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| Shear-Deformable Finite Element for Free Vibrations of Laminated Composite Beams with Arbitrary Lay-Up  Volkan Kahya1a[[1]](#footnote-1)\*, Sebahat Karaca1b, Thuc P. Vo2c  *1Karadeniz Technical University, Faculty of Engineering, Department of Civil Engineering, 61080 Trabzon, Turkey*  *2School of Engineering and Mathematical Sciences, La Trobe University, Bundoora, VIC 3086, Australia*  *(Received*  *keep as blank , Revised*  *keep as blank ,* *Accepted*  *keep as blank )9pt* |
| **Abstract.** A shear-deformable finite element model (FEM) with five nodes and thirteen degrees of freedom (DOFs) for free vibrations of laminated composite beams with arbitrary lay-up is presented. This model can be capable of considering the elastic couplings among the extensional, bending and torsional deformations, and the Poisson’s effect. Lagrange’s principle is employed in derivation of the equations of motion, and thus the element matrices are obtained. Comparisons of the present element’s results with those in experiment, available literature and the 3D finite element analysis software (ANSYS®) are made to show its accuracy. Some further results are given as referencing for the future studies in vibrations of laminated composite beams.  **Keywords:** laminated composite beam; free vibration; finite element method; anisotropy |

**1. Introduction**

Fibre reinforced composite materials have a wide range of applications in civil, mechanical, aerospace engineering and the related areas. The interest is due to their high strength, lightness, corrosion resistance, good thermal properties, design flexibility, etc. The mechanical behaviour of such materials under loading greatly depends on ply-stacking sequences. Due to anisotropic properties of composite materials, their structural analysis is more complicated than the metallic ones.

Since the laminated composite beams are often considered as important load-carrying elements of structures, an accurate model to predict their dynamic characteristics is necessary. In some cases, angle-ply and unsymmetric laminates may be essential from better design viewpoints. For angle-ply laminates, elastic couplings among extension, bending and torsion deformations due to anisotropy will become important. Furthermore, in one-dimensional analysis of laminated composites with no stresses in the width direction, neglecting the Poisson’s effect causes the loss of some stiffness coefficients (Shao et al. 2017). Hence, the elastic couplings and the Poisson’s effect should be considered in analysis of laminated composite beams with arbitrary lay-ups.

Various analytical and numerical models, which are based on various beam theories to consider the effect of shear deformation, rotary inertia, warping, and so forth, have been developed for laminated composite beams. A comprehensive literature review on their structural behaviours was summarized in Sayyad et al. (2017). Among the solution methods, the finite element method (FEM) has been widely used by researchers due to its flexibility to define the unknown displacement variables by various polynomial expressions. In literature, various finite element models based on the first-order/Timoshenko and higher-order as well as layer-wise and zig-zag beam theories for the free vibration analysis of laminated composite and sandwich beams have been developed. Some papers related to each theory are briefly summarized in the following separate paragraphs.

Based on the first-order beam theory, Kadivar and Mohebpour (1998) proposed a beam element having 16, 20 and 24 degrees of freedom (DOFs) for dynamic analysis of unsymmetric laminated composite beams subjected to moving loads. Chakraborty et al. (2002) developed a refined locking-free beam element for free vibration and wave propagation analyses of asymmetric laminated composite beams. Goyal and Kapania (2007) developed a 21-DOF beam element for analysis of laminated composite beams with arbitrary lay-ups, in which an accurate model for the shear correction factor was used. Jun et al. (2008) derived a dynamic FEM, which incorporated the Poisson’s effect and couplings among extensional, flexural and torsional deformations, to perform free vibration analysis of generally laminated composite beams. Mohebpour et al. (2011) studied the dynamic response of laminated composite beams under the action of a moving oscillator. They accounted for the complete dynamic interaction between the beam and oscillator. Kahya (2012) studied the dynamic response of laminated composite beams subjected to moving loads by using a multi-layered finite element. This model considered separate rotational DOFs for each lamina but did not require any additional axial or transversal DOFs. Jafari-Talookolaei et al. (2017) proposed a FEM for in-plane and out-of-plane vibrations of laminated composite beams. They indicated the importance of out-of-plane displacement component in calculation of torsional modes.

Based on higher-order beam theories, Shi and Lam (1999) developed a two-noded beam element for free vibration of laminated composite beams. They studied the effect of mass components due to the higher-order displacement and the coupling of the different order axial displacement components on the accuracy of analysis. Subramanian (2006) presented the free vibration analysis of laminated composite beams by two-noded *C*1 beam elements. In-plane and out-of-plane displacements were, respectively, assumed as a quantic and quartic variation through-the-thickness. Lezgy-Nazargah et al. (2011) developed a refined high-order global-local theory for laminated and sandwich beams, that satisfies all the kinematic and stress continuity conditions at the layer interfaces and considers the effects of transverse normal stress and transverse flexibility. Vo and co-workers proposed a two-noded beam element for vibration and buckling of laminated composite beams with arbitrary lay-ups (Vo and Thai 2012a,b; Vo et al. 2013). They took into account the parabolical and sinusoidal variation of shear strains through-the-thickness. Later, they included both shear and normal deformations to study free vibrations of composite beams with axial loads (Vo et al. 2017).

This paragraph reviews few papers related to layer-wise and zig-zag beam theories for free vibrations of laminated composite beams. Ramtekkar et al. (2002) developed a six-noded, plane-stress mixed FEM by using Hamilton’s principle. Vidal and Polit (2010) proposed a family of sinus models for the free vibration analysis of laminated composite beams. Chalak et al. (2012) investigated the free vibration of soft-core sandwich beams using a *C*0 beam element based on a higher-order zig-zag theory, in which the cubic and quadratic distribution of axial and transverse displacement were considered. Filippi et al. (2016) proposed 1D layer-wise theories using the higher-order zig-zag functions defined over fictitious/mathematical layers of the cross-sectional area. Wimmer and Gherlone (2017) presented explicit expressions for the linear and geometric stiffness matrix, the mass matrix and the equivalent nodal force vector of a simple planar beam element based on the refined zig-zag theory. Kahya and Turan (2018) presented a multilayer finite element for buckling and free vibration of laminated composite and sandwich beams based on a higher-order layer-wise theory. They gave some comparisons for buckling loads and natural frequencies of beams with different end conditions and lamina stacking to show the accuracy of proposed element.

Although the elastic couplings due to anisotropy and the Poisson’s effect are well-studied in laminated composite beams by various analytical/semi-analytical methods, according to above-given literature survey, the numerical solutions based on FEM accounting for both bending-extension, bending-twist and extension-twist couplings and the Poisson’s effect in free vibrations of such structural elements has not been adequately studied. To fill this gap, we proposed a five-noded finite element with 13 DOFs based on the first-order shear deformation theory. This higher-order element is capable of considering the aforementioned couplings due to anisotropy and the Poisson’s effect. By using Lagrange’s principle, governing equations of motion and the element mass and stiffness matrices are derived. Natural frequencies and corresponding mode shapes of laminated composite beams are then obtained by solving the standard eigenvalue problem under various boundary conditions, lay-ups, orthotropy ratio and slenderness. In order to show accuracy of the present element, comparisons with the experimental study, available literature and finite element analyses in ANSYS® (2014) are made. Some numerical results are presented for the first time as a reference for the future studies in the area.

**2. Governing equations of motion**

A laminated composite beam of length *L*, width *b* and height *h* is considered as illustrated in Fig. 1. The beam is made of *n* orthotropic layers with different fibre angle measured in counter-clockwise direction to the *x*-axis.

Based on the first-order beam theory, the displacement field can be assumed as:

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| --- | --- |
|  | (1) |

where *u*, *v* and *w* are the displacement components in the *x*-, *y*- and *z*-directions at any point of the beam, respectively. *u0* and *w0* are the displacements in the *x*- and *z*-directions at a point on the midplane, *θ* and *ψ* are the rotations of the normal to the midplane about the *y*- and *x*-axes, respectively, and *t* is time. The displacement field in Eq. (1) allows that the beam can stretch along the *z*-axis, bend in the *x*-*z* plane, and twist around *x*-axis. Note that there is no bending in the *y*-*z* plane. With the use of such displacement field, one can take into account the bending-stretching, bending-torsion and torsion-stretching couplings due to anisotropy in the formulation. The displacement field given by Eq. (1) has been previously used by Jun et al. (2008) and Jafari-Talokolaei et al. (2012) for the free vibration analysis of generally laminated composite beams, by Mohebpour et al. (2011) for the dynamic response analysis of laminated beams subjected to moving oscillators, and by Wang et al. (2015) for the buckling analysis of laminated composite beams with general lamina layup. For the present study, the displacement field given by Eq. (1) is used to develop a higher-order finite element for laminated composite beams shown in Fig. 2.

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| Fig. 1 Geometry and coordinate system for laminated composite beam with rectangular cross-section |
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| Fig. 2 13-DOF beam finite element |

From Eq. (1), the strain-displacement relationships can be written as follows:

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|  | (2) |

where,andare the midplane strain, bending and twisting curvatures, respectively, (,*x*) denotes the derivative with respect to *x*.

The constitutive relations of the laminate can be given by (Reddy, 1999):

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|  | (3) |
|  | (4) |

where *Nx*, *Ny* and *Nxy* are the in-plane forces, *Mx*, *My* and *Mxy* are the bending and twisting moments, *Qyz* and *Qxz* are the transverse shear forces per unit length, , , and *γxy* are the normal and shear strains, *κx*, *κy* and *κxy* are the bending and twisting curvatures, respectively. The laminate stiffness coefficients *A*, *B* and *D* are given by

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|  | (5) |

where *K* is the shear correction factor, *n* is the number of layers in the laminate, and  are the transformed reduced stiffness constants for the *k*th layer obtained by considering the transverse normal stress  be negligible. They can be found in the explicit form in Reddy (1999).

To include the Poisson’s effect, the strain components  and *γyz*, and the curvature *κy* are assumed as non-zero while the in-plane forces *Ny* and *Nxy*, the bending moment *My* and the lateral shearing force *Qyz* are equal to zero. Therefore, Eqs. (3) and (4) can be rewritten as

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|  | (6) |
|  | (7) |

where

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|  | (8) |
|  | (9) |

Eqs. (6) and (7) are the constitutive relations for a laminated composite beam including the Poisson’s effect. The coefficients ,andare replaced by *A*, *B* and *D* to ignore this effect.

The strain energy *U* of the system is

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|  | (10) |

Substituting Eqs. (2), (6) and (7) into Eq. (10) yields

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|  | (11) |

The kinetic energy *T* of system is

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|  | (12) |

whereis the mass density, and dot denotes the derivative with respect to time. Substituting Eqs. (1) into Eq. (12) and performing integration with respect to *z* yields

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|  | (13) |

where

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|  | (14) |

By using the Lagrange’s principle given by

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|  | (15) |

where *L = T – U* is the Lagrangian functional and *q* denotes the unknown variables *u*0, *w*0, *θ* and *ψ*, the governing equations of motion for the laminated composite beam can be derived.

**3. Finite element model**

The present finite element has five nodes and thirteen DOFs as shown in Fig. 2. The element can be capable of taking into account the shear deformation, rotary inertia, Poisson’s effect and elastic couplings due to anisotropy. The additional DOFs at the inner nodes provides to better represent the complex behaviour of the generally layered composite beams.

The solutions are assumed to be:

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|  | (16) |

where  and  are, respectively, quadratic and cubic Langrange polynomials. According to Eq. (2), the axial deformation is dependent on first spatial derivatives of *u*0 and *θ*. Thus, the degree of polynomials of *u*0 and *θ* must be same. Also, the shear strain *γxz* is a linear function of *w*0,*x* and *θ*, thus the degree of polynomial used for *w*0 must be one order higher than those used for *u*0 and *θ* to ensure compatibility. Therefore, the cubic polynomial used for *w*0 requires that quadratic functions for both *u*0 and *θ* for consistency. Such choice also prevents the shear locking. In addition, the quadratic polynomial for *ψ* is appropriate since the twisting DOFs are located at the same nodes with extensional and rotational DOFs. Explicit expressions of the shape functions are given in Appendix

The solutions given by Eq. (16) are substituted into the energy expressions given by Eqs. (11) and (13), the result is substituted into Eq. (15), after some arrangements, the following expression can be obtained:

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|  | (17) |

where **Me** and **Ke** are the element mass and stiffness matricesand **ue** is a vector including the nodal displacements, respectively, which are given in Appendix.

The element matrices are assembled into the global ones such as  and , where *m* is the number of elements. Thus, the total number of nodal displacements in the discretized finite element model of the beam is (9*m* + 4). The global matrix equation of motion for free vibrations of the whole system becomes

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|  | (18) |

where **M** and **K** are the global mass and stiffness matrices, and **U** is the nodal displacements of the entire system. Assuming the solution of Eq. (18) to be in the form  yields

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|  | (19) |

where  is the natural frequency, U0 is the mode shapes vector. The nontrivial solution of Eq. (19) requires solving the standard eigenvalue problem as  for

the natural frequencies. The corresponding mode shapes can then be obtained by Eq. (19) with back-substituting.

**4. Results and discussion**

Results for natural frequencies and mode shapes of the laminated composite beams with various configurations including boundary conditions and lay-ups are presented in this section. Comparisons with the experimental study, available literature and the finite element analyses in ANSYS® are made to show the accuracy, reliability and feasibility of the present element. Further, some new results, which may be used as reference data for future, are presented. In numerical examples, the shear correction factor *K* is taken to be 5/6, and all layers within the laminate is assumed to have equal thickness. Five boundary conditions are considered such as clamped-clamped (C-C), simply supported (S-S), clamped-free (C-F), clamped-simply supported (C-S) and free-free (F-F) at its both ends of laminated composite beams.

*4.1* *Convergence study*

A convergence study is performed to determine the number of elements that will be sufficient in the numerical examples. To this aim, an unsymmetric [30/50]2 laminated composite beam with clamped-clamped end conditions is considered. Geometry and material properties of the beam are *E*1 = 144.8 GPa, *E*2 = 9.65 GPa, *G*12 = *G*13 = 4.14 GPa, *G*23 = 3.45 GPa, *ν* = 0.3, *ρ* = 1389.23 kg/m3, *L* = 0.381 m, *b* = *h* = 25.4 mm. The first five natural frequencies (in Hz) for bending vibrations of the laminated composite beam are presented in Table 1 with various number of elements along with the analytical results given by Jun et al. (2008). As can be seen, a rapid convergence is obtained, and *m* =20 elements are sufficient to guarantee the numerical convergence. Besides, an excellent agreement with the analytical results given by Jun et al. (2008) can be observed.

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| Table 1 Convergence study of the present element for a clamped-clamped laminated beam with [30/50]2 lay-up | | | | | |
| Number of elements (*m*) | Natural frequencies (Hz) | | | | |
| 1st mode | 2nd mode | 3rd mode | 4th mode | 5th mode |
| 2 | 647.416 | 2059.921 | 3927.237 | 5104.989 | 5385.672 |
| 4 | 638.749 | 1660.087 | 3052.057 | 3913.302 | 5061.198 |
| 8 | 638.036 | 1648.654 | 3005.153 | 3911.176 | 4605.382 |
| 12 | 637.997 | 1647.981 | 3001.087 | 3911.058 | 4591.068 |
| 16 | 637.991 | 1647.866 | 3000.383 | 3911.038 | 4588.517 |
| **20** | **637.989** | **1647.835** | **3000.188** | **3911.032** | **4587.807** |
| 24 | 637.988 | 1647.824 | 3000.118 | 3911.030 | 4587.550 |
| 30 | 637.988 | 1647.817 | 3000.079 | 3911.029 | 4587.408 |
| Jun et al. (2008) | 637.9 | 1647.8 | 3000.0 | 3911.0 | 4587.3 |

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| Table 2 Comparison of natural frequencies (in Hz) for in-plane bending vibrations of a cantilever laminated beam with experiment | | | | |
| Mode | Experimental | ANSYS® | Present model | |
| Poisson’s effect excluded | Poisson’s effect included |
| 1 | 5.37 | 5.1664 | 5.1075 | 5.1933 |
| 2 | 32.52 | 32.5517 | 31.9799 | 32.5158 |
| 3 | 96.64 | 90.5035 | 89.4149 | 90.9090 |

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| Fig. 3 The laboratory model of the cantilever composite beam for experimental measurements |

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| Fig. 4 EFDD and SSI results for undamaged beam |

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| Fig. 5 SHELL181 element geometry |
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| Fig. 6 Representative FE model of a laminated composite beam in ANSYS® |

*4.2 Verification with experimental result*

In this study, the ambient vibration test of a cantilever composite beam with the lamination scheme of [90/±45/0/90/±45/0/±45/90] is carried out. The test beam has the length *L* = 1 m, width *b* = 80 mm, thickness *h* = 8 mm. The material properties are given by *E*1 = 30.6GPa, *E*2 = 8.5GPa, *G*12 = *G*13 = *G*23 = 3.26GPa, *ν*12 = 0.34, *ρ* = 1920 kg/m3.

B&K3560 data acquisition system with 17 channels, B&K8340-type uniaxial accelerometers and uniaxial signal cables are used in measurements. Recording data is processed in PULSE (2006) and OMA (2006) software for extracting the dynamic characteristics. Ten sensitive accelerometers are located on the beam in vertical direction shown in Fig. 3, and measurements are performed during 10 minutes. Frequency span, FFT analysers and multi-buffer are 0-128Hz, 800 lines and 100 averages, 50 size and 500m update, respectively. Enhanced Frequency Domain Decomposition (EFDD) method and Stochastic Subspace Identification (SSI) method are used to obtain the modal characteristics. Singular values of spectral density matrices (SVSDM) and the average of auto spectral densities (AASD) of data set obtained by EFDD method, stabilization diagram of estimated state space models and select-link modes across data sets obtained by SSI method are shown in Fig. 4 for undamaged beam. The peaks of the curves in the figures indicate the natural frequencies.

Experimental results are compared with those calculated by the present model and ANSYS® in Table 2. In the ANSYS® modelling, SHELL181 element which allows for layered shell definition is used. SHELL181 is a four-node element with six DOFs at each node: translations in the *x*-, *y*-, and *z*-directions, and rotations about the *x*-, *y*-, and *z*-axes (see Fig. 5), and is capable of specifying the thickness, material, fibre orientation and the number of integration points through the thickness of layers. In FE discretization, 150 elements in the length direction and 10 elements in the width direction are used. Fig. 6 shows the representative finite element model for a laminated composite beam in ANSYS®. As seen in Table 2, the present element shows a good agreement with both the experimental and ANSYS® results. The table also shows that the Poisson’s effect is more noticeable on the results when it is taken into account. As previously explained, to ignore this effect the barred stiffness coefficients in Eqs. (8) and (9) are replaced by ones in Eqs. (5).

*4.3 Verification with previous results*

The first example is taken from Goyal and Kapania (2007) for a laminated composite beam with the geometry and material properties as *E*1/*E*2 = 25, *G*12/*E*2 = *G*13/*E*2 = 0.5, *G*23/*E*2 = 0.2, *E*2 = 1.9584×108 psf, *ν*12 = 0.3, *ρ* = 0.250387 slugs/ft3, *h*/*b* = 0.5, *b* = 0.1 ft. Three slenderness ratios of *L*/*h*=10, 20, and 100 are considered. The natural frequencies of bending vibrations of the beam for different end conditions and lay-ups are presented in Table 3 with the non-dimensional form of . It can be seen that the present solutions agree well those of Goyan and Kapania (2007) when the Poisson’s effect is included. It should be noted that the Poisson’s effect is more noticeable for angle-ply lay-up even though the slenderness ratio increases.

As a second example, the fundamental frequencies with the non-dimensional form of for bending vibrations of different symmetric and anti-symmetric laminated beams with and without the Poisson’s effect are calculated and tabulated in Table 4. Geometry and material properties are taken from Goyal and Kapania (2007) as: *E*1/*E*2 = 13.7088, *G*12/*E*2 = 0.5471, *G*13/*E*2 = 0.45679, *G*23/*E*2 = 0.269641, *E*2 = 9.42512 GPa, *ν*12 = 0.3, *ρ* = 1550.0666 kg/m3, *h*/*b* = 0.3175, *b* = 0.01m, *L*/*h* = 60. As can be seen, the present element is in good agreement with that of Goyal and Kapania (2007). As the fibre angle increases, the differences between the present results with and without the Poisson’s effect decrease for the symmetric angle-ply laminates and increase for the anti-symmetric ones.

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| Table 3Non-dimensional fundamental frequencies () for bending vibrations of laminated composite beams with different boundary conditions | | | | | | | | | |
| Beam | *L*/*h*=10 | | | *L*/*h*=20 | | | *L*/*h*=100 | | |
| FEMa | Present | | FEM | Present | | FEM | Present | |
| (-) | (+) | (-) | (+) | (-) | (+) |
| C-F beam |  |  |  |  |  |  |  |  |  |
| 0 | 4.560 | 4.564 | 4.560 | 4.931 | 4.936 | 4.931 | 5.070 | 5.075 | 5.070 |
| 90 | 1.002 | 1.002 | 1.002 | 1.012 | 1.012 | 1.012 | 1.015 | 1.016 | 1.015 |
| [0/90]S | 4.178 | 4.179 | 4.178 | 4.597 | 4.598 | 4.597 | 4.758 | 4.759 | 4.758 |
| [45/-45]S | 1.332 | 1.962 | 1.324 | 1.337 | 2.004 | 1.337 | 1.341 | 2.018 | 1.341 |
|  |  |  |  |  |  |  |  |  |  |
| C-C beam |  |  |  |  |  |  |  |  |  |
| 0 | 17.215 | 17.218 | 17.212 | 25.336 | 25.346 | 25.327 | 31.916 | 31.938 | 31.899 |
| 90 | 5.764 | 5.767 | 5.761 | 6.264 | 6.267 | 6.260 | 6.453 | 6.458 | 6.450 |
| [0/90]S | 14.839 | 14.838 | 14.837 | 22.679 | 22.677 | 22.672 | 29.873 | 29.867 | 29.857 |
| [45/-45]S | 7.623 | 10.240 | 7.616 | 8.280 | 12.014 | 8.280 | 8.531 | 12.811 | 8.526 |
| S-S beam |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
| 0 | 11.636 | 11.645 | 11.635 | 13.431 | 13.444 | 13.430 | 14.211 | 14.227 | 14.210 |
| 90 | 2.771 | 2.774 | 2.771 | 2.829 | 2.832 | 2.829 | 2.849 | 2.852 | 2.848 |
| [0/90]S | 10.488 | 10.489 | 10.488 | 12.435 | 12.438 | 12.434 | 13.335 | 13.338 | 13.333 |
| [45/-45]S | 3.752 | 6.522 | 3.752 | 3.834 | 6.934 | 3.834 | 3.861 | 7.084 | 3.861 |
| (-) Poisson’s effect excluded, (+) Poisson’s effect included  a Goyal and Kapania (2007) | | | | | | | | | |

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| Table 4 Non-dimensional fundamental frequencies () for bending vibrations of laminated composite beams with different boundary conditions | | | | | | | | | |
| Lamina  stacking | C-F beam | | | C-C beam | | | S-S beam | | |
| FEMa | Present | | FEM | Present | | FEM | Present | |
| (-) | (+) | (-) | (+) | (-) | (+) |
| [0/30/0] | 12.858 | 12.921 | 12.858 | 80.354 | 80.682 | 80.310 | 36.051 | 36.227 | 36.042 |
| [0/45/0] | 12.792 | 12.858 | 12.792 | 79.902 | 80.244 | 79.860 | 35.840 | 36.031 | 35.835 |
| [0/60/0] | 12.769 | 12.821 | 12.769 | 79.703 | 79.972 | 79.662 | 35.760 | 35.909 | 35.757 |
| [0/90/0] | 12.778 | 12.808 | 12.777 | 79.692 | 79.853 | 79.651 | 35.780 | 35.865 | 35.777 |
| [0/90]2 | 8.853 | 8.861 | 8.853 | 55.753 | 55.776 | 55.724 | 26.352 | 24.839 | 24.816 |
| [0/±30/0] | 12.403 | 12.709 | 12.403 | 77.580 | 79.387 | 77.542 | 34.745 | 35.595 | 34.741 |
| [0/±45/0] | 12.270 | 12.457 | 12.270 | 76.703 | 77.852 | 76.663 | 34.373 | 34.917 | 34.366 |
| (-) Poisson’s effect excluded, (+) Poisson’s effect included  a Goyal and Kapania (2007) | | | | | | | | | |

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| Table 5. Non-dimensional fundamental frequencies () for bending vibrations of symmetric angle ply  laminated beams with different boundary conditions | | | | | | | | |
| Beam | Method | *θ* |  |  |  |  |  |  |
| 0 | 15 | 30 | 45 | 60 | 75 | 90 |
| C-C | Present | 4.841 | 3.958 | 2.852 | 1.929 | 1.628 | 1.605 | 1.615 |
|  | Analyticala | 4.841 | 3.959 | 2.853 | 1.929 | 1.628 | 1.605 | 1.615 |
|  |  |  |  |  |  |  |  |  |
| C-S | Present | 3.723 | 3.009 | 2.082 | 1.359 | 1.137 | 1.121 | 1.128 |
|  | Analytical | 3.723 | 3.011 | 2.084 | 1.359 | 1.137 | 1.121 | 1.128 |
|  |  |  |  |  |  |  |  |  |
| C-F | Present | 0.979 | 0.729 | 0.485 | 0.314 | 0.263 | 0.259 | 0.261 |
|  | Analytical | 0.979 | 0.729 | 0.485 | 0.314 | 0.263 | 0.259 | 0.261 |
|  |  |  |  |  |  |  |  |  |
| S-S | Present | 2.649 | 2.307 | 1.505 | 0.902 | 0.736 | 0.725 | 0.730 |
|  | Analytical | 2.649 | 2.310 | 1.508 | 0.902 | 0.736 | 0.725 | 0.730 |
| a Jafari-Talookolaei et al. (2012) | | | | | | | | |

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| Table 6. The first three non-dimensional fundamental frequencies () for bending vibrations of the anti-symmetric angle-ply [±45]2 laminated beams with different boundary conditions | | | | |
| Beam | Method | 1st mode | 2nd mode | 3rd mode |
| C-C | Present | 1.977 | 5.185 | 9.587 |
|  | Analyticala | 1.977 | 5.184 | 9.585 |
|  | IGAb | 1.976 | 5.184 | 9.584 |
|  | DQMc | 1.844 | 4.987 | 9.539 |
|  | HSDTd | 1.980 | 5.216 | 9.691 |
|  |  |  |  |  |
| S-S | Present | 0.901 | 3.502 | 7.537 |
|  | Analytical | 0.901 | 3.501 | 7.535 |
|  | IGA | 0.901 | 3.500 | 7.535 |
|  | HSDT | 0.827 | 3.233 | 7.014 |
|  |  |  |  |  |
| C-F | Present | 0.323 | 1.967 | 5.285 |
|  | Analytical | 0.323 | 1.967 | 5.283 |
| a Series solution with Lagrange multipliers by Jafari-Talookolaei et al. (2012)  b Isogeometric FEM by Wang et al. (2015)  c State-space-based differential quadrature method by Chen et al. (2004)  d Higher-order shear deformation theory by Chandrashekhara and Bangera (1992) | | | | |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 7 Non-dimensional frequencies () of cross-ply laminated beams with various end conditions for different orthotropy ratio *E*1/*E*2 (*L*/*h* = 10) | | | | | | | | | | |
| Lay-up | Mode |  | *E*1/*E*2=10 |  |  | *E*1/*E*2=20 |  |  | *E*1/*E*2=40 |  |
| F-F | C-C | S-S | F-F | C-C | S-S | F-F | C-C | S-S |
| [0/90] | 1 | 3.1513 | 2.8588 | 1.4258 | 2.6551 | 2.3087 | 1.2050 | 2.3097 | 1.8773 | 1.0532 |
|  | 2 | 7.0248 | 6.7679 | 5.0825 | 4.9673 | 4.9672 | 4.1531 | 3.5124 | 3.5124 | 3.4432 |
|  | 3 | 7.7034 | 7.0248 | 7.0248 | 6.2672 | 5.2777 | 4.9673 | 5.1553 | 4.1100 | 3.5124 |
|  | 4 | 13.1008 | 11.5098 | 9.5761 | 9.9347 | 8.7945 | 7.7719 | 7.0249 | 6.7067 | 6.2278 |
|  | 5 | 14.0497 | 14.0497 | 11.2529 | 10.3416 | 9.9346 | 9.8739 | 8.1703 | 7.0248 | 7.0249 |
|  |  |  |  |  |  |  |  |  |  |  |
| [0/90/0] | 1 | 5.5086 | 4.2676 | 2.5124 | 4.9672 | 3.4257 | 2.3004 | 3.5124 | 2.6178 | 2.0009 |
|  | 2 | 7.0248 | 7.0248 | 7.0248 | 4.9789 | 4.9672 | 4.9672 | 4.2547 | 3.5124 | 3.5124 |
|  | 3 | 11.8530 | 9.1669 | 7.9853 | 9.6189 | 7.0397 | 6.5576 | 7.0248 | 5.2563 | 5.1012 |
|  | 4 | 14.0497 | 14.0497 | 13.1808 | 9.9346 | 9.9346 | 9.9346 | 7.4024 | 7.0248 | 7.0248 |
|  | 5 | 18.4587 | 14.8303 | 14.0497 | 14.2652 | 11.1671 | 10.9433 | 10.5375 | 8.2073 | 8.1386 |
|  |  |  |  |  |  |  |  |  |  |  |
| [0/90]2 | 1 | 4.0823 | 3.4478 | 1.8506 | 3.7095 | 2.8545 | 1.6942 | 3.2974 | 2.2684 | 1.5244 |
|  | 2 | 7.0248 | 7.0248 | 6.3106 | 4.9672 | 4.9672 | 4.9672 | 3.5124 | 3.5124 | 3.5124 |
|  | 3 | 9.4589 | 7.7700 | 7.0248 | 7.9265 | 6.1053 | 5.3393 | 6.3666 | 4.6586 | 4.3388 |
|  | 4 | 14.0497 | 12.8514 | 11.3007 | 9.9346 | 9.8602 | 9.4070 | 7.0248 | 7.0248 | 7.0248 |
|  | 5 | 15.4342 | 14.0497 | 12.0456 | 12.2819 | 9.9346 | 9.9346 | 9.4320 | 7.3888 | 7.2368 |
|  |  |  |  |  |  |  |  |  |  |  |
| [0/90]S | 1 | 5.2541 | 4.0368 | 2.3982 | 4.7225 | 3.2246 | 2.1841 | 3.5124 | 2.4569 | 1.8912 |
|  | 2 | 7.0248 | 7.0248 | 7.0248 | 4.9672 | 4.9672 | 4.9672 | 4.0169 | 3.5124 | 3.5124 |
|  | 3 | 11.2221 | 8.6373 | 7.5666 | 9.0593 | 6.6126 | 6.1807 | 6.9497 | 4.9294 | 4.7919 |
|  | 4 | 14.0497 | 13.9501 | 11.6847 | 9.9346 | 9.9346 | 9.9346 | 7.0248 | 7.0248 | 7.0248 |
|  | 5 | 17.4050 | 14.0497 | 13.3565 | 13.4017 | 10.4782 | 10.2794 | 9.9865 | 7.6902 | 7.6291 |

Third example considers the laminated composite beams with different boundary condition and different angle-ply lamina sequences. Geometry and material properties are the same with those used in the convergence study. Natural frequencies are presented in a non-dimensional form as  in Tables 5 and 6.

In Table 5, non-dimensional fundamental frequencies for bending vibrations of symmetric angle ply  laminates obtained by the present method are compared with those of the series load solution in conjunction with Lagrange multipliers given by Jafari-Talookolaei et al. (2012). As seen, the results are in excellent agreement.

Table 6 gives the non-dimensional natural frequencies of the laminated beams with anti-symmetric angle-ply [±45]2 lamina stacking and different boundary conditions. Present results are compared with the analytical solution by Jafari-Talookolaei et al. (2012), isogeometric finite element method by Wang et al. (2015), state-space-based differential quadrature method by Chen et al. (2004) and the finite element based on a higher-order shear deformation theory by Chandrashekhara and Bangera (1992) for testing the validity of the present method. According to the table, the present element very close results to the analytical ones compared to other methods.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 8 Non-dimensional frequencies () of angle-ply laminated beams with various end conditions for different orthotropy ratio *E*1/*E*2 (*L*/*h* = 10) | | | | | | | | | | |
| Lay-up | Mode |  | *E*1/*E*2=10 |  |  | *E*1/*E*2=20 |  |  | *E*1/*E*2=40 |  |
| F-F | C-C | S-S | F-F | C-C | S-S | F-F | C-C | S-S |
|  |  |  |  |  |  |  |  |  |  |  |
| 30 | 1 | 3.7253 | 3.3092 | 1.6817 | 3.0561 | 2.6098 | 1.3832 | 2.4081 | 1.9867 | 1.0926 |
|  | 2 | 9.0202 | 7.7588 | 5.9974 | 7.1584 | 5.9361 | 4.7769 | 5.4759 | 4.4164 | 3.6663 |
|  | 3 | 13.0994 | 13.0994 | 9.9093 | 11.7953 | 9.8646 | 8.2946 | 8.8273 | 7.2557 | 6.6264 |
|  | 4 | 15.2823 | 13.1164 | 11.6825 | 12.3513 | 12.3513 | 9.0295 | 11.5680 | 10.2584 | 6.7645 |
|  | 5 | 20.6728 | 18.9228 | 13.5814 | 16.4769 | 14.0623 | 12.9268 | 12.1405 | 11.5680 | 9.9685 |
|  |  |  |  |  |  |  |  |  |  |  |
| 45 | 1 | 2.5212 | 2.3668 | 1.1345 | 1.8473 | 1.7249 | 0.8315 | 1.3356 | 1.2428 | 0.6013 |
|  | 2 | 6.3869 | 5.8434 | 4.2263 | 4.6596 | 4.2352 | 3.0847 | 3.3594 | 3.0405 | 2.2246 |
|  | 3 | 11.3103 | 10.2354 | 6.4763 | 8.2141 | 7.3871 | 4.7454 | 5.9046 | 5.2892 | 3.4279 |
|  | 4 | 12.5565 | 12.5565 | 8.6292 | 9.3848 | 9.3848 | 6.2680 | 6.8212 | 6.8212 | 4.5062 |
|  | 5 | 15.5570 | 15.1513 | 13.7758 | 12.1270 | 10.9001 | 9.9617 | 8.6939 | 7.7889 | 7.1412 |
|  |  |  |  |  |  |  |  |  |  |  |
| 60 | 1 | 2.0593 | 1.9548 | 0.9262 | 1.4571 | 1.3830 | 0.6553 | 1.0308 | 0.9784 | 0.4636 |
|  | 2 | 5.2634 | 4.8840 | 3.4798 | 3.7240 | 3.4551 | 2.4620 | 2.6344 | 2.4440 | 1.7417 |
|  | 3 | 9.4112 | 8.6360 | 5.2317 | 6.6581 | 6.1091 | 3.7005 | 4.7098 | 4.3211 | 2.6172 |
|  | 4 | 10.4324 | 10.4324 | 7.1778 | 7.3933 | 7.3933 | 5.0781 | 5.2321 | 5.2321 | 3.5921 |
|  | 5 | 13.5095 | 12.8782 | 11.5705 | 9.9609 | 9.1094 | 8.1851 | 7.0458 | 6.4430 | 5.7897 |
|  |  |  |  |  |  |  |  |  |  |  |
| 30 | 1 | 3.3756 | 3.0834 | 1.6927 | 2.7598 | 2.4356 | 1.3876 | 2.2200 | 1.8856 | 1.0935 |
|  | 2 | 8.2827 | 7.3303 | 5.4603 | 6.5997 | 5.6366 | 4.3723 | 5.1576 | 4.2560 | 3.4364 |
|  | 3 | 12.7622 | 12.5013 | 10.0553 | 10.8015 | 9.4573 | 8.3856 | 8.3996 | 7.0452 | 6.5065 |
|  | 4 | 14.5338 | 12.9390 | 10.9676 | 12.0766 | 11.7095 | 8.5393 | 10.7922 | 10.0050 | 6.6703 |
|  | 5 | 20.1106 | 18.1635 | 12.3728 | 15.6287 | 13.5923 | 10.9907 | 11.7395 | 10.7284 | 9.4333 |
|  |  |  |  |  |  |  |  |  |  |  |
| 45 | 1 | 2.4421 | 2.3052 | 1.1368 | 1.7989 | 1.6874 | 0.8322 | 1.3121 | 1.2246 | 0.6014 |
|  | 2 | 6.2039 | 5.7143 | 4.1001 | 4.5506 | 4.1587 | 3.0102 | 3.3073 | 3.0041 | 2.1892 |
|  | 3 | 10.9950 | 10.0404 | 6.4921 | 8.0412 | 7.2745 | 4.7531 | 5.8249 | 5.2365 | 3.4311 |
|  | 4 | 12.9841 | 12.9841 | 8.4342 | 9.5061 | 9.5061 | 6.1545 | 6.8622 | 6.8622 | 4.4527 |
|  | 5 | 13.2656 | 13.2187 | 12.7314 | 11.9114 | 10.7559 | 9.7438 | 8.5954 | 7.7228 | 7.0549 |
|  |  |  |  |  |  |  |  |  |  |  |
| 60 | 1 | 2.0509 | 1.9480 | 0.9264 | 1.4528 | 1.3796 | 0.6554 | 1.0290 | 0.9769 | 0.4636 |
|  | 2 | 5.2436 | 4.8695 | 3.4662 | 3.7141 | 3.4478 | 2.4553 | 2.6301 | 2.4408 | 1.7388 |
|  | 3 | 9.3768 | 8.6137 | 5.2332 | 6.6420 | 6.0979 | 3.7011 | 4.7030 | 4.3163 | 2.6175 |
|  | 4 | 10.4664 | 10.4664 | 7.1562 | 7.4022 | 7.4022 | 5.0674 | 5.2349 | 5.2349 | 3.5876 |
|  | 5 | 11.5043 | 11.4998 | 11.3466 | 9.9399 | 9.0949 | 8.1647 | 7.0369 | 6.4368 | 5.7820 |

*4.4 Parametric study*

In this section, the non-dimensional natural frequencies and corresponding mode shapes of laminated composite beams with various configurations are presented. The beams have the slenderness *L*/*h* = 10, and cross-sectional dimensions are assumed to be unity. The Poisson’s effect is considered. All frequencies are given in the dimensionless form as . Some new results which can be used a referencing data for future studies are presented. Unless otherwise stated, the material properties of the beam are *E*1/ *E*2 = Open, *G*12 / *E*2 = *G*13/*E*2 = 0.5, *G*23/*E*2 = 0.2, *ν* = 0.3, *ρ* = 1550 kg/m3.

Tables 7-9 show the results of laminated composite beams with different lay-ups and boundary conditions (C-C, S-S and F-F) for various orthotropy ratio such as *E*1/*E*2 = 10, 20, and 40. As seen, for both cross-ply and angle-ply laminates, when the orthotropy ratio increases, the non-dimensional natural frequencies decrease. F-F beam has the greatest natural frequencies compared to the others. According to Tables 8 and 9, for angle-ply laminates, the non-dimensional frequencies decrease with increasing the fibre angle.

In Figs. 7-9, the first five mode shapes of vibration for unsymmetric cross-ply, antisymmetric and unsymmetric angle-ply laminated composite beams with C-C, S-S and F-F boundary conditions and the orthotropy ratio *E*1/*E*2 = 40 are presented to show the effect of material coupling among the displacement components of motion. As seen in Fig. 7, for [0/90] unsymmetric cross-ply laminated beams, bending and torsional vibration components are dominant in the lowest modes. For the lower modes considered, bending-torsion coupling does not appear; however, a small amount of extension-bending coupling is observed. According to Fig. 8, for [45/-45] antisymmetric angle-ply laminated beams, bending modes of vibration are generally observed, and there is no coupling between the bending and torsional components. In some modes, e.g., 3rd mode of S-S beam, 4th mode of F-F and C-C beams, the extensional and torsional components are greatly coupled. For [0/45] unsymmetric angle-ply laminated beams, as can be seen in Fig. 9, the mode shapes show great complexity and involve various couplings among the extensional, bending and torsional components of the motion.

**5. Conclusion**

This study presents a finite element model with five nodes and 13 DOFs for free vibration analysis of laminated composite beams with arbitrary lay-ups using the first-order shear deformation theory. The model considers simultaneously the material couplings among extensional, bending and torsional deformations due to anisotropy and the Poisson’s effect. Applying Lagrange’s principle gives the equations of motion and thus element matrices of the laminated composite beam. The free vibration problem is solved as a standard eigenvalue problem to obtain the natural frequencies and corresponding mode shapes. The current element is verified by comparing with experimental study, the results reported in the literature as well as those obtained by ANSYS® finite element analysis software to show its accuracy. Some parametric results can be used for benchmarking for the future studies.

The present element can capture vibration modes of the laminated composite beams in a good accuracy. According to the study, for unsymmetric angle-ply laminated beams, the material coupling greatly affects the vibration modes.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Table 9 Non-dimensional frequencies () of unsymmetric laminated beams with various end conditions for different orthotropy ratio *E*1/*E*2 (*L*/*h* = 10) | | | | | | | | | | |
| Lay-up | Mode |  | *E*1/*E*2=10 |  |  | *E*1/*E*2=20 |  |  | *E*1/*E*2=40 |  |
| F-F | C-C | S-S | F-F | C-C | S-S | F-F | C-C | S-S |
|  |  |  |  |  |  |  |  |  |  |  |
| 30 | 1 | 3.7884 | 3.4203 | 1.8578 | 3.0803 | 2.6935 | 1.5497 | 2.5562 | 2.1228 | 1.2961 |
|  | 2 | 9.0361 | 7.8864 | 5.8893 | 7.1505 | 6.0149 | 4.6317 | 5.7027 | 4.5874 | 3.7123 |
|  | 3 | 9.0840 | 8.9985 | 8.9643 | 7.1601 | 7.0978 | 7.0435 | 5.7067 | 5.6539 | 5.5696 |
|  | 4 | 15.3445 | 13.2944 | 11.3589 | 11.8260 | 9.9750 | 8.9260 | 9.1575 | 7.4914 | 6.9546 |
|  | 5 | 17.7096 | 17.6819 | 12.5303 | 13.7819 | 13.7906 | 11.0756 | 10.8193 | 10.5570 | 9.6712 |
|  |  |  |  |  |  |  |  |  |  |  |
| 45 | 1 | 3.3755 | 3.0726 | 1.5703 | 2.7997 | 2.4558 | 1.3078 | 2.3986 | 1.9812 | 1.1203 |
|  | 2 | 8.2389 | 7.2573 | 5.4104 | 6.6225 | 5.6145 | 4.3599 | 5.1581 | 4.3507 | 3.5804 |
|  | 3 | 8.8447 | 8.8304 | 8.7937 | 6.7681 | 6.7550 | 6.7329 | 5.4027 | 5.1477 | 5.1350 |
|  | 4 | 14.0380 | 12.3351 | 10.2446 | 10.9832 | 9.3686 | 8.2449 | 8.6388 | 7.1224 | 6.5699 |
|  | 5 | 17.5443 | 17.5700 | 11.7685 | 13.3933 | 13.3970 | 10.3045 | 10.1981 | 10.0463 | 9.0011 |
|  |  |  |  |  |  |  |  |  |  |  |
| 60 | 1 | 3.2071 | 2.9200 | 1.4558 | 2.6913 | 2.3540 | 1.2247 | 2.3355 | 1.9146 | 1.0667 |
|  | 2 | 7.8581 | 6.9297 | 5.1793 | 5.8948 | 5.3988 | 4.2206 | 4.2990 | 4.2020 | 3.4970 |
|  | 3 | 8.0350 | 8.0335 | 8.0312 | 6.3789 | 5.8945 | 5.8944 | 5.2475 | 4.3045 | 4.3008 |
|  | 4 | 13.3980 | 11.8056 | 9.7589 | 10.5644 | 9.0166 | 7.9253 | 8.3548 | 6.8814 | 6.3608 |
|  | 5 | 16.0572 | 16.0573 | 11.4032 | 11.7803 | 11.7791 | 9.9943 | 8.5914 | 8.5902 | 8.5863 |
|  |  |  |  |  |  |  |  |  |  |  |
| 30 | 1 | 3.1936 | 2.9300 | 1.4390 | 2.5567 | 2.2868 | 1.1537 | 2.0523 | 1.7786 | 0.9280 |
|  | 2 | 7.9414 | 7.0704 | 5.2652 | 6.2261 | 5.3931 | 4.1371 | 4.8674 | 4.0891 | 3.2436 |
|  | 3 | 9.5108 | 9.5108 | 8.6637 | 7.9852 | 7.9852 | 7.1855 | 7.0409 | 6.8347 | 5.9821 |
|  | 4 | 13.7969 | 12.1784 | 10.5345 | 10.6033 | 9.1499 | 8.1036 | 8.1010 | 7.0409 | 6.1983 |
|  | 5 | 19.0217 | 17.8039 | 11.6387 | 15.1942 | 13.2335 | 9.9392 | 11.4008 | 9.7827 | 8.4868 |
|  |  |  |  |  |  |  |  |  |  |  |
| 45 | 1 | 2.3844 | 2.2558 | 1.0726 | 1.7527 | 1.6494 | 0.7886 | 1.2843 | 1.2023 | 0.5780 |
|  | 2 | 6.0782 | 5.6156 | 4.0195 | 4.4488 | 4.0827 | 2.9433 | 3.2463 | 2.9600 | 2.1486 |
|  | 3 | 10.0238 | 9.9003 | 6.3632 | 7.8948 | 7.1655 | 4.6768 | 5.7352 | 5.1733 | 3.3949 |
|  | 4 | 10.8360 | 10.0238 | 8.2653 | 8.0375 | 8.0375 | 6.0228 | 6.2886 | 6.2886 | 4.3761 |
|  | 5 | 14.6862 | 14.6862 | 11.2906 | 11.6768 | 10.6233 | 9.6350 | 8.4845 | 7.6451 | 6.9698 |
|  |  |  |  |  |  |  |  |  |  |  |
| 60 | 1 | 2.0444 | 1.9424 | 0.9194 | 1.4486 | 1.3760 | 0.6515 | 1.0267 | 0.9750 | 0.4617 |
|  | 2 | 5.2293 | 4.8580 | 3.4569 | 3.7044 | 3.4403 | 2.4489 | 2.6250 | 2.4369 | 1.7354 |
|  | 3 | 9.3576 | 8.5971 | 5.2214 | 6.6274 | 6.0868 | 3.6952 | 4.6950 | 4.3104 | 2.6149 |
|  | 4 | 9.7018 | 9.7018 | 7.1367 | 7.2457 | 7.2457 | 5.0546 | 5.1925 | 5.1925 | 3.5808 |
|  | 5 | 11.0895 | 11.0895 | 10.2595 | 8.9759 | 8.9759 | 8.1526 | 7.0266 | 6.4293 | 5.7740 |

|  |  |  |
| --- | --- | --- |
| Ω1 = 2.3097 | Ω1 = 1.8773 | Ω1 = 1.0532 |
| Ω2 = 3.5124 | Ω2 = 3.5124 | Ω2 = 3.4432 |
| Ω3 = 5.1553 | Ω3 = 4.1100 | Ω3 = 3.5124 |
| Ω4 = 7.0249 | Ω4 = 6.7067 | Ω4 = 6.2278 |
| Ω5 = 8.1703    F-F Beam | Ω5 = 7.0248    C-C Beam | Ω5 = 7.0249    S-S Beam |
| Fig. 7 The lowest five mode shapes of [0/90] unsymmetric cross-ply laminated composite beams with *L*/*h*=10 and *E*1/*E*2=40 | | |

|  |  |  |
| --- | --- | --- |
| Ω1 = 1.2843 | Ω1 = 1.2023 | Ω1 = 0.5780 |
| Ω2 = 3.2463 | Ω2 = 2.9600 | Ω2 = 2.1486 |
| Ω3 = 5.7352 | Ω3 = 5.1733 | Ω3 = 3.3949 |
| Ω4 = 6.2886 | Ω4 = 6.2886 | Ω4 = 4.3761 |
| Ω5 = 8.4845    F-F Beam | Ω5 = 7.6451    C-C Beam | Ω5 = 6.9698    S-S Beam |
| Fig. 8 The lowest five mode shapes of [45/-45] antisymmetric angle-ply laminated composite beams with *L*/*h*=10 and *E*1/*E*2=40 | | |

|  |  |  |
| --- | --- | --- |
| Ω1 = 2.3986 | Ω1 = 1.9812 | Ω1 = 1.1203 |
| Ω2 = 5.1581 | Ω2 = 4.3507 | Ω2 = 3.5804 |
| Ω3 = 5.4027 | Ω3 = 5.1477 | Ω3 = 5.1350 |
| Ω4 = 8.6388 | Ω4 = 7.1224 | Ω4 = 6.5699 |
| Ω5 = 10.1981    F-F Beam | Ω5 = 10.0463    C-C Beam | Ω5 = 9.0011    S-S Beam |
| Fig. 9 The lowest five mode shapes of [0/45] unsymmetric angle-ply laminated composite beams with *L*/*h*=10 and *E*1/*E*2=40 | | |

**Appendix**

The shape functions for the present finite element model are

|  |  |
| --- | --- |
|  | (A1) |

where  and *l* is the element length.

The element mass and stiffness matrices are in the form

|  |  |
| --- | --- |
|  | (A2) |

where

|  |  |
| --- | --- |
|  | (A3) |

|  |  |
| --- | --- |
|  | (A4) |

where the stiffness and inertia coefficients are given by Eqs. (5), (8) and (9), respectively.

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