}+\rho I\frac{d\ddot{w}_b}{dx}\\&+\mu\left(-\frac{dq(x)}{dx}+\rho A\frac{d\ddot{w}_b}{dx}-\rho I\frac{d^3\ddot{w}_b}{dx^3}+\underline{\frac{6\rho^2I}{5G}\frac{d\ddddot{w}_b}{dx}-\frac{6\rho IE}{5G}\frac{d^3\ddot{w}_b}{dx^3}-\frac{6\mu\rho^2I}{5G}\frac{d^3\ddddot{w}_b}{dx^3}}\right)\end{aligned} \tag{16a}$$

$$\begin{aligned}M=&-EI\frac{d^2w_b}{dx^2}\\&+\mu\left(-q(x)+\rho A\ddot{w}_b-\rho I\frac{d^2\ddot{w}_b}{dx^2}+\underline{\frac{6\rho^2I}{5G}\ddddot{w}_b-\frac{6\rho IE}{5G}\frac{d^2\ddot{w}_b}{dx^2}-\frac{6\mu\rho^2I}{5G}\frac{d^2\ddddot{w}_b}{dx^2}}\right)\end{aligned} \tag{16b}$$

In Eqs. (15) and (16) above, the underlined terms represent for the shear components of the present theory. If these terms are eliminated, Eqs. (15) and (16) convert to the counterparts of nonlocal EBT (see Reddy and Pang [6]).

4. Analytical solutions

In this section, analytical solutions for transverse deflections and natural frequencies of nonlocal beams with length of L are presented. Four types of boundary conditions are considered including simply supported-simply supported (S-S), clamped-clamped (C-C), cantilever or clamped-free (C-F) and propped cantilever or clamped-simply supported (C-S). Each type of boundary conditions is given as follow:

For S-S beam:

$$u_3 = 0 \,\text{at}\, x = 0, L \tag{17a}$$

$$M = 0 \,\text{at}\, x = 0, L \tag{17b}$$

For C-C beam:

$$u_3 = 0 \,\text{and}\, \left.\frac{du_1}{dz}\right|_{z=0} = 0 \,\text{at}\, x = 0, L \tag{18}$$

For C-F beam:

$$u_3 = 0 \,\text{and}\, \left.\frac{du_1}{dz}\right|_{z=0} = 0 \,\text{at}\, x = 0 \tag{19a}$$

$$M = 0 \,\text{and}\, V = 0 \,\text{at}\, x = L \tag{19b}$$

For C-S beam:

$$u_3 = 0 \,\text{at}\, x = 0, L \tag{20a}$$

$$\left.\frac{du_1}{dz}\right|_{z=0} = 0 \,\text{at}\, x = 0 \tag{20b}$$

$$M = 0 \,\text{at}\, x = L \tag{20c}$$

It should be noted that the clamped support is described by $\frac{du_1}{dz}\big|_{z=0} = 0$ instead of prescribing a zero value for the slope $\frac{dw}{dx} = 0$. In general, these two kinds of clamped support are satisfied for thin beams, where the so-called *effect of shearing force* is relative small and can be neglected. However, when the shear deformation is taken into account, the slope is fixed at clamped support could lead to an underestimate of transverse deflections [45]. Therefore, the boundary condition concerning the slope $\frac{du_1}{dz}\big|_{z=0} = 0$ is employed in this study henceforth to describe the clamped support.

4.1. Bending solutions

In case of static bending analysis, the governing equation is obtained by eliminating time derivative terms in Eq. (15) as follow

$$EI\frac{d^4 w_b}{dx^4} = -\mu\frac{d^2 q\left(x\right)}{dx^2} + q\left(x\right) \tag{21}$$

The expressions for natural boundary conditions in Eq. (16) are reduced to

$$V = -EI\frac{d^3 w_b}{dx^3} - \frac{\mu dq\left(x\right)}{dx} \tag{22a}$$

$$M = -EI\frac{d^2 w_b}{dx^2} - \mu q\left(x\right) \tag{22b}$$

A remarkable notice from the present bending governing equation is that it is virtually identical to that of EBT in [5, 6]. Once the bending component w_b in Eq. (21) is calculated, the total transverse deflections of the present study is obtained by using Eq. (14b).

Consider a nonlocal beam under a distributed load of intensity $q\left(x\right)$ acting in the z-direction, the analytical solutions of linear bending and expressions for shear force V and bending moment M are obtained by integrating Eqs. (21) and (22) as follow

$$EIw_b = -\mu \int^{x}\int^{\eta} q\left(\xi\right) d\eta d\xi + \int^{x}\int^{\eta}\int^{\xi}\int^{\varsigma} q\left(\zeta\right) d\eta d\xi d\varsigma d\zeta + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \tag{23a}$$

$$V = -\int_0^x q\left(\eta\right) d\eta - c_1 - \frac{\mu dq\left(x\right)}{dx} \tag{23b}$$

$$M = \int^{x}\int^{\eta} q\left(\xi\right) d\eta d\xi - c_1 x - c_2 \tag{23c}$$

where $c_1 - c_4$ are constants of integration, which are determined using boundary conditions. In case of uniform distributed load of intensity $q\left(x\right) = q_0$, Eq. (23) can be rewritten as

$$EIw_b = -\mu q_0\frac{x^2}{2} + q_0\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \tag{24a}$$

$$V = -q_0 x - c_1 \tag{24b}$$

$$M = -q_0\frac{x^2}{2} - c_1 x - c_2 \tag{24c}$$

Therefore, the transverse deflection is obtained by substituting Eq. (24a) into Eq. (14b) as follow

$$\begin{aligned} u_3 = \frac{1}{EI}&\left(-\mu q_0\frac{x^2}{2} + q_0\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4\right) \\ &- \frac{6}{5GA}\left(-\mu q_0 + q_0\frac{x^2}{2} + c_1 x + c_2\right) \end{aligned} \tag{25}$$

4.1.1. S-S beam

By employing the boundary conditions given in Eq. (17), the integrating constants are obtained as follow

$$c_1 = -\frac{q_0 L}{2};\ c_2 = 0;\ c_3 = \frac{\mu q_0 L}{2} + \frac{q_0 L^3}{24};\ c_4 = -\frac{6EI\mu q_0}{5GA} \tag{26}$$

Substituting these constants into the expression of transverse deflection and bending moment in Eqs. (24) and (25), the analytical solutions for simply supported beam are given by

$$w = \frac{q_0 L^4}{24EI}\left(\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \frac{x}{L}\right) + \left(\mu + \underline{\frac{6EI}{5GA}}\right)\frac{q_0 L^2}{2EI}\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right) \tag{27a}$$

$$M = -\frac{q_0 L^2}{2}\left(\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right) \tag{27b}$$

It can be seen that the present solutions for transverse deflection and bending moment turn out to those of EBT [6] if the shear component in the underlined term is dismissed. It also shows that the inclusion of nonlocal parameter has a tendency to increase transverse deflections while the bending moment is totally insensitive to the size effect. The maximum transverse deflection and bending moment occur at $x = L/2$ and are given by

$$w_{\max} = \frac{q_0 L^4}{384EI}\left[5 + \frac{48\mu}{L^2} + \underline{\frac{288EI}{5GAL^2}}\right] \tag{28a}$$

$$M_{\max} = \frac{q_0 L^2}{8} \tag{28b}$$

4.1.2. C-C beam

The constants of integration are obtained by using the boundary conditions in Eq. (18) as follow

$$\begin{gathered} c_1 = -\frac{Lq_0}{2};\ c_2 = q_0\left(\frac{L^2}{12} + \mu - \frac{3EI}{10GA}\right); \\ c_3 = \frac{3q_0 L}{20}\frac{EI}{GA};\ c_4 = q_0\frac{6EI}{5GA}\left(\frac{L^2}{12} - \frac{3EI}{10GA}\right) \end{gathered} \tag{29}$$

Using these constants in Eqs. (24) and (25), the expressions of transverse deflection and bending moment are given by

$$w = \frac{q_0 L^4}{24EI}\left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^2\right] + \underline{\frac{3q_0 L^2}{4GA}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right]} \tag{30a}$$

$$M = -\frac{q_0 L^2}{12}\left[1 - 6\frac{x}{L} + 6\left(\frac{x}{L}\right)^2\right] - q_0\mu + \underline{q_0\frac{3EI}{10GA}} \tag{30b}$$

It is observed that the present expressions for transverse deflection and bending moment become identical to those given by nonlocal EBT [6] if the terms involving shear modulus as underlined are omitted. It also seen that the transverse deflection of C-C beam is totally insensitive to the size effect. The maximum transverse deflection and bending moment occur at $x = 0$ and $x = L/2$ respectively, and are given by

$$w_{\max} = \frac{q_0 L^4}{384 J_2} + \underline{\frac{3 q_0 L^2}{16 GA}} \tag{31a}$$

$$M_{\max} = -\frac{q_0 L^2}{12}\left[1 + \frac{12\mu}{L^2} - \underline{\frac{18 EI}{5 GA L^2}}\right] \tag{31b}$$

4.1.3. C-F beam

The constants of integration corresponding with boundary conditions Eq. (19) are obtained as follow

$$c_1 = -q_0 L;\ c_2 = \frac{q_0 L^2}{2};\ c_3 = q_0 L \frac{3EI}{10GA};\ c_4 = \frac{3EI q_0}{5GA}\left(L^2 - 2\mu\right) \tag{32}$$

Then, the transverse deflection of beam and bending moment are given by

$$\begin{aligned} w = {} & \frac{q_0 L^4}{24 EI}\left(\left(\frac{x}{L}\right)^4 - 4\left(\frac{x}{L}\right)^3 + 6\left(\frac{x}{L}\right)^2\right) \\ & -\frac{q_0 L^2}{2EI}\left[\mu\left(\frac{x}{L}\right)^2 + \underline{\frac{6EI}{5GA}\left(\frac{x}{L}\right)^2 + \frac{3EI}{2GA}\left(\frac{x}{L}\right)}\right] \end{aligned} \tag{33a}$$

$$M = -\frac{q_0 L^2}{2}\left(1 - \frac{x}{L}\right)^2 \tag{33b}$$

It is seen that the solutions for cantilever nonlocal EBT is a subset of present solution if the terms with shear modulus as underlined are dismissed. Besides, the bending moment is independent of size effects while transverse deflection tends downwards with the inclusion of nonlocal parameter. The maximum transverse deflection and bending moment are obtained when $x = L$ and $x = 0$, respectively.

$$w_{\max} = \frac{q_0 L^4}{8EI}\left(1 - \frac{4\mu}{L}\right) - \underline{\frac{9 q_0 L^2}{10 GA}} \tag{34a}$$

$$M_{\max} = -\frac{q_0 L^2}{2} \tag{34b}$$

4.1.4. C-S beam

The constants of integration corresponding with boundary conditions Eq. (20) are obtained as follow

$$
\begin{aligned}
c_1 &= -\frac{q_0 L \left(5GAL^2 + 72EI + 120GA\mu\right)}{4\left(10GAL^2 + 36EI + 9EI\right)};\\
c_2 &= \frac{q_0 L^2 \left(5GAL^2 - 18EI + 60GA\mu)\right)}{4\left(10GAL^2 + 36EI + 9EI\right)};\\
c_3 &= \frac{3q_0 EIL \left(25GAL^2 + 72EI + 60GA\mu\right)}{40GA\left(10GAL^2 + 36EI + 9EI\right)};\\
c_4 &= -\frac{3q_0 EI \left(140\mu EI - 5GAL^4 + 18EIL^2 - 20GA\mu L^2\right)}{10GA\left(10GAL^2 + 36EI + 9EI\right)}
\end{aligned}
\tag{35}
$$

The expressions for the transverse deflection and bending moment are derived by substituting those constants into Eqs. (24a) and (24c), respectively. The locations of maximum deflection and bending moment are given by the roots of following equations

$$
\frac{dw}{dx} = \frac{q_0}{EI}\frac{x^3}{6} + \frac{c_1}{EI}\frac{x^2}{2} + \left(\frac{c_2}{EI} - \frac{6q_0}{5GA} - \frac{\mu q_0}{EI}\right)x - \frac{6c_1}{5GA} + \frac{c_3}{EI} = 0 \tag{36a}
$$

$$
\frac{dM}{dx} = -c_1 - q_0 x = 0 \tag{36b}
$$

4.2. Free vibration solutions

For free vibration of the present nonlocal theory, the governing equation is obtained by neglecting loading terms in Eq. (15) as follow

$$
\begin{aligned}
&EI\frac{d^4 w_b}{dx^4} + \rho A\ddot{w}_b + \frac{6\rho^2 I}{5G}\ddddot{w}_b - \rho\left(\mu A + I + \frac{6EI}{5G}\right)\frac{d^2\ddot{w}_b}{dx^2} + \\
&\mu\rho I\left(1 + \frac{6E}{5G}\right)\frac{d^4\ddot{w}_b}{dx^4} - \frac{12\mu\rho^2 I}{5G}\frac{d^2\ddddot{w}_b}{dx^2} + \frac{6\mu^2\rho^2 I}{5G}\frac{d^4\ddddot{w}_b}{dx^4} = 0
\end{aligned}
\tag{37}
$$

The periodic solutions are employed by assuming $w_b(x,t) = W_b e^{i\omega t}$, in which W_b is the mode shape, ω is the natural frequency and $i = \sqrt{-1}$. Then, substituting this expression into Eq. (37) to obtain

$$
p\frac{d^4 W_b}{dx^4} + q\frac{d^2 W_b}{dx^2} - rW_b = 0 \tag{38}
$$

where

$$
p = EI - \omega^2\rho I\mu\left(1 + \frac{6E}{5G}\right) + \frac{6\mu^2\rho^2 I}{5G}\omega^4 \tag{39a}
$$

$$
q = \omega^2\rho\left(\mu A + I + \frac{6EI}{5G}\right) - \frac{12\mu\rho^2 I}{5G}\omega^4 \tag{39b}
$$

$$
r = \omega^2\rho A - \frac{6\rho^2 I}{5G}\omega^4 \tag{39c}
$$

The general solution for the differential equation given in Eq. (38) is

$$W_b = c_1 \sin \alpha x + c_2 \cos \alpha x + c_3 \sinh \beta x + c_4 \cosh \beta x \tag{40}$$

in which

$$\alpha^2 = \frac{1}{2p}\left(q + \sqrt{q^2 + 4pr}\right) \tag{41a}$$

$$\beta^2 = \frac{1}{2p}\left(-q + \sqrt{q^2 + 4pr}\right) \tag{41b}$$

and $c_1 - c_4$ are the integration constants. It should be noted that $\alpha = \beta$ only when the rotary inertia and size effect are neglected. By using Eqs. (14b), (16) and (40), the expressions for transverse deflection, shear force and bending moment are given by

$$\begin{aligned} w = e^{i\omega t}\Bigg[&\left(1 - \omega^2\frac{6\rho I}{5GA} + \alpha^2\left(\frac{6EI}{5GA} - \omega^2\mu\frac{6\rho I}{5GA}\right)\right)(c_1 \sin \alpha x + c_2 \cos \alpha x) + \\ &\left(1 - \omega^2\frac{6\rho I}{5GA} - \beta^2\left(\frac{6EI}{5GA} - \omega^2\mu\frac{6\rho I}{5GA}\right)\right)(c_3 \sinh \beta x + c_4 \cosh \beta x)\Bigg] \end{aligned} \tag{42a}$$

$$V = e^{i\omega t}\left[\alpha\left(p\alpha^2 - s\right)(c_1 \cos \alpha x - c_2 \sin \alpha x) - \beta\left(p\beta^2 + s\right)(c_3 \cosh \beta x + c_4 \sinh \beta x)\right] \tag{42b}$$

$$M = e^{i\omega t}\left[\left(p\alpha^2 - \mu r\right)(c_1 \sin \alpha x + c_2 \cos \alpha x) - \left(p\beta^2 + \mu r\right)(c_3 \sinh \beta x + c_4 \cosh \beta x)\right] \tag{42c}$$

where $s = \omega^2 I_2 + \mu r$. From Eq. (41a), it can be deduced that

$$\alpha^4 p - \alpha^2 q - r = 0 \tag{43}$$

Substituting Eq. (39) into Eq. (43), the quadratic equation of ω^2 is given as follow

$$P\omega^4 - Q\omega^2 + R = 0 \tag{44}$$

in which

$$P = \alpha^4\frac{6\mu^2\rho^2 I}{5G} + \alpha^2\frac{12\mu\rho^2 I}{5G} + \frac{6\rho^2 I}{5G} \tag{45a}$$

$$Q = \alpha^4\mu\rho I\left(1 + \frac{6\rho}{5G}\right) + \alpha^2\rho\left(\mu A + I + \frac{6EI}{5G}\right) + \rho A \tag{45b}$$

$$R = EI \tag{45c}$$

The natural frequencies are given by the roots of Eq. (44) as follow

$$\omega^2 = \frac{Q \pm \sqrt{Q^2 - 4PR}}{2P} \tag{46}$$

In this study, for simplification purpose, the rotary inertial is neglected. Therefore, the solution for natural frequencies is simplified to

$$\omega = \alpha^2 \sqrt{\frac{EI}{\rho A}} \sqrt{\frac{1}{(1+\mu\alpha^2)\left(1+\underline{\alpha^2 \frac{6EI}{5GA}}\right)}} \tag{47}$$

and relationship between α and β is given by

$$\alpha^2 \sqrt{\frac{1}{(1+\mu\alpha^2)\left(1+\underline{\alpha^2 \frac{6EI}{5GA}}\right)}} = \beta^2 \sqrt{\frac{1}{(1-\mu\beta^2)\left(1-\underline{\beta^2 \frac{6EI}{5GA}}\right)}} \tag{48}$$

It is observed that the solution for the natural frequencies of nonlocal EBT beams [6] is obtained when the shear components in the underlined terms are ignored in Eqs. (47) and (48).

4.2.1. S-S beam

By using the boundary conditions in Eq. (17), the following coefficient equations are obtained

$$\Lambda c_2 + \Xi c_4 = 0 \tag{49a}$$

$$\Lambda (\sin \alpha L) c_1 + \Lambda (\cos \alpha L) c_2 + \Xi (\sinh \beta L) c_3 + \Xi (\cosh \beta L) c_4 = 0 \tag{49b}$$

$$\Pi c_2 - \Phi c_4 = 0 \tag{49c}$$

$$\Pi (\sin \alpha L) c_1 + \Pi (\cos \alpha L) c_2 - \Phi (\sinh \beta L) c_3 + \Phi (\cosh \beta L) c_4 = 0 \tag{49d}$$

where

$$\Lambda = 1 + \alpha^2 \frac{6EI}{5GA}; \; \Xi = 1 - \beta^2 \frac{6EI}{5GA} \tag{50a}$$

$$\Pi = p\alpha^2 - \mu r \, ; \;\; \Phi = p\beta^2 + \mu r \tag{50b}$$

The first and third equations give $c_2 = c_4 = 0$, therefore Eq. (49) is reduced to

$$c_1 A \sin \alpha L + c_3 B \sinh \beta L = 0 \tag{51a}$$

$$c_1 P \sin \alpha L - c_3 Q \sinh \beta L = 0 \tag{51b}$$

The determinant of the coefficient matrix of above equations must zero to obtain non-trivial solutions, which means

$$\left(\beta^2 + \alpha^2\right) \left(p + \mu r \frac{6EI}{5GA}\right) \sin \alpha L \sinh \beta L = 0 \tag{52}$$

For nonzero values of α and β

$$\sin \alpha L = 0 \rightarrow \alpha = n\pi/L \tag{53}$$

The natural frequencies of S-S beam are given by

$$\omega = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \sqrt{\frac{1}{\left(1 + \mu\left(\frac{n\pi}{L}\right)^2\right)\left(1 + \left(\frac{n\pi}{L}\right)^2 \frac{6EI}{5GA}\right)}} \tag{54}$$

4.2.2. C-C beam

The following coefficient equations are obtained

$$\Lambda c_2 + \Xi c_4 = 0 \tag{55a}$$

$$c_1 \Lambda \sin \alpha L + c_2 \Lambda \cos \alpha L + c_3 \Xi \sinh \beta L + c_4 \Xi \cosh \beta L = 0 \tag{55b}$$

$$\alpha \Psi c_1 + \beta \Gamma c_3 = 0 \tag{55c}$$

$$c_1 \alpha \Psi \cos \alpha L - c_2 \alpha \Psi \sin \alpha L + c_3 \beta \Gamma \cosh \beta L + c_4 \beta \Gamma \sinh \beta L = 0 \tag{55d}$$

where

$$\Psi = 1 - \alpha^2 \frac{3EI}{10GA}\,; \;\; \Gamma = 1 + \beta^2 \frac{3EI}{10GA} \tag{56}$$

Eliminating c_3 and c_4 with the relations obtained from the first and the third equations in Eq. (55), then one can be reduced to

$$\left(\Lambda \sin \alpha L - \frac{\alpha}{\beta}\frac{\Psi \Xi}{\Gamma} \sinh \beta L\right) c_1 + \Lambda \left(\cos \alpha L - \cosh \beta L\right) c_2 = 0 \tag{57a}$$

$$\alpha \Psi \left(\cos \alpha L - \cosh \beta L\right) c_1 - \left(\alpha \Psi \sin \alpha L + \beta \frac{\Lambda \Gamma}{\Xi} \sinh \beta L\right) c_2 = 0 \tag{57b}$$

For non-trivial solutions of α and β, the determination of coefficient matrix of above equations should be zero, which gives

$$2 - \left(\frac{\alpha}{\beta}\frac{\Psi \Xi}{\Lambda \Gamma} - \frac{\beta}{\alpha}\frac{\Lambda \Gamma}{\Psi \Xi}\right) \sin \alpha L \sinh \beta L - 2 \cos \alpha L \cosh \beta L = 0 \tag{58}$$

When the value of α is obtained from Eqs. (48) and (54), the natural frequencies of C-C beam are given by Eq. (47).

4.2.3. C-F beam

The following coefficient equations are obtained

$$\Lambda c_2 + \Xi c_4 = 0 \tag{59a}$$

$$\alpha \Pi c_1 + \beta \Phi c_3 = 0 \tag{59b}$$

$$\Psi (\sin \alpha L)\, c_1 + \Psi (\cos \alpha L)\, c_2 - \Gamma (\sinh \beta L)\, c_3 - \Gamma (\cosh \beta L)\, c_4 = 0 \tag{59c}$$

$$\alpha \Psi (\cos \alpha L)\, c_1 - \alpha \Psi (\sin \alpha L)\, c_2 - \beta \Gamma (\cosh \beta L)\, c_3 - \beta \Gamma (\sinh \beta L)\, c_4 = 0 \tag{59d}$$

Using the first and second equations in the remaining ones to obtain

$$\left(\Psi \sin \alpha L + \frac{\alpha}{\beta} \frac{\Pi \Gamma}{\Phi} \sinh \beta L \right) c_1 + \left(\Psi \cos \alpha L \ + \frac{\Gamma \Lambda}{\Xi} \cosh \beta L \right) c_2 = 0 \tag{60a}$$

$$\left(\alpha \Psi \cos \alpha L + \alpha \frac{\Gamma \Pi}{\Phi} \cosh \beta L \right) c_1 - \left(\alpha \Psi \sin \alpha L - \beta \frac{\Lambda \Gamma}{\Xi} \sinh \beta L \right) c_2 = 0 \tag{60b}$$

A transcendental equation is obtained by setting the determinant of coefficient matrix of above equations as follow

$$\begin{aligned} \Psi^2 \Xi \Pi + \Gamma^2 \Lambda \Phi + \left(\frac{\alpha}{\beta} \Pi \Xi - \frac{\beta}{\alpha} \Lambda \Phi \right) \Psi \Gamma \sin \alpha L \sinh \beta L \\ + (\Lambda \Phi + \Pi \Xi)\, \Psi \Gamma \cos \alpha L \cosh \beta L = 0 \end{aligned} \tag{61}$$

The natural frequencies of cantilever nonlocal beams are obtained from Eq. (47) when the value of α is determined from Eqs. (48) and (61).

4.2.4. C-S beam

The following coefficient equations are obtained

$$\Lambda c_2 + \Xi c_4 = 0 \tag{62a}$$

$$c_1 \Lambda \sin \alpha L + c_2 \Lambda \cos \alpha L + c_3 \Xi \sinh \beta L + c_4 \Xi \cosh \beta L = 0 \tag{62b}$$

$$\alpha \Pi c_1 + \beta \Phi c_3 = 0 \tag{62c}$$

$$c_1 \Psi \sin \alpha L + c_2 \Psi \cos \alpha L - c_3 \Gamma \sinh \beta L - c_4 \Gamma \cosh \beta L = 0 \tag{62d}$$

Using the first and the third equations to eliminate c_3 and c_4, then

$$\left(\Lambda \sin \alpha L - \frac{\alpha}{\beta} \frac{\Pi \Xi}{\Phi} \sinh \beta L \right) c_1 + \Lambda (\cos \alpha L - \cosh \beta L)\, c_2 = 0 \tag{63a}$$

$$\left(\Psi\sin\alpha L+\frac{\alpha}{\beta}\frac{\Gamma\Pi}{\Phi}\sinh\beta L\right)c_1+\left(\Psi\cos\alpha L+\frac{\Gamma\Lambda}{\Xi}\cosh\beta L\right)c_2=0 \tag{63b}$$

For non-trivial solutions, the determinant of the coefficient matrix of above equations should be zero, which leads to

$$\frac{\Lambda}{\Xi}\tan\alpha L-\frac{\alpha}{\beta}\frac{\Pi}{\Phi}\tanh\beta L=0 \tag{64}$$

After determining α from Eqs. (48) and (64), the natural frequencies are obtained by using Eq. (47).

5. Numerical examples

In this section, the numerical results for transverse deflections and fundamental natural frequencies of nanobeams are presented and then compared with those calculated form nonlocal EBT and TBT formulations reported by Reddy and Pang [6] to validate the accuracy. Four types of boundary conditions are considered, such as (S-S), (C-C), (C-F) and (C-S). A wide range of nonlocal parameters and thickness ratios are also taken into investigation to illustrate their effects. The length of nanobeam L is assumed to be 10 nm, while the nonlocal parameters $\mu=(e_0a)^2$ have the values of 0, 1, 2, 3 and 4. It should be noted that $\mu=0$ refers to local beams and value of shear correction factor used for TBT is 5/6. For the shake of convenience, the following normalized quantities are employed

$$\bar{w}=100\frac{EIw}{q_0L^4};\ \bar{\omega}=\omega L^2\sqrt{\frac{\rho A}{EI}} \tag{65}$$

Tables 1 and 2 present the results for normalized maximum deflections and fundamental natural frequencies of nanobeams with various boundary conditions. In general, the results obtained from the EBT are close to those of TBT and present theory for thin beams. However, when thick beams are considered, which implies the shear deformation effect becomes significant, the present results are in close agreement with those of TBT. Since EBT neglects the shear deformation effect, there are discrepancies between present results and those from the EBT. It should be noted that the present theory is analogous to the ETB as it only requires only one variable; however, its predictions are comparable to those of the TBT, which involves two variables and requires a shear correction factor. The effects of shear deformation and nonlocal parameters on transverse deflections and fundamental frequencies of nanobeams is depicted in Figs. 1 and 2. Here $\bar{w}/\bar{w}_e$ and $\bar{\omega}/\bar{\omega}_e$ denote the ratios of maximum deflections and fundamental frequencies obtained by the present theory to those predicted by the local EBT, respectively.

As expected, when the shear deformation effect increases, the ratios of deflections increase and ratios of fundamental frequencies decrease with all considered boundary conditions. The most significant effect is seen for C-C beams and the least one is for C-F beams.

In this part, the the effect of nonlocal parameters on bending and vibration behaviour of nanobeams is investigated. For the bending behaviour, this effect increases the deflection of S-S beams and C-S beams, reduces the deflection of C-F beams but has no change on C-C beams (see Fig. 1). It is also seen that the size effects are most pronounced for S-S beams, as the differences between the nonlocal ($\mu = 4\,\text{nm}$) and local ($\mu = 0$) results are 38.39% and 34.91% for thin beams ($h/L = 0.01$) and thick beams ($h/L = 0.2$), respectively. The corresponding figures for other beams are 26.28% and 21.26% (for C-S beams with $h/L = 0.01$ and $h/L = 0.2$, respectively), 16% and 15.6% (for C-F beams with $h/L = 0.01$ and $h/L = 0.2$, respectively). For the vibration behaviour, the effect of nonlocal parameters on fundamental frequencies is different. As this effect increases, considerable decreases in the values of fundamental natural frequencies of S-S, C-C and C-S beams are observed, while the results for C-F beams slightly increase as depicted in Fig. 2. Specifically, this effect is most significant for C-C beams and lessened for C-S beams and S-S beams, in which the discrepancies between the nonlocal ($\mu = 4\,\text{nm}$) and local ($\mu = 0$) results are 18.24% ($h/L = 0.01$) and 16.14% ($h/L = 0.2$) for C-C beams; 17.33% ($h/L = 0.01$) and 16.3% ($h/L = 0.2$) for C-S beams; 15.33% ($h/L = 0.01$) and 15.33% ($h/L = 0.2$) for S-S beams. However, only small amounts of discrepancies of 1.8% ($h/L = 0.01$) and 1.5% ($h/L = 0.2$) are seen for C-F beams.

6. Conclusions

In this study, a simple shear deformation beam theory involving one variable is proposed for static bending and free vibration analyses of nanobeams. Nonlocal theory is utilized to capture size effects. The equilibrium equations of elasticity theory are employed to developed the governing equation. Analytical solutions for transverse deflections and natural frequencies with respect to four types boundary conditions are presented. Numerical results are also given and compared well with those predicted by the nonlocal EBT and TBT. It is shown that the present nonlocal theory has ability to account for the size effect and shear deformation without requirement of shear correction factor, although it has close similarities with the nonlocal EBT in terms of governing equation and boundary conditions. Furthermore, the numerical results point out that the inclusion of nonlocal parameters has tendencies to increase the transverse deflection and decrease the fundamental natural frequencies in simply supported beams and propped cantilever beams, whereas the inverse effects are observed in cantilever beams. For clamped beams, although there is a considerable effect on their fundamental vibration, their bending behaviour is totally insensitive to the nonlocal parameters.

Acknowledgements

This research study was supported by a Postgraduate Research Scholarship at La Trobe University. This financial support is gratefully acknowledged.

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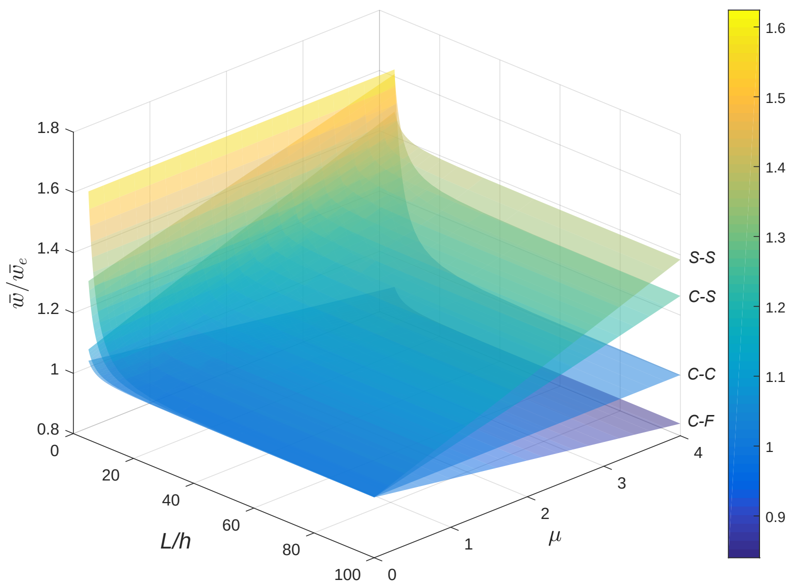

Figure 1: Effects of shear deformation and nonlocal parameters on the deflection of nanobeams.

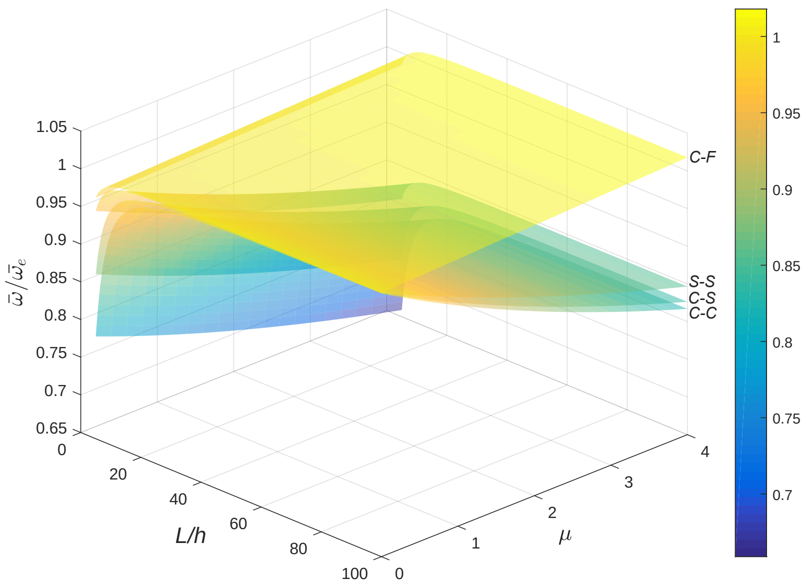

Figure 2: Effects of shear deformation and nonlocal parameters on the fundamental frequency of nanobeams.

Table 1: Normalized deflections $\bar{w}$ of nanobeams under uniform distribution load.

		Simply supprted beam (S-S)			Clamped beam (C-C)			Cantilever beam (C-F)			Propped canteliver beam (C-S)		
h/L	μ	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present
0.01	0	1.3021	1.3024	1.3024	0.2604	0.2607	0.2608	12.5000	12.5013	12.5020	0.5416	0.5420	0.5421
	1	1.4271	1.4274	1.4274	0.2604	0.2607	0.2608	12.0000	12.0013	12.0020	0.5416	0.5774	0.5774
	2	1.5521	1.5524	1.5524	0.2604	0.2607	0.2608	11.5000	11.5013	11.5020	0.5416	0.6129	0.6130
	3	1.6771	1.6774	1.6774	0.2604	0.2607	0.2608	11.0000	11.0013	11.0020	0.5416	0.6486	0.6487
	4	1.8021	1.8024	1.8024	0.2604	0.2607	0.2608	10.5000	10.5013	10.5020	0.5416	0.6844	0.6845
0.05	0	1.3021	1.3102	1.3102	0.2604	0.2685	0.2706	12.5000	12.5325	12.5487	0.5416	0.5509	0.5527
	1	1.4271	1.4352	1.4352	0.2604	0.2685	0.2706	12.0000	12.0325	12.0487	0.5416	0.5864	0.5882
	2	1.5521	1.5602	1.5602	0.2604	0.2685	0.2706	11.5000	11.5325	11.5487	0.5416	0.6221	0.6239
	3	1.6771	1.6852	1.6852	0.2604	0.2685	0.2706	11.0000	11.0325	11.0487	0.5416	0.6579	0.6598
	4	1.8021	1.8102	1.8102	0.2604	0.2685	0.2706	10.5000	10.5325	10.5487	0.5416	0.6939	0.6958
0.1	0	1.3021	1.3346	1.3346	0.2604	0.2929	0.3010	12.5000	12.6300	12.6950	0.5416	0.5790	0.5860
	1	1.4271	1.4596	1.4596	0.2604	0.2929	0.3010	12.0000	12.1300	12.1950	0.5416	0.6148	0.6220
	2	1.5521	1.5846	1.5846	0.2604	0.2929	0.3010	11.5000	11.6300	11.6950	0.5416	0.6509	0.6581
	3	1.6771	1.7096	1.7096	0.2604	0.2929	0.3010	11.0000	11.1300	11.1950	0.5416	0.6871	0.6945
	4	1.8021	1.8346	1.8346	0.2604	0.2929	0.3010	10.5000	10.6300	10.6950	0.5416	0.7235	0.7310
0.2	0	1.3021	1.4321	1.4321	0.2604	0.3904	0.4229	12.5000	13.0200	13.2800	0.5416	0.6910	0.7185
	1	1.4271	1.5571	1.5571	0.2604	0.3904	0.4229	12.0000	12.5200	12.7800	0.5416	0.7285	0.7565
	2	1.5521	1.6821	1.6821	0.2604	0.3904	0.4229	11.5000	12.0200	12.2800	0.5416	0.7661	0.7946
	3	1.6771	1.8071	1.8071	0.2604	0.3904	0.4229	11.0000	11.5200	11.7800	0.5416	0.8039	0.8329
	4	1.8021	1.9321	1.9321	0.2604	0.3904	0.4229	10.5000	11.0200	11.2800	0.5416	0.8418	0.8713

Table 2: Normalized fundamental frequencies $\bar{\omega}$ of nanobeams.

		Simply supprted beam (S-S)			Clamped beam (C-C)			Cantilever beam (C-F)			Propped canteliver beam (C-S)		
h/L	μ	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present	EBT[6]	TBT[6]	Present
0.01	0	9.8696	9.8683	9.8683	22.3733	22.3589	22.3562	3.5160	3.5158	3.5157	15.4182	15.4128	15.4120
	1	9.4159	9.4147	9.4147	21.1090	21.0960	21.0938	3.5313	3.5311	3.5310	14.5992	14.5942	14.5935
	2	9.0195	9.0183	9.0183	20.0328	20.0208	20.0190	3.5469	3.5467	3.5466	13.8962	13.8915	13.8909
	3	8.6693	8.6682	8.6682	19.1029	19.0917	19.0902	3.5630	3.5627	3.5627	13.2843	13.2799	13.2794
	4	8.3569	8.3558	8.3558	18.2894	18.2789	18.2776	3.5794	3.5792	3.5791	12.7458	12.7416	12.7412
0.05	0	9.8696	9.8381	9.8381	22.3733	22.0217	21.9572	3.5160	3.5107	3.5091	15.4182	15.2835	15.2643
	1	9.4159	9.3858	9.3858	21.1090	20.7896	20.7367	3.5313	3.5258	3.5242	14.5992	14.4751	14.4589
	2	9.0195	8.9907	8.9907	20.0328	19.7384	19.6941	3.5469	3.5413	3.5397	13.8962	13.7806	13.7666
	3	8.6693	8.6416	8.6416	19.1029	18.8286	18.7909	3.5630	3.5571	3.5555	13.2843	13.1758	13.1635
	4	8.3569	8.3302	8.3302	18.2894	18.0316	17.9990	3.5794	3.5734	3.5717	12.7458	12.6431	12.6323
0.1	0	9.8696	9.7454	9.7454	22.3733	21.0564	20.8311	3.5160	3.4949	3.4888	15.4182	14.8990	14.8279
	1	9.4159	9.2973	9.2973	21.1090	19.9087	19.7204	3.5313	3.5095	3.5033	14.5992	14.1202	14.0596
	2	9.0195	8.9059	8.9059	20.0328	18.9241	18.7641	3.5469	3.5244	3.5181	13.8962	13.4496	13.3971
	3	8.6693	8.5601	8.5601	19.1029	18.0681	17.9304	3.5630	3.5397	3.5333	13.2843	12.8646	12.8184
	4	8.3569	8.2517	8.2517	18.2894	17.3157	17.1957	3.5794	3.5554	3.5489	12.7458	12.3486	12.3075
0.2	0	9.8696	9.3990	9.3990	22.3733	18.1546	17.5790	3.5160	3.4337	3.4103	15.4182	13.6035	13.3889
	1	9.4159	8.9669	8.9669	21.1090	17.2291	16.7253	3.5313	3.4463	3.4226	14.5992	12.9177	12.7298
	2	9.0195	8.5894	8.5894	20.0328	16.4259	15.9812	3.5469	3.4591	3.4352	13.8962	12.3236	12.1571
	3	8.6693	8.2559	8.2559	19.1029	15.7211	15.3253	3.5630	3.4723	3.4481	13.2843	11.8027	11.6537
	4	8.3569	7.9585	7.9585	18.2894	15.0964	14.7418	3.5794	3.4858	3.4613	12.7458	11.3414	11.2068